The Truncated geometric election algorithm: duration of the game

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June 16, 2015

Abstract

1 Motivation

This is a sequel paper to Kalpathy and Ward [2]. The earlier paper introduces a randomized leader election algorithm, in which a truncated geometric number of participants survive from round to round. (Louchard and Prodinger [4] constitutes a good starting point for readers who are unfamiliar with randomized leader election algorithms; they also have extensive references, to provide a broader context.)

Kalpathy and Ward[2] precisely analyze the number of rounds that a *particular* contestant survives in the election. That analysis, however, had three key shortcomings. (1) It also focused on only the duration of a particular contestant; it did not analyze the length of the entire election (which should be more interesting and more useful in practice). (2) It only analyzed the mean and variation of the duration of a particular contestant (who was not necessarily the winner), but it did not discuss the asymptotic distribution of the duration (the analysis contained in the present investigation is more informative and useful). (3) The method of election in the earlier paper did not guarantee a unique winner.

The present paper addresses all three of these issues with the earlier paper. We focus on the duration of the entire election, not just of one participant. We go beyond the mean and variance, and analyze also the asymptotic distribution of the entire election. We also analyze two variants of the election, namely, the version from the original paper, and also a new style of election in which a unique winner is guaranteed to appear at the end of the election process.

As another point of motivation for this sequel paper, we discovered that the total length of both of these styles of election, when starting with n participants, can be decomposed into a sum of $n-1$ independent random variables that do not have identical distributions. This surprising point about the decomposition of the length of the whole election is discussed further in the analysis below, using moment generating functions.

2 Definitions

We consider elections for which, if n contestants are present in a round, then K_n contestants proceed to the next round, where K_n is a *truncated geometric random variable* with parameters p and $q := 1-p$. So the mass of K_n is

$$\mathbb{P}(K_n = \ell) = \frac{pq^\ell}{1-q^{n+1}}, \qquad \text{for } \ell = 0, 1, \ldots, n.$$

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We study the number of rounds needed for the election is two distinct situations. The difference occurs in how the one of the base conditions is handled, namely, what happens when $K_n = 0$.

1. **Model** 1. In the literature, it is very common that, when all of the current participants fail to advance to the next round, all of them are given another chance to participate in one more (renewal) round.
2. **Model** 2. As in the Kalpathy–Ward paper [2], if $K_n = 0$ in one of the rounds, the election stops, and the remaining n participants can all be treated as winners, or (alternatively) all considered as losers, but the main point is that no additional rounds of the election take place.

We define X_n as the number of rounds (duration of the game), when starting with n participants, for the elections given in **Model** 1 and 2, respectively.

3 Generating function, Model 1

Let $X(n)$ be the duration of the game.

Notations

$$P(n,j) := \mathbb{P}(X(n)=j), \qquad M(n) := \mathbb{E}(X(n)),$$
$$M^{(2)}(n) := \mathbb{E}(X(n)^2) \qquad \text{and} \qquad \phi(n) := \mathbb{E}(e^{tX(n)}).$$

We have

$$\phi(n) = e^t\left[\frac{p}{1-q^{n+1}}\phi(n) + \frac{pq}{1-q^{n+1}}.1 + \sum_{l=2}^{n}\frac{pq^l}{1-q^{n+1}}\phi(l)\right], \quad \phi(1)=1,$$
$$\phi(n)\left(1-\frac{pe^t}{1-q^{n+1}}\right) = e^t\sum_{l=2}^{n}\frac{pq^l}{1-q^{n+1}}\phi(l) + \frac{pqe^t}{1-q^{n+1}},$$
$$\phi(n)(1-q^{n+1}-pe^t) = e^t\sum_{l=2}^{n}pq^l\phi(l) + pqe^t,$$
$$\phi(n-1)(1-q^{n}-pe^t) = e^t\sum_{l=2}^{n-1}pq^l\phi(l) + pqe^t.$$

Substracting, this gives

$$\phi(n) = \phi(n-1)\frac{-1+q^n+pe^t}{-1+q^{n+1}+pe^t+pe^tq^n},$$
$$\phi(n) = \prod_{i=2}^{n}\phi_i,$$
$$\phi_2 = \phi(2) = \frac{pqe^t}{1-q^3-pe^t-pe^tq^2},$$
$$\phi_i := \frac{-1+q^i+pe^t}{-1+q^{i+1}+pe^t+pe^tq^i}, \quad i>2,$$
$$\phi(2)|_{t=0} = 1, \quad \phi_i|_{t=0} = 1.$$

This corresponds to a sum of independent, not equi-distributed random variables. It is indeed easy to check that, with $e^t = z$, $\phi_i(z)$ is an honest probability generating function, corresponding to a non-negative random variable.

