A Lattice Theory for Solving Games of Imperfect Information

Martin De Wulf, Laurent Doyen, and Jean-François Raskin

HSCC'06 - March 31st - Santa Barbara, CA

Motivation

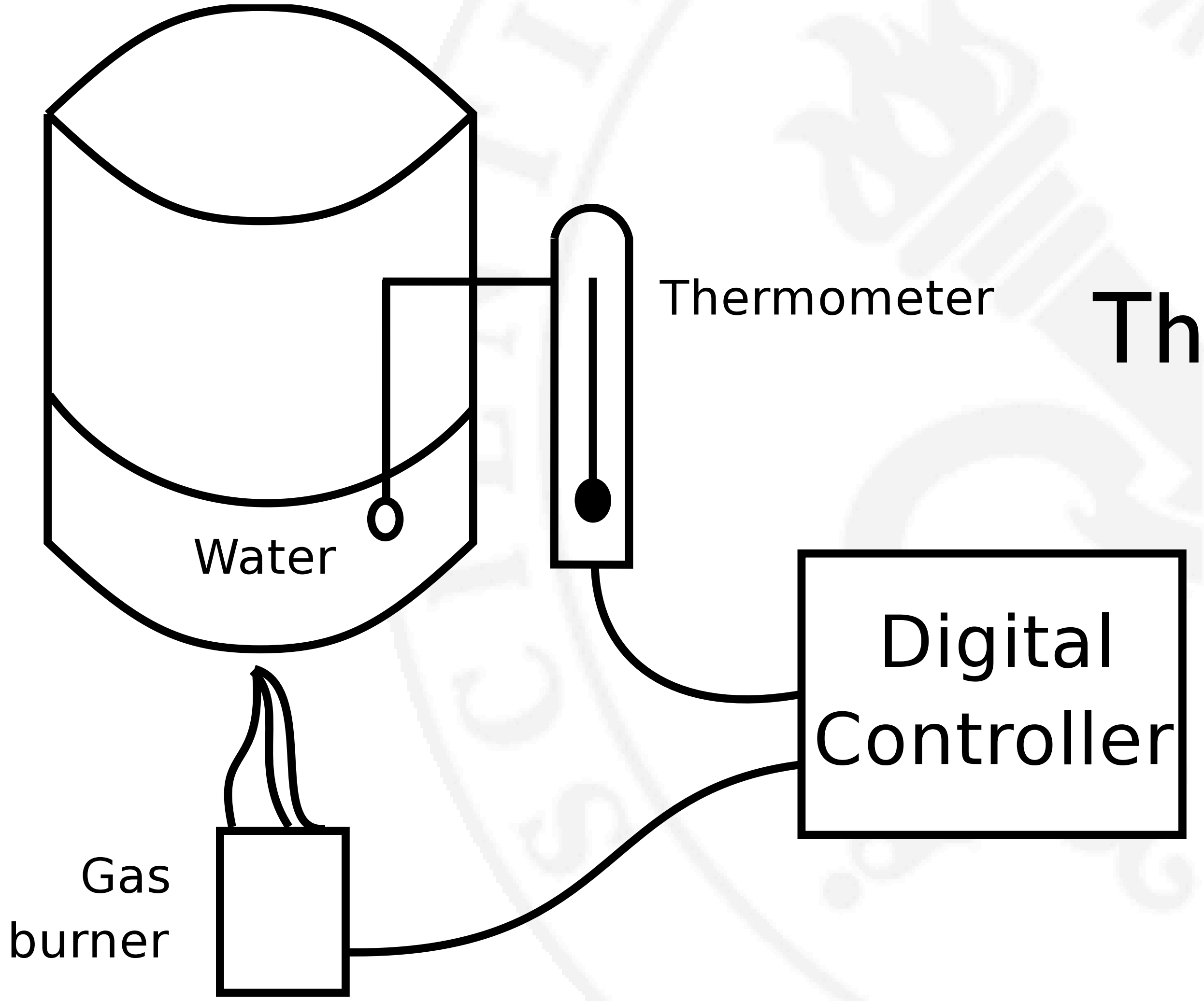
- Controller Generation

$$\boxed{\ ?\ \|\ Env \text{ avoid } Bad\ }$$

Controller

- Two-Player Games :
  - Controller = Player 1
  - Environment = Player 2
- We are looking for a winning strategy for Player 1