4 Moments, Model 1

We have

$$
\begin{aligned}
M(n) &= \left.\frac{d\phi(n)}{dt}\right|_{t=0} = \frac{q^2+q+1}{q} - \sum_{i=3}^{n} \frac{pq^i}{q^i-q} \\
&= \frac{q^2+q+1}{q} - \sum_{i=3}^{\infty} \frac{pq^i}{q^i-q} + \sum_{i=n+1}^{\infty} \frac{pq^i}{q^i-q} \\
&= C_1 + \sum_{k=1}^{\infty} \frac{pq^{n+k}}{q^{n+k}-q}, \\
C_1 &:= \frac{q^2+q+1}{q} - \sum_{i=3}^{\infty} \frac{pq^i}{q^i-q}, \\
&\sum_{k=1}^{\infty} \frac{pq^{n+k}}{q^{n+k}-q} \sim -\sum_{k=1}^{\infty} pq^{n-1+k} = -q^n + \Theta(q^{2n}), \\
M(n) &\sim C_1 - q^n.
\end{aligned}
$$

The next term is of order q^{2n} and is easily computed by Maple (with some human guidance). All moments can be similarly computed. For instance, for the variance, we use the fact that we have a sum of independent, not equi-distributed random variables. This leads to

$$
\begin{aligned}
\mathbb{V}(X(n)) &= \left.\frac{d^2\phi(2)}{dt^2}\right|_{t=0} - \left[\left.\frac{d\phi(2)}{dt}\right|_{t=0}\right]^2 + \sum_{i=3}^{n} \left(\left.\frac{d^2\phi_i}{dt^2}\right|_{t=0} - \left[\left.\frac{d\phi_i}{dt}\right|_{t=0}\right]^2 \right), \\
\left.\frac{d^2\phi(2)}{dt^2}\right|_{t=0} &= \frac{(q^2+q+1)(2q^2+q+2)}{q^2}, \\
\left.\frac{d^2\phi_i}{dt^2}\right|_{t=0} &= -\frac{pq^i(q^{i+1}-1-p-pq^i)}{(q^i-q)^2}, \\
\mathbb{V}(X(n)) &= \frac{(q^2+q+1)(1+q^2)}{q^2} - \sum_{i=3}^{\infty} \frac{pq^i(q^i+q-2)}{(q^i-q)^2} + \frac{q-2}{q}q^n + \Theta(q^{2n}).
\end{aligned}
$$

5 $P(\infty, j)$, Model 1

It doesn't seem possible to derive explicitly $P(n,j)$ from $\phi(n)$. We should need a kind of Euler formulae. We should compute

$$
[z^j]\left[\frac{pqz}{1-q^3-pz-pzq^2} \prod_{i=3}^{n} \frac{-1+q^i+pz}{-1+q^{i+1}+pz+pzq^i}\right], \tag{1}
$$

which leads to complicated formulae. We have

$$
\begin{aligned}
\frac{pqz}{1-q^3-pz-pzq^2} &\sim \frac{pq}{1-q^3}z + \ldots, \\
\frac{-1+q^i+pz}{-1+q^{i+1}+pz+pzq^i} &\sim \frac{1-q^i}{1-q^{i+1}} + \frac{p(q^{i+1}-q^{2i})}{(1-q^{i+1})^2}z + \ldots
\end{aligned}
$$

Note that $\phi(n), n \geq 2$ starts with z, as expected. This leads to

$$
P(n,1) = \frac{pq}{1-q^{n+1}},
$$

which is the correct value.

To analyze $P(\infty, j)$, we shall proceed as in Lavault, Louchard [3], Sec.3, with the same notations, and use singularity anaysis (see for instance, Flajolet and Sedgewick [1], chap.6). We have

$$P(n,1) = \frac{pq}{1-q^{n+1}}, \quad P(\infty,1) = pq,$$
$$P(n,j) = \frac{p}{1-q^{n+1}} P(n,j-1) + \sum_{l=2}^{n} \frac{pq^l}{1-q^{n+1}} P(l,j-1), \quad j>1, \quad n \geq 2. \tag{2}$$

This leads to

$$P(\infty,j) = pP(\infty,j-1) + D(j-1), j \geq 2, \text{ with} \tag{3}$$
$$D(j) := \sum_{l=2}^{\infty} pq^l P(l,j), j \geq 1.$$

Equ.(3) leads, by iteration, to

$$P(\infty,j) = p^{j-1}P(\infty,1) + \sum_{k=1}^{j-1} p^{k-1} D(j-k).$$

Set

$$H(z) := \sum_{1}^{\infty} P(\infty,j) z^j,$$
$$G(z) := \sum_{1}^{\infty} D(j) z^j.$$

Equ.(3) leads to

$$H(z) - pqz = pzH(z) + zG(z),$$
$$H(z) = \frac{zG(z)+pqz}{1-pz}, \text{ with a first singularity } z_1^* = 1/p. \tag{4}$$

Set

$$\Pi(k,z) := \sum_{1}^{\infty} P(k,j) z^j.$$

We have, by (2),

$$\Pi(k,z) - \frac{pqz}{1-q^{k+1}} = \frac{pz}{1-q^{k+1}} \Pi(k,z) + \sum_{l=2}^{k} \frac{pq^l}{1-q^{k+1}} z\Pi(l,z), \quad k \geq 2, \tag{5}$$
$$\Pi(2,z) = \frac{pqz}{1-q^3-zp(1+q^2)} = \frac{pqz}{(1-q^3)(1-z/z_2^*)}, \quad z_2^* := \frac{1-q^3}{p(1+q^2)}, \quad 1 < z_2^* < z_1^*,$$
$$\Pi(2,z) \asymp \frac{pqz_2^*}{(1-q^3)(1-z/z_2^*)},$$
$$\Pi(2,z) \asymp R(2) \frac{1}{(1-z/z_2^*)}, \text{ with}$$
$$R(2) = \frac{q}{1+q^2}. \tag{6}$$

By (5), we have

$$\Pi(k,z) \asymp R(k)\frac{1}{(1-z/z_2^*)}, \text{ with}$$

$$R(k) = \frac{p}{1-q^{k+1}}z_2^*R(k) + \sum_{l=2}^{k}\frac{pq^l}{1-q^{k+1}}z_2^*R(l), \quad k \geq 2,$$

or

$$R(k) = \prod_{i=3}^{k}\frac{-1+q^i+pz_2^*}{-1+q^{i+1}+pz_2^*+pz_2^*q^i}R(2), \quad \text{by (1)}.$$

Indeed the singularity z_i^* of

$$\frac{-1+q^i+pz}{-1+q^{i+1}+pz+pzq^i}$$

is given by

$$z_i^* = \frac{1-q^{i+1}}{p(1+q^i)},$$

which is an increasing function of i and $z_i^* > z_2^*$. So z_2^* is the dominant simple pole.

Notice that this leads to

$$P(k,j) \sim R(k)z_2^{*-j}, \quad j \to \infty.$$

Now

$$G(z) = \sum_{l=2}^{\infty} pq^l\Pi(l,z), \text{ and with}$$

$$\rho = \sum_{l=2}^{\infty} pq^l R(l) \text{ to be numerically computed,}$$

$$G(z) \asymp \frac{\rho}{1-z/z_2^*}.$$

Equ.(4) leads now to

$$H(z) \asymp \frac{\rho z_2^*}{(1-z/z_2^*)(1-pz_2^*)}.$$

Hence

$$P(\infty,j) \sim \frac{\rho z_2^*}{1-pz_2^*}z_2^{*-j}, \quad j \to \infty. \tag{7}$$

Let us summarize our results in the following theorem

Theorem 5.1 ***Model*** **1**

Asymptotically, mean, variance and distribution are given by the following expressions

$$M(n) \sim C_1 - q^n + \Theta(q^{2n}),$$

$$\mathbb{V}(X(n)) = \frac{(q^2+q+1)(1+q^2)}{q^2} - \sum_{i=3}^{\infty}\frac{pq^i(q^i+q-2)}{(q^i-q)^2} + \frac{q-2}{q}q^n + \Theta(q^{2n}),$$

$$P(k,j) \sim R(k)z_2^{*-j}, \quad j \to \infty,$$

$$P(\infty,j) \sim \frac{\rho z_2^*}{1-pz_2^*}z_2^{*-j}, \quad j \to \infty.$$

$P(\infty,j)$ is rapidly decreasing. For instance, for $p = 0.3$, we have $z_2^* := 1.469798658\ldots, \rho = 0.3074863705$. We give in figure 1 a plot of $P(\infty,j)$. In figure 2, we give a comparison between $P(\infty,j), j = 20..40$ (circle) and the asymptotic (7) (line)