Imperfect Information = Finite Precision

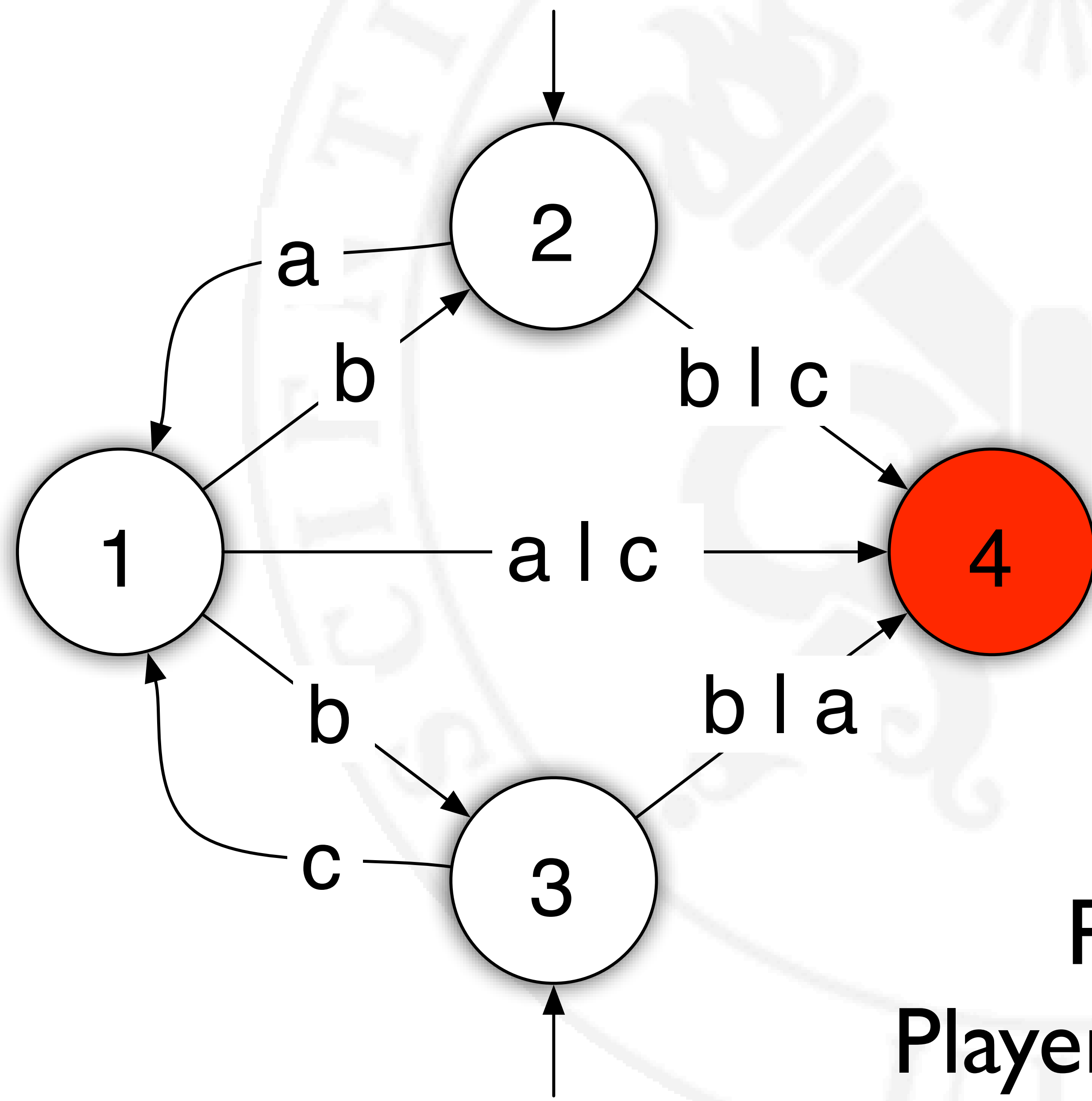


The temperature is in the interval

$$(c-1, c+1)$$

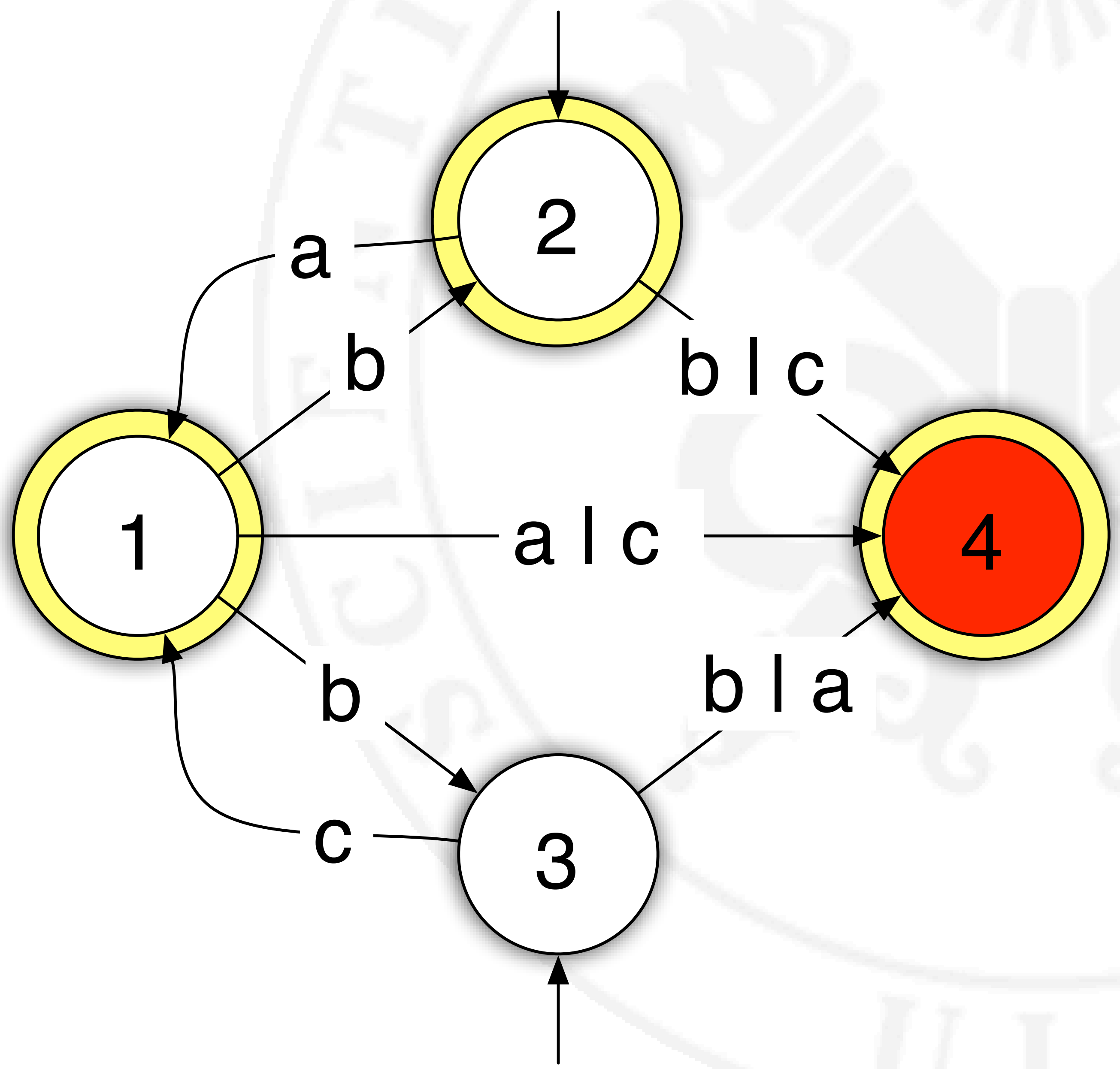
Typical Hybrid System

A Game Example



Player 1 chooses a letter
Player 2 resolves nondeterminism

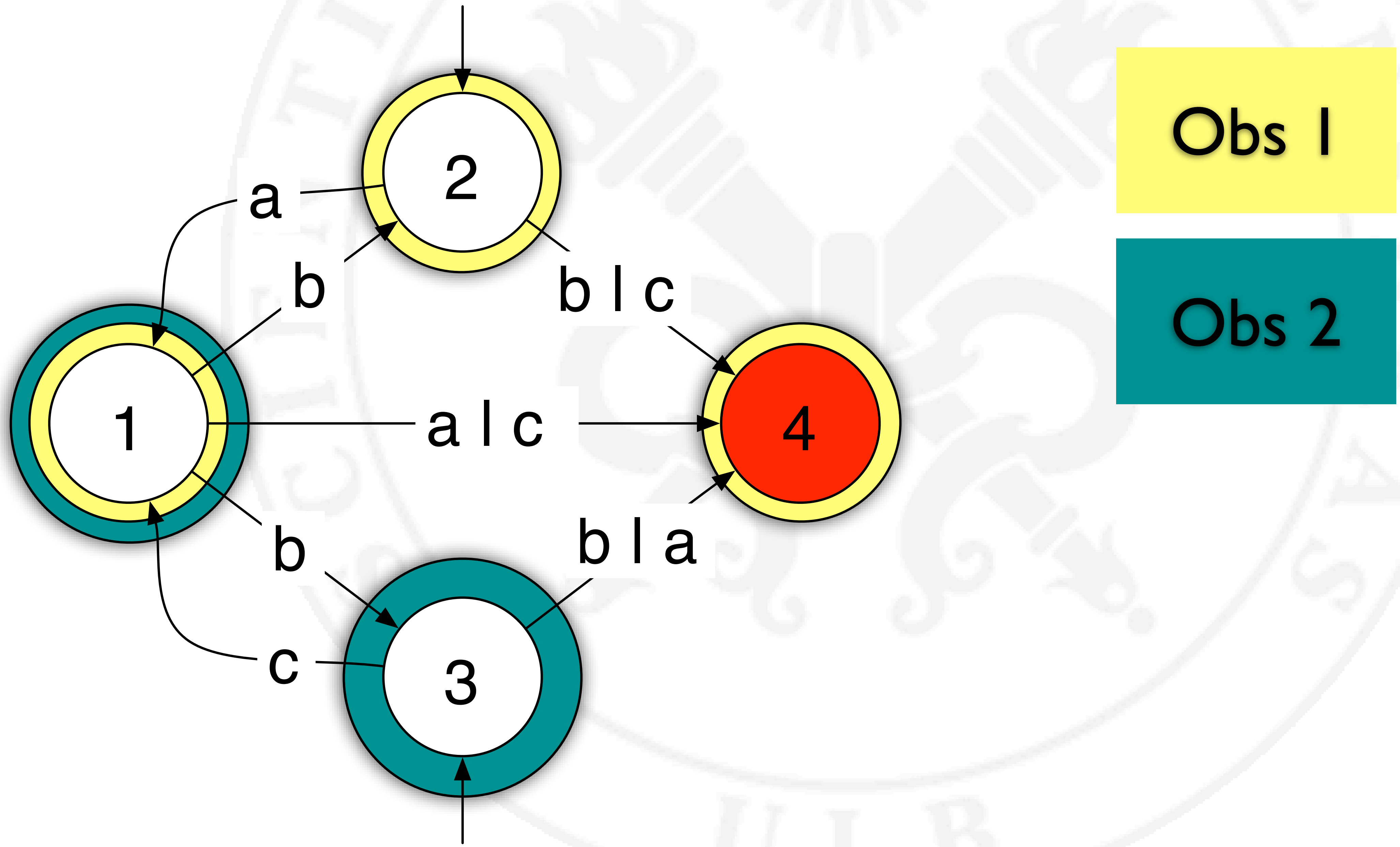
Imperfect Observations



Obs 1

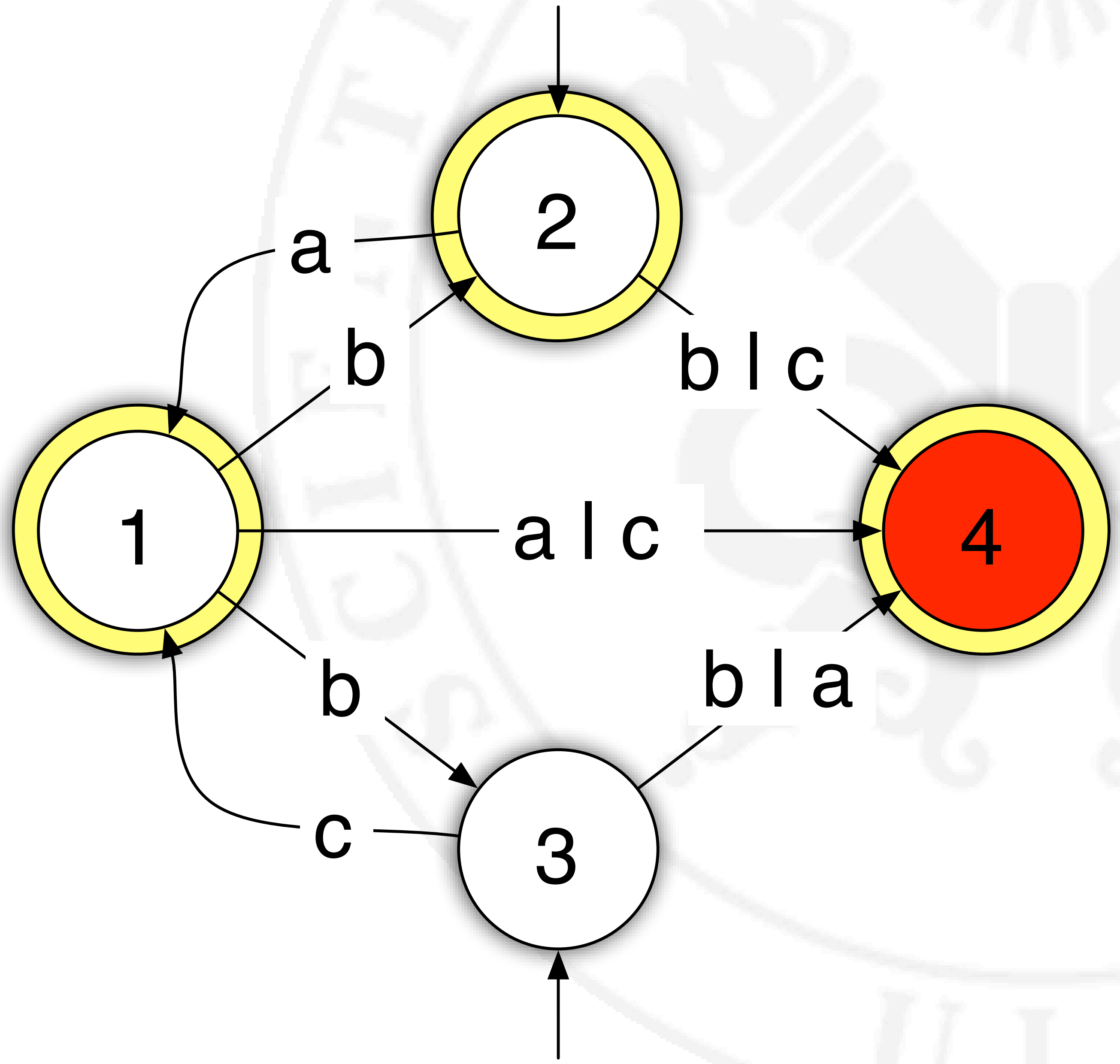
One observation, many possible states

Imperfect Observations



One state, many possible observations

Memory is needed to win



Obs 1

Player 1 cannot win with a strategy based only on the current observation

Games / Strategies

Definition 1 [Two-player games] A *two-player game* is a tuple $\langle S, S_0, \Sigma, \rightarrow \rangle$.

Definition 2 [Observation set] An *observation set* of S is a couple (Obs, γ) where $\gamma : \mathsf{Obs} \rightarrow 2^S$.

Definition 3 [Observation based strategy] An *observation based strategy* is a function $\lambda : \mathsf{Obs}^+ \rightarrow \Sigma$.

Our objective is to find an algorithm to construct strategies that avoid a set of Bad states.

Games / Strategies

Definition 1 [Two-player games] A *two-player game* is a tuple $\langle S, S_0, \Sigma, \rightarrow \rangle$.

Definition ... $\!$ple (Obs, γ)
where $\gamma : \mathsf{Ob}$ ...

Definition ... *strategy* is a
function $\lambda :$ ...

Those games generalize games of *perfect information* where $\mathsf{Obs} = S$ and γ is the identity function

Our objective is to find an algorithm to construct strategies that avoid a set of Bad states.

Games / Strategies

Definition 1 [Two-player games] A *two-player game* is a tuple $\langle S, S_0, \Sigma, \rightarrow \rangle$.

Definition ... ple (Obs, γ)
where γ : Ob...

Definition ... is a
function λ : ...

Those games generalize games
of p...
whe...
iden...

Those games generalize games
of *incomplete information*:
in that case **Obs partitions**
the state space S. [Rei84]

Our objective is to find an algorithm to construct strategies
that avoid a set of Bad states.

Classical Approaches

- For games of perfect information :
  - fixed point algorithm using a controllable predecessor operator
- For games of incomplete information
  - [Reif84] build a game of perfect information using a kind of subset construction

A fixed point algorithm

We define a *controllable predecessor* operator for a *set of sets of states* q

$$\mathsf{CPre}(q) = \{s \subseteq \overline{\mathsf{Bad}} \mid \exists \sigma \in \Sigma \cdot \forall \mathsf{obs} \in \mathsf{Obs} \cdot \exists s' \in q : \mathsf{Post}_\sigma(s) \cap \gamma(\mathsf{obs}) \subseteq s'\}$$

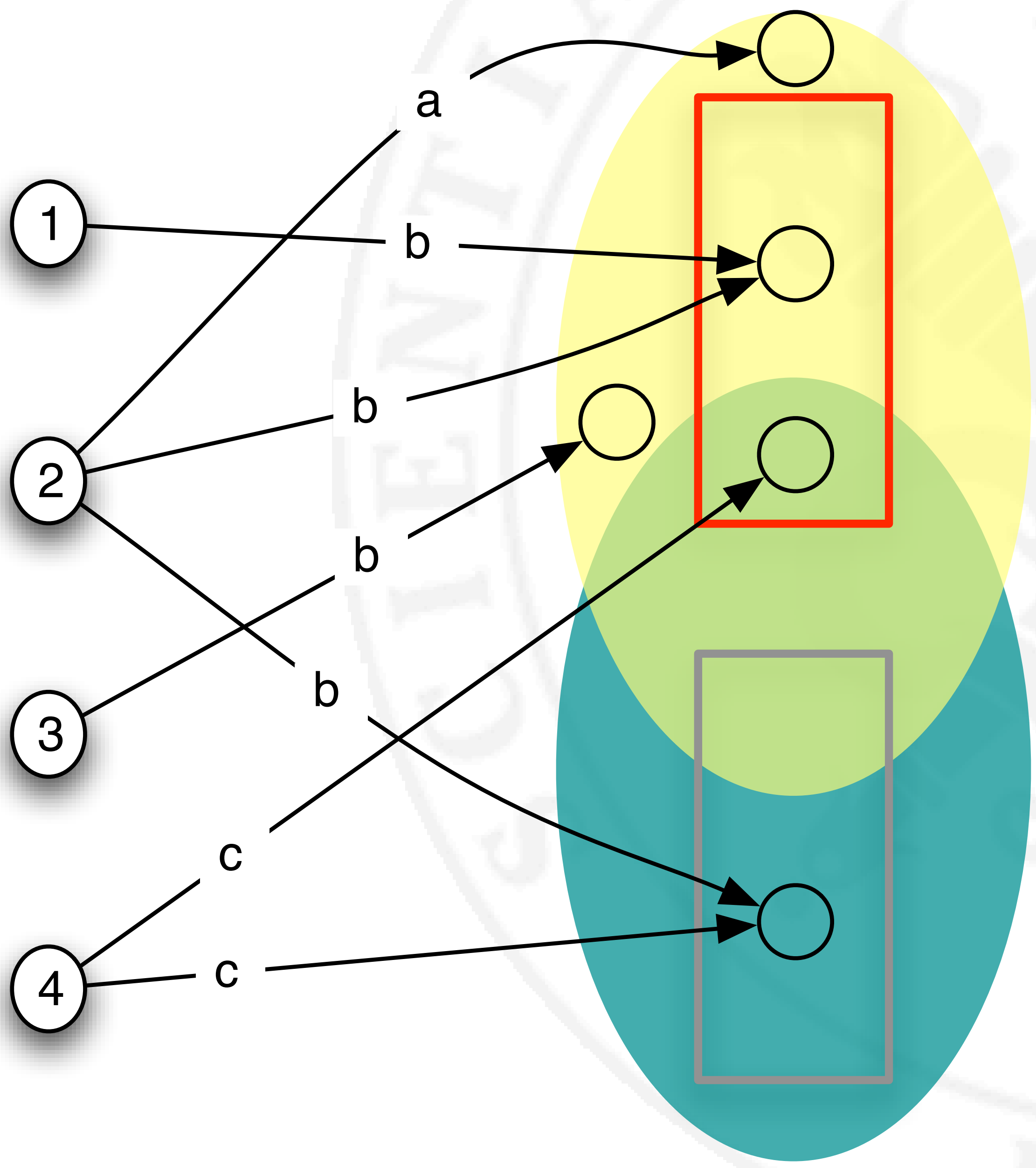
(i) s does not intersect with **Bad**,

(ii) there exists σ s.t. the set of possible successors of s by σ is covered by q