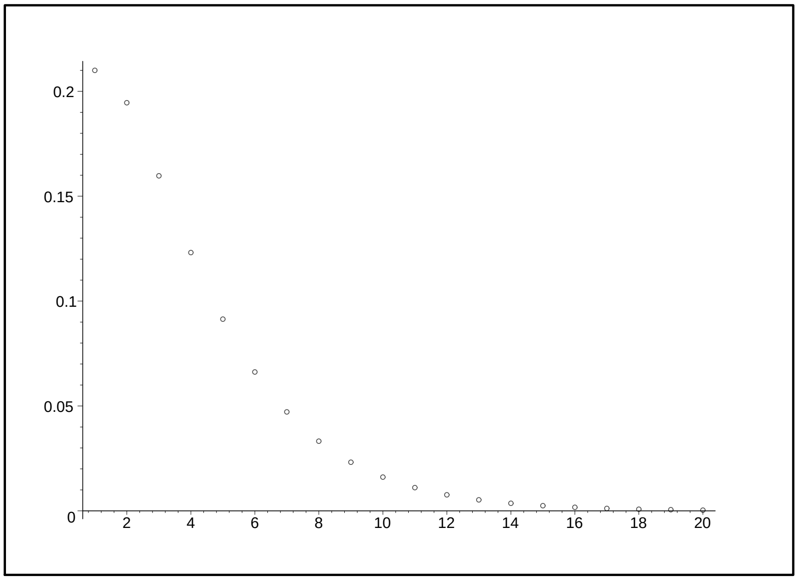

Figure 1: $P(\infty, j), j = 1..20$.

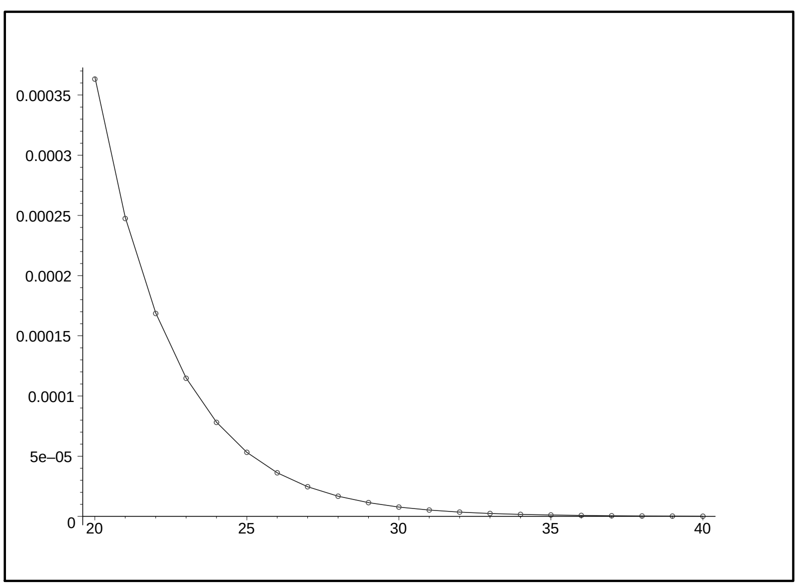

Figure 2: Comparison between $P(\infty, j), j = 20..40$ (circle) and the asymptotic (7) (line).

6 Survivors, Model 1

6.1 Survivors near the beginning of the game

Let $Q(n,i,l)$ be the probability that l players survive at step i, starting with n players. If only 1 player survive, she is the winner, the game is over, and this player is not counted a a survivor. Let $S(n,l)$ be the probability that l players survive after 1 step, starting with n players. We have

$$S(n,l) = \frac{pq^l}{1-q^{n+1}}, \quad l = 2..n-1,$$
$$S(n,n) = \frac{pq^n}{1-q^{n+1}} + \frac{p}{1-q^{n+1}} = \frac{p(1+q^n)}{1-q^{n+1}}.$$

This gives

$$Q(n,1,l) = S(n,l), \quad l = 2..n,$$
$$Q(n,i,l) = \sum_{u=l}^{n} Q(n,i-1,u)S(u,l) = S^i(n,l), \quad l = 2..n, \quad \text{in matrix notation.}$$

Note that, in the matrix, indices start from 2.