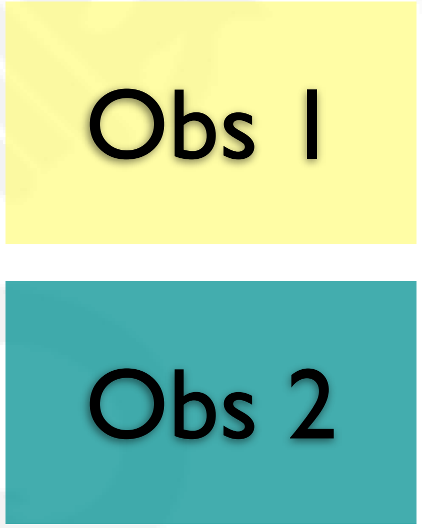
(a) no matter how the adversary resolves non-determinism,

(b) no matter the compatible observation **Obs**

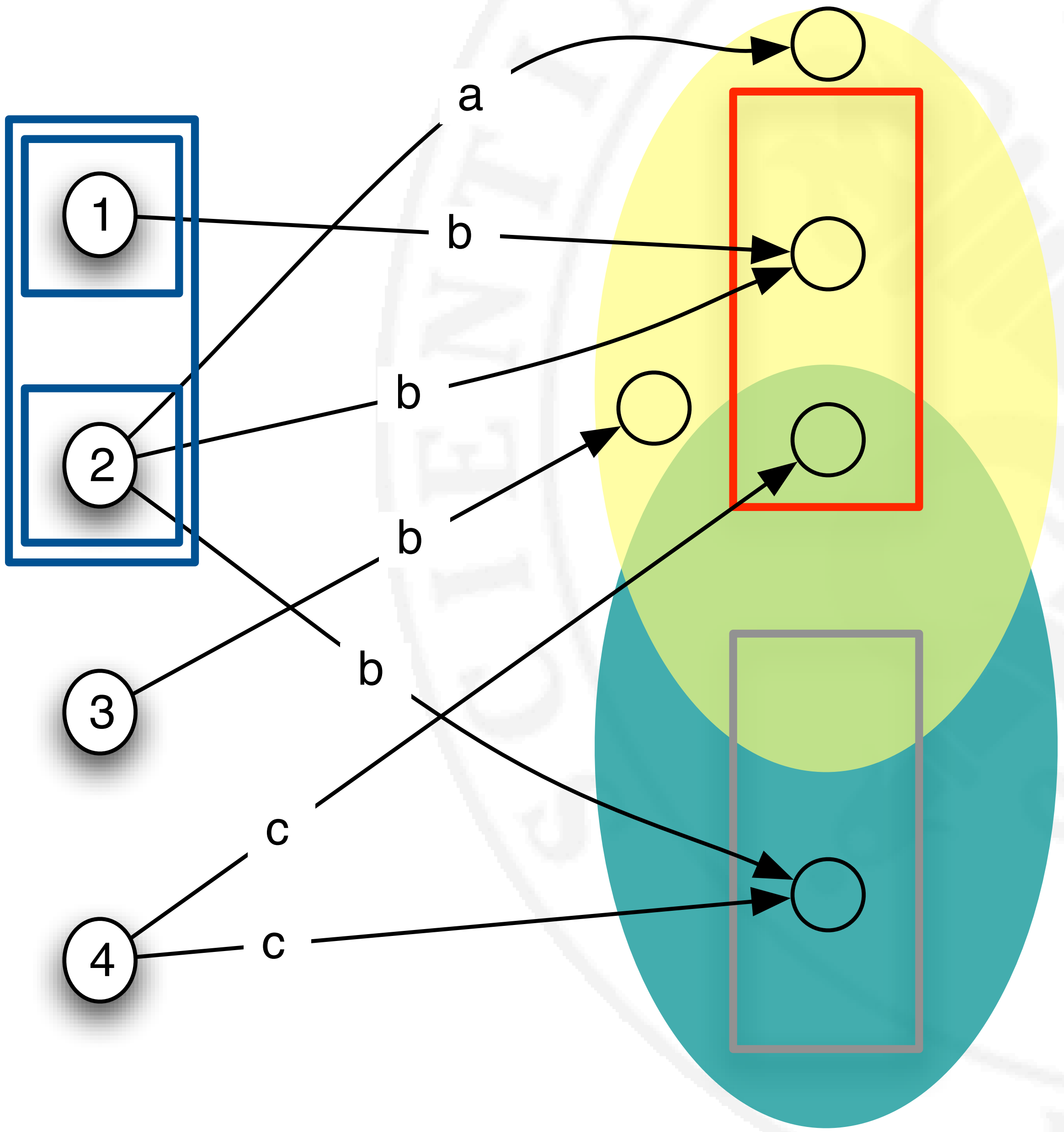
Example



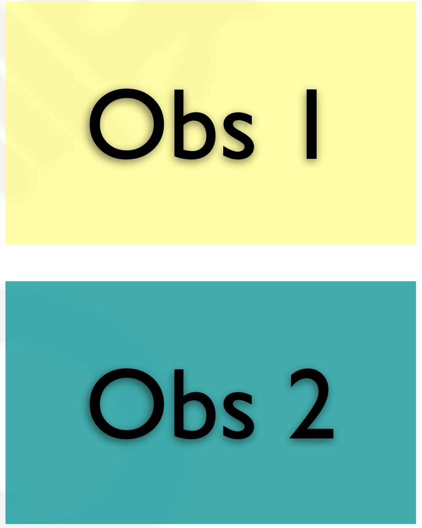
q ={A, B}



Example



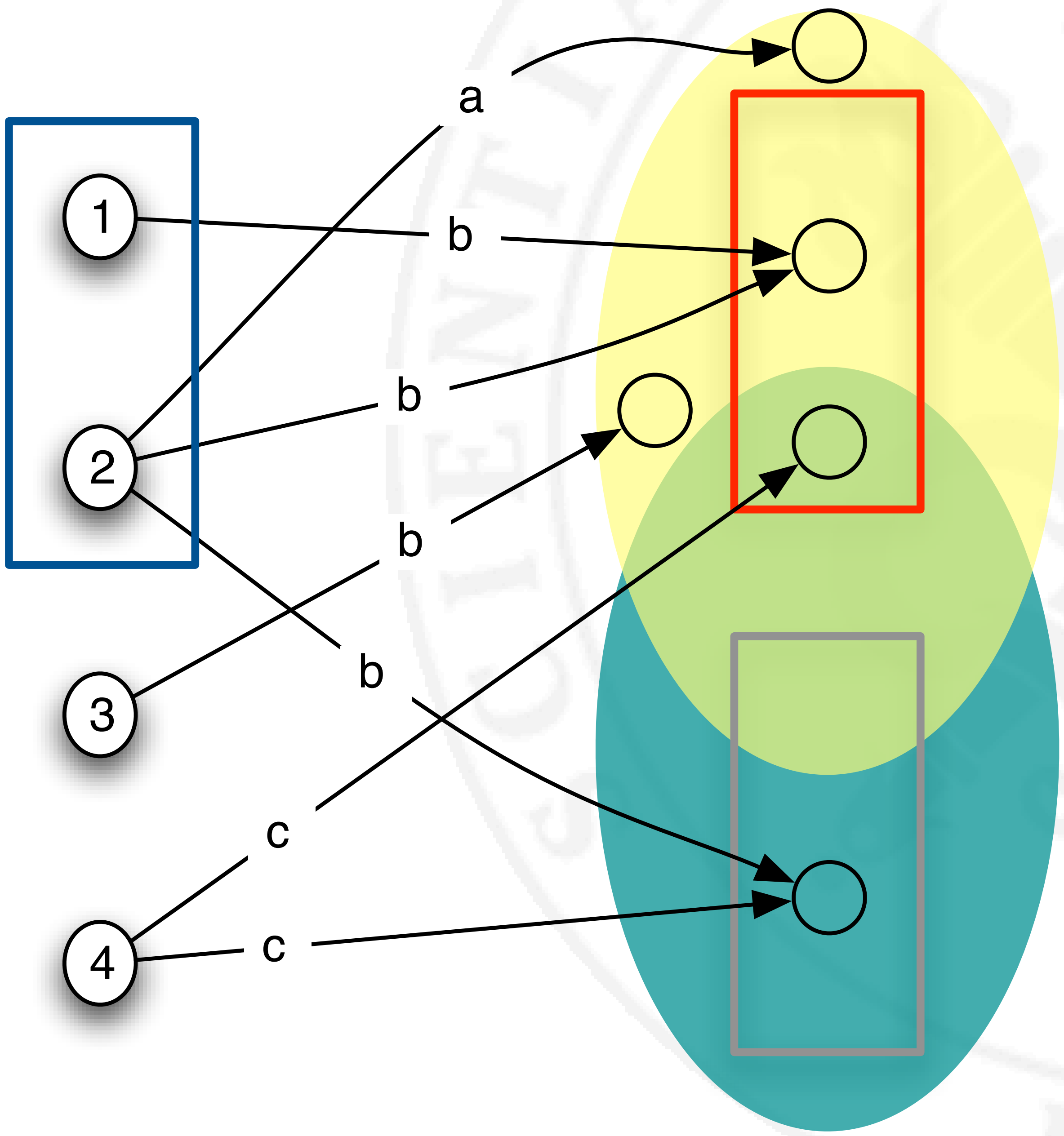
q ={A, B}



Cpre({A,B})= Blue sets

Maximal sets

If there is a strategy for set A,
there is a strategy for any B included in A



It is enough to keep only
the maximal sets

$$\mathsf{CPre}(q) = [\{s \subseteq \overline{\mathsf{Bad}} \mid \exists \sigma \in \Sigma \cdot \forall \mathsf{obs} \in \mathsf{Obs} \cdot \exists s' \in q : \mathsf{Post}_\sigma(s) \cap \gamma(\mathsf{obs}) \subseteq s'\}]$$

Antichains

Definition 4 [Antichain of sets of states] An *antichain* on the partially ordered set $\langle 2^S, \subseteq \rangle$ is a set $q \subseteq 2^S$ such that for any $A, B \in q$ we have $A \not\subset B$.

Let us call L the set of antichains on S.

Definition 5 [$\sqsubseteq$] Let $q, q' \in 2^{2^S}$ and define $q \sqsubseteq q'$ if and only if

$$\forall A \in q : \exists A' \in q' : A \subseteq A'$$

$\langle L, \sqsubseteq \rangle$ is a complete lattice.

The minimal element is $\emptyset$, the maximal element $\{S\}$.

CPre over antichains

$$\mathsf{CPre}(q) = \color{red}{[}\{s \subseteq \overline{\mathsf{Bad}} \mid \exists \sigma \in \Sigma \cdot \forall \mathsf{obs} \in \mathsf{Obs} \cdot \exists s' \in q : \mathsf{Post}_\sigma(s) \cap \gamma(\mathsf{obs}) \subseteq s'\}\color{red}{]}$$

- CPre is a **monotone** function over the lattice of antichains
- CPre has a *least* and a *greatest* fixed point

Advantage : we only keep the needed information to find a strategy

Main theorem

Let $G = \langle S, S_0, \Sigma, \rightarrow, \mathsf{Obs}, \gamma \rangle$

be a two-player game of imperfect information. Player 1 has a winning observation based strategy to avoid Bad, **iff**

$$\{S_0 \cap \gamma(\mathsf{obs}) \mid \mathsf{obs} \in \mathsf{Obs}\} \sqsubseteq \bigsqcup \{q \mid q = \mathsf{CPre}(q)\}.$$

We can extract a strategy from the fixed point

Complexity for finite state games

- The imperfect information control problem is *EXPTIME-complete*
- There exist finite state games of incomplete information for which the algorithm of [Rei84] requires an exponential time where our algorithm needs only polynomial time

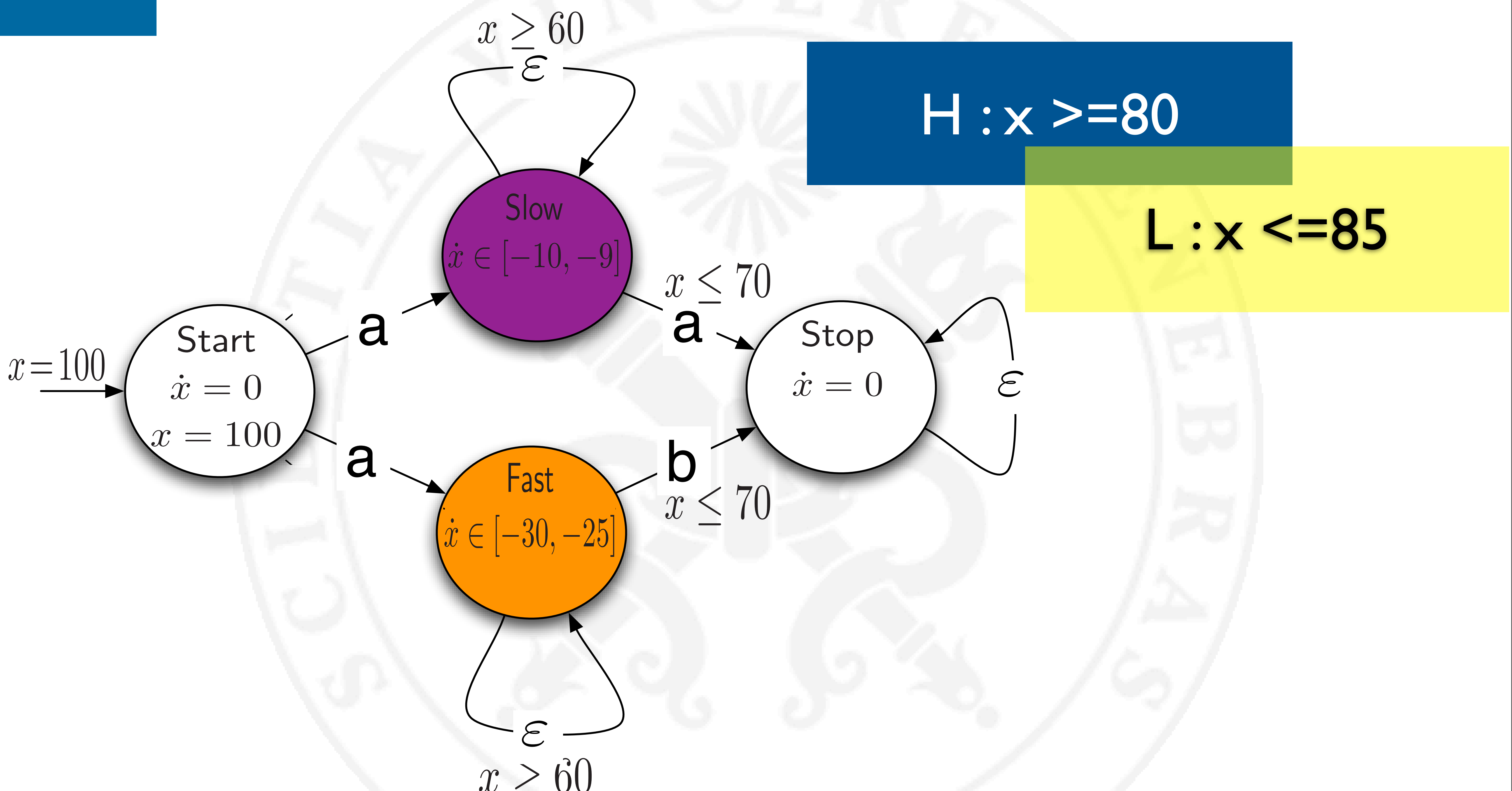
Infinite state games

- We drop the assumption that S if finite

- We can use our fixed point algorithm if
  - There exists a finite quotient of the state space
  - Post, Enabled, γ are definable using this quotient