6.2 Survivors near the end of the game

Let $\Pi(n,i,j,l)$ be the probability that l players survive at step $j-i$, starting with n players, where j denotes the duration of the game. Let us first consider the case of 1 step before the end of the game i.e. i=1. Then

$$\Pi(n,1,2,l) = S(n,l)S(l,1), \quad l = 1..n,$$
$$\Pi(n,1,j,l) = \sum_{u=l}^{n} S(n,u)\Pi(u,1,j-1,l). \tag{8}$$

Now set

$$V(n,1,l) := \sum_{j=2}^{\infty} \Pi(n,1,j,l).$$

We have, from (8)

$$V(n,1,l) - \Pi(n,1,2,l) = \sum_{u=l}^{n} S(n,u)V(u,1,l)$$
$$= S(n,n)V(n,1,l) + \sum_{u=l}^{n-1} S(n,u)V(u,1,l), \quad n \geq l+2,$$
$$V(n,1,l)[1-q^{n+1}-p(1+q^n)] = \sum_{u=l}^{n-1} pq^u V(u,1,l) + pq^l S(l,1),$$
$$V(n-1,1,l)[1-q^n-p(1+q^{n-1})] = \sum_{u=l}^{n-2} pq^u V(u,1,l) + pq^l S(l,1),$$
$$V(n,1,l) = V(n-1,1,l), \quad n \geq l+2.$$

This should have a simple probabilistic explanation!

Now we must write down the boundary relations.

$$\begin{aligned}
V(l+1,1,l)[1-q^{l+2}-p(1+q^{l+1})] &= pq^l V(l,1,l)+pq^l S(l,1), \\
V(l,1,l) &= \Pi(l,1,2,l)+S(l,l)V(l,1,l), \\
V(l,1,l) &= S(l,l)S(l,1)+S(l,l)V(l,1,l), \\
V(l,1,l)[1-q^{l+1}-p(1+q^l)] &= p(1+q^l)\frac{pq}{1-q^{l+1}}, \\
V(l,1,l) &= \frac{p^2(q^l+1)q}{(-1+q^{l+1})/(-1+q^{l+1}+pq^l+p)},
\end{aligned}$$

$$V(l+1,1,l) = \frac{p^2q^l(q^l-2q^{l+2}+q^{2l+3})}{(-p+1-q^{l+2}+q^{2l+2}p-q^{l+1}+q^{2l+3})/(-1+q^{l+1})/(-1+q^{l+1}+pq^l+p)}.$$

Let us now consider the case of 2 steps before the end of the game i.e. $i=2$. Now we have

$$\begin{aligned}
\Pi(n,2,3,l) &= S(n,l)\sum_{u=2}^{l} S(l,u)S(u,1) = S(n,l)S^2(l,1), \quad l=2..n, \\
\Pi(n,2,j,l) &= \sum_{u=l}^{n} S(n,u)\Pi(u,2,j-1,l), \\
V(n,2,l)-\Pi(n,2,3,l) &= \sum_{u=l}^{n} S(n,u)V(u,2,l) \\
&= S(n,n)V(n,2,l)+\sum_{u=l}^{n-1} S(n,u)V(u,2,l), \quad n \geq l+2.
\end{aligned}$$

The rest of the computation proceeds as in the previous subsection, we omit the details.

7 Moments, Model 2

We have

$$\begin{aligned}
\phi_2 = \phi(2) &= \frac{p(1+q)e^t}{1-q^3-pe^tq^2}, \\
\phi_i &:= \frac{1-q^i}{1-q^{i+1}-pe^tq^i}, \quad i>2, \\
\phi(2)|_{t=0} &= 1, \quad \phi_i|_{t=0}=1, \\
M(n) &= \left.\frac{d\phi(n)}{dt}\right|_{t=0} = \frac{q^2+q+1}{q+1}+\sum_{i=3}^{n}\frac{pq^i}{1-q^i}, \\
&= \frac{q^2+q+1}{q+1}+\sum_{i=3}^{\infty}\frac{pq^i}{1-q^i}-\sum_{i=n+1}^{\infty}\frac{pq^i}{1-q^i} \\
&= C_2-\sum_{k=1}^{\infty}\frac{pq^{n+k}}{1-q^{n+k}}, \\
C_2 &:= \frac{q^2+q+1}{q+1}+\sum_{i=3}^{\infty}\frac{pq^i}{1-q^i}, \\
&-\sum_{k=1}^{\infty}\frac{pq^{n+k}}{1-q^{n+k}} \sim -qq^n+\Theta(q^{2n}), \\
M(n) &\sim C_2-qq^n+\Theta(q^{2n}).
\end{aligned}$$