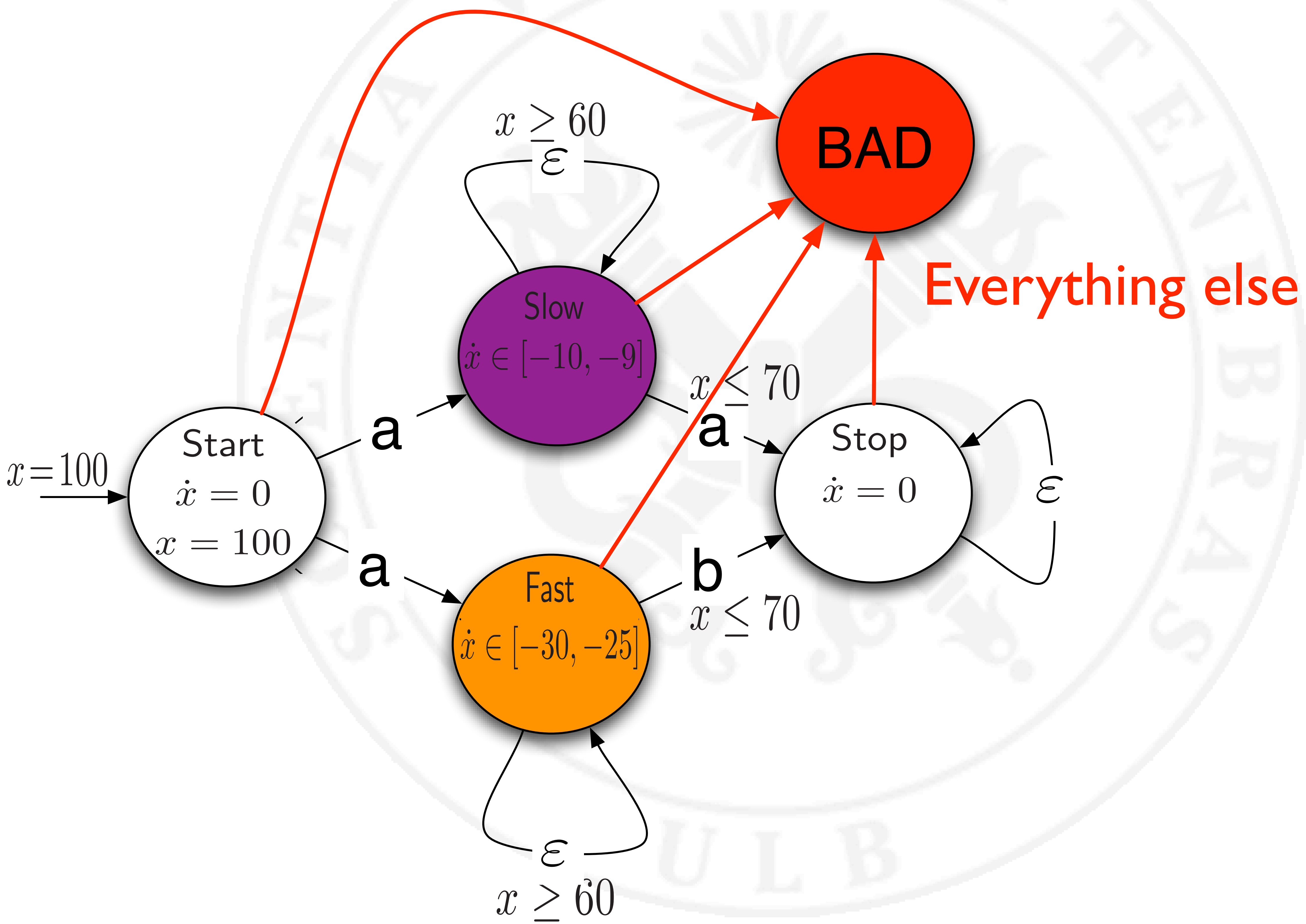
Application : Discrete Time Control of RHA

Discrete time control of RHA

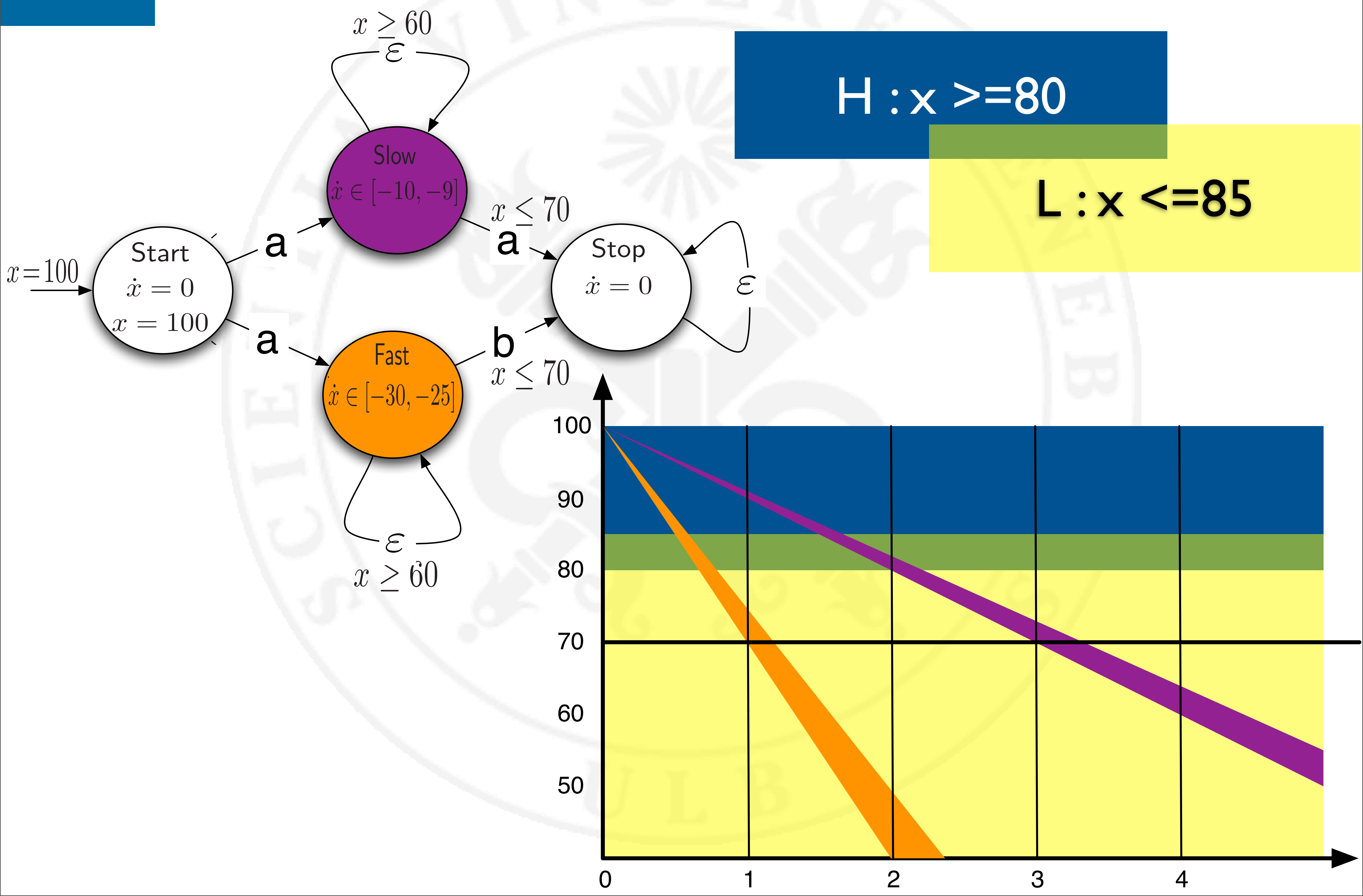


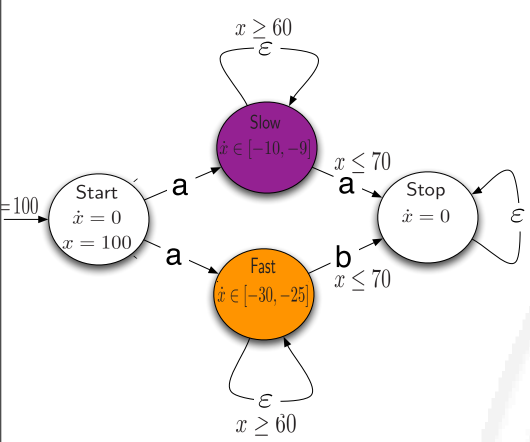
- Player 1 (contr.) must choose an action every 1 time unit
- Player 2 (env.) resolves nondeterminism (in discrete and continuous steps).

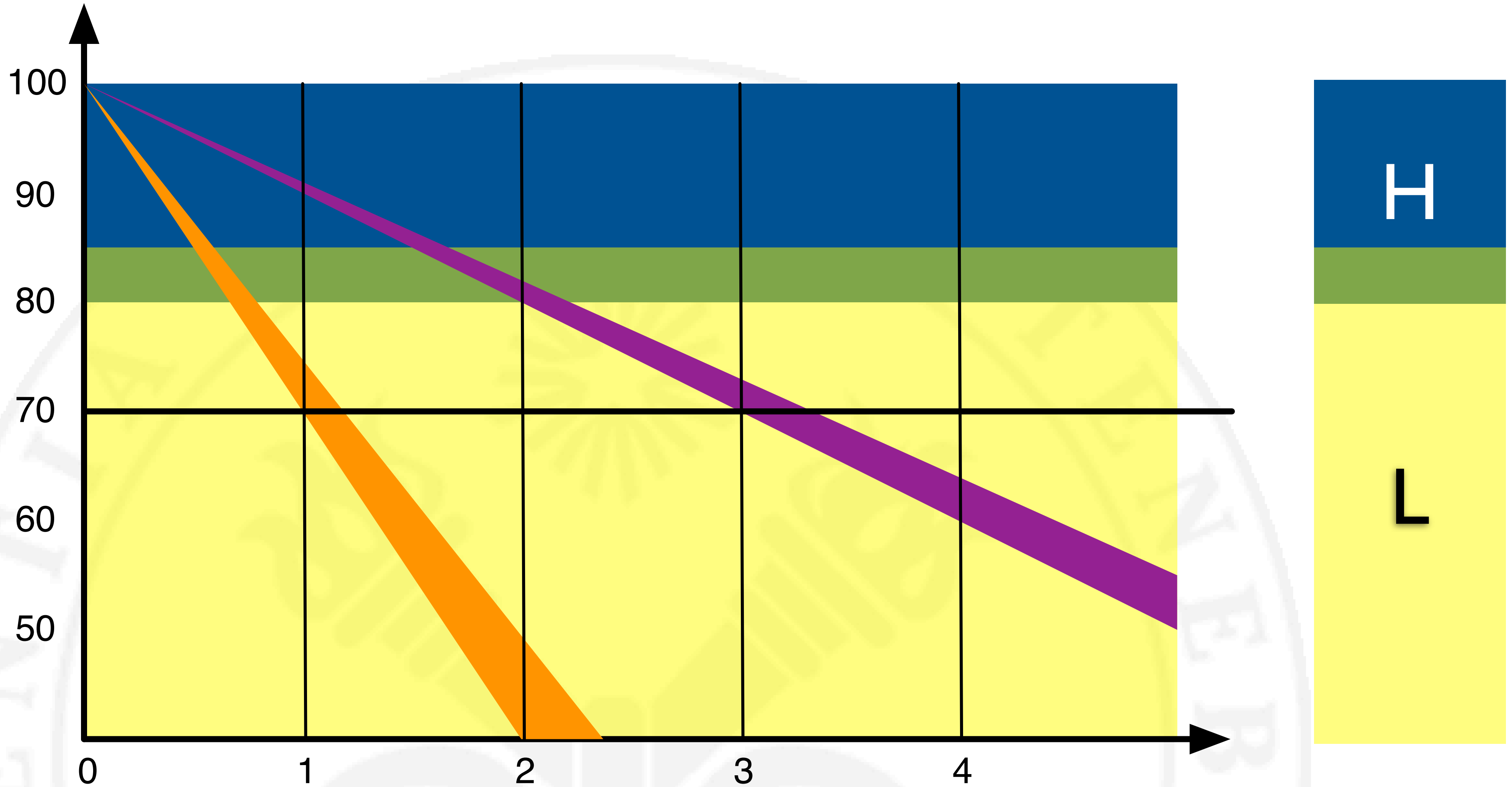
Discrete time control of RHA

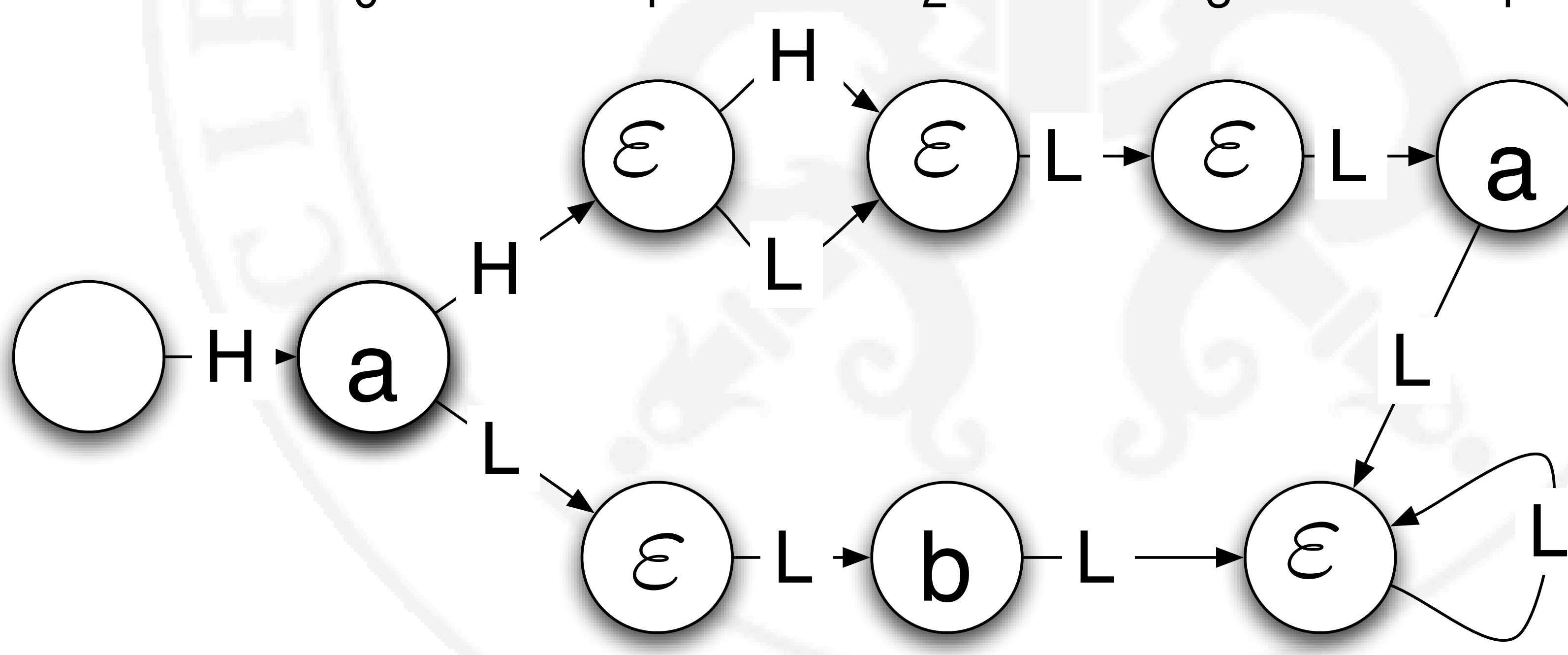


Discrete time control of RHA

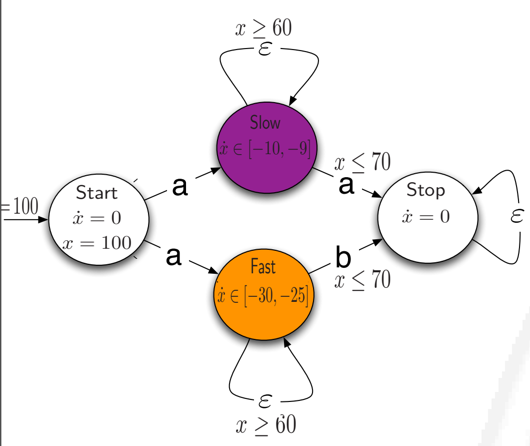


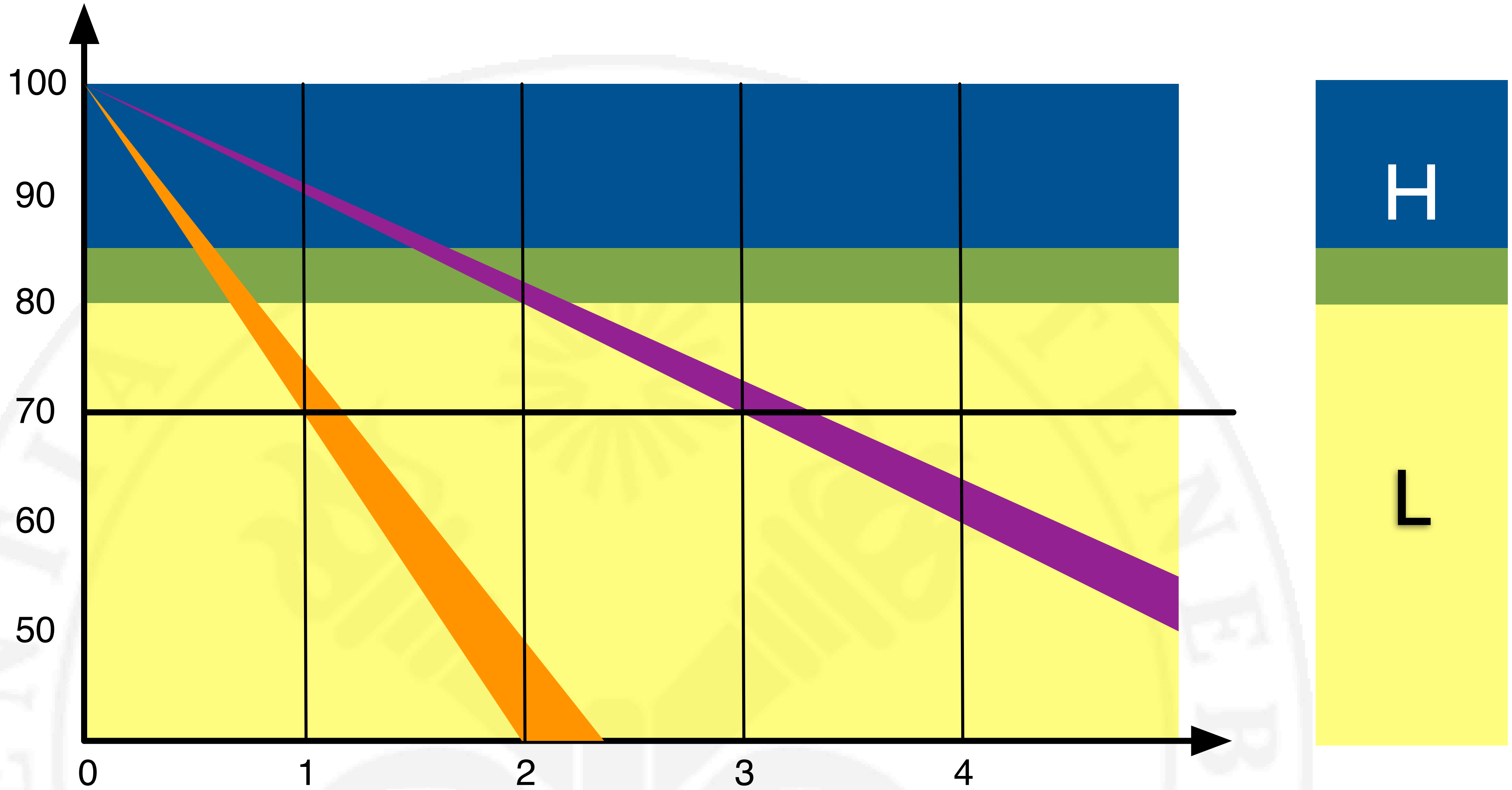


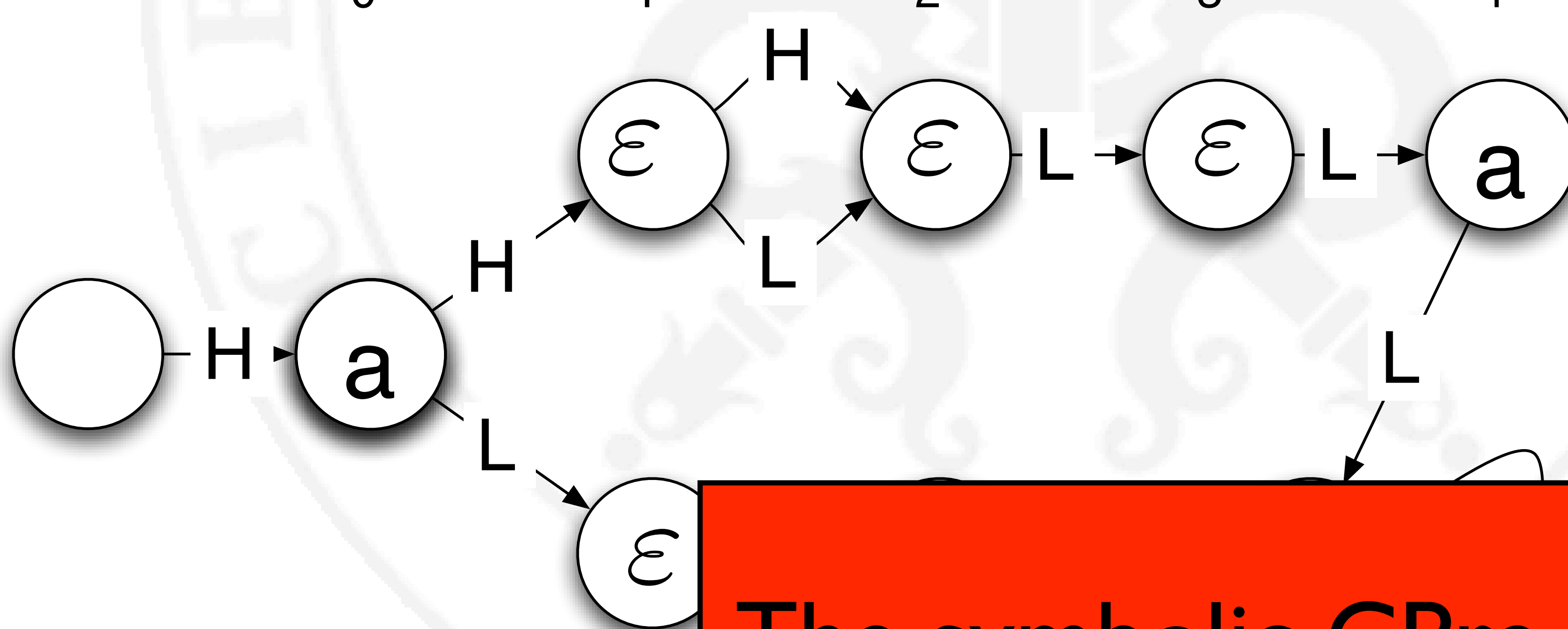




The Strategy







The symbolic CPre can be encoded in the script language of HyTech

Conclusion/Perspectives

- A lattice theory to solve games of imperfect information, those games are needed to make the synthesis of robust controllers
(= finite precision).
- Our technique computes only the needed information to find a strategy
- Applicable to discrete time control of RHA

- Define games whose controllers are implementable
- Are antichains useful for other purposes? The answer is YES

[ACH+95] R. Alur, C. Courcoubetis, N. Halbwachs, T.A. Henzinger, P.-H. Ho, X. Nicollin, A. Olivero, J. Sifakis, and S. Yovine. The algorithmic analysis of hybrid systems. *Theoretical Computer Science*, 138:3–34, 1995.

[AD94] Rajeev Alur and David L. Dill. A theory of timed automata. *Theoretical Computer Science*, 126(2):183–235, 1994.

[AHK02] R. Alur, T.A. Henzinger, and O. Kupferman. Alternating-time temporal logic. *Journal of the ACM*, 49:672–713, 2002.

[CC77] Patrick Cousot and Radhia Cousot. Abstract interpretation: A unified lattice model for static analysis of programs by construction or approximation of fixpoints. In *POPL*, pages 238–252, 1977.

[DDR06] M. De Wulf, L. Doyen, and J.-F. Raskin. A lattice theory for solving games of imperfect information (extended version). Technical Report 58, U.L.B. – Federated Center in Verification, 2006. http://www.ulb.ac.be/di/ssd/cfv/publications.html.

[HHWT95] T.A. Henzinger, P.-H. Ho, and H. Wong-Toi. A user guide to HyTech. In *TACAS 95: Tools and Algorithms for the Construction and Analysis of Systems*, Lecture Notes in Computer Science 1019, pages 41–71. Springer-Verlag, 1995.

[HK99] T.A. Henzinger and P.W. Kopke. Discrete-time control for rectangular hybrid automata. *Theoretical Computer Science*, 221:369–392, 1999.

[MPS95] O. Maler, A. Pnueli, and J. Sifakis. On the synthesis of discrete controllers for timed systems. In *STACS'95*, volume 900 of *Lecture Notes in Computer Science*, pages 229–242. Springer, 1995.

[Rei84] John H. Reif. The complexity of two-player games of incomplete information. *Journal of Computer and System Sciences*, 29(2):274–301, 1984.