For the variance, we use again the fact that we have a sum of independent, not equi-distributed random variables. This leads to

$$\begin{aligned}
\mathbb{V}(X(n)) &= \left.\frac{d^2\phi(2)}{dt^2}\right|_{t=0} - \left[\left.\frac{d\phi(n)}{dt}\right|_{t=0}\right]^2 + \sum_{i=3}^{n}\left(\left.\frac{d^2\phi_i}{dt^2}\right|_{t=0} - \left[\left.\frac{d\phi_i}{dt}\right|_{t=0}\right]^2\right),\\
\left.\frac{d^2\phi(2)}{dt^2}\right|_{t=0} &= \frac{(q^2+q+1)(2q^2+q+1)}{(q+1)^2},\\
\left.\frac{d^2\phi_i}{dt^2}\right|_{t=0} &= -\frac{pq^i(q^{i+1}-1-pq^i)}{(1-q^i)^2},\\
\mathbb{V}(X(n)) &= \frac{(q^2+q+1)q^2}{(q+1)^2} - \sum_{i=3}^{\infty}\frac{pq^i(q^{i+1}-1)}{(1-q^i)^2} - qq^n + \Theta(q^{2n}).
\end{aligned}$$

8 $P(\infty, j)$, Model 2

We again shall proceed as in Lavault, Louchard [3], Sec.3, with the same notations We have

$$
\begin{aligned}
P(n,1) &= \frac{p(1+q)}{1-q^{n+1}}, \quad P(\infty,1) = p(1+q), \\
P(n,j) &= \sum_{l=2}^{n} \frac{pq^l}{1-q^{n+1}} P(l,j-1), \quad j>1, \quad n \geq 2, \\
P(\infty,j) &= D(j-1), \\
D(j) &:= \sum_{l=2}^{\infty} pq^l P(l,j), \quad j \geq 1, \\
H(z) - p(1+q)z &= zG(z), \\
&\quad \text{there is no first singularity here ,} \\
\Pi(k,z) - \frac{p(1+q)z}{1-q^{k+1}} &= \sum_{l=2}^{k} \frac{pq^l}{1-q^{k+1}} z\Pi(l,z), \quad k \geq 2, \\
\Pi(2,z) &= \frac{p(1+q)z}{1-q^3-zpq^2} = \frac{p(1+q)z}{(1-q^3)(1-z/z^*)}, \quad z^* := \frac{1-q^3}{pq^2}, \\
\Pi(2,z) &\asymp \frac{p(1+q)z^*}{(1-q^3)(1-z/z^*)}, \\
R(2) &= \frac{1+q}{q^2}, \\
R(k) &= \sum_{l=2}^{k} \frac{pq^l}{1-q^{l+1}} z^* R(l), \\
R(k) &= \prod_{i=3}^{k} \frac{1-q^i}{1-q^{i+1}-z^*pq^i} R(2), \\
\Pi(k,z) &\asymp R(k) \frac{1}{(1-z/z^*)}, \\
P(k,j) &\sim R(k) z^{*-j}, \quad j \to \infty, \\
G(z) &= \sum_{l=2}^{\infty} pq^l \Pi(l,z), \\
\rho &= \sum_{l=2}^{\infty} pq^l R(l) \text{ to be numerically computed,} \\
G(z) &\asymp \frac{\rho}{1-z/z^*}, \\
H(z) &\asymp \frac{\rho z^*}{(1-z/z^*)}, \\
P(\infty,j) &\sim \rho z^* z^{*-j}, \quad j \to \infty.
\end{aligned}
$$

$P(\infty, j)$ is rapidly decreasing, faster than in **Model** 1. For instance, for $p = 0.3$, we have $z_2^* = 4.469387755\ldots, \rho = 5.073566431\ldots$. We give in figure 3 a plot of $P(\infty, j)$. In figure 4, we give a comparison between $P(\infty, j), j = 10..15$ (circle) and the asymptotic (7) (line)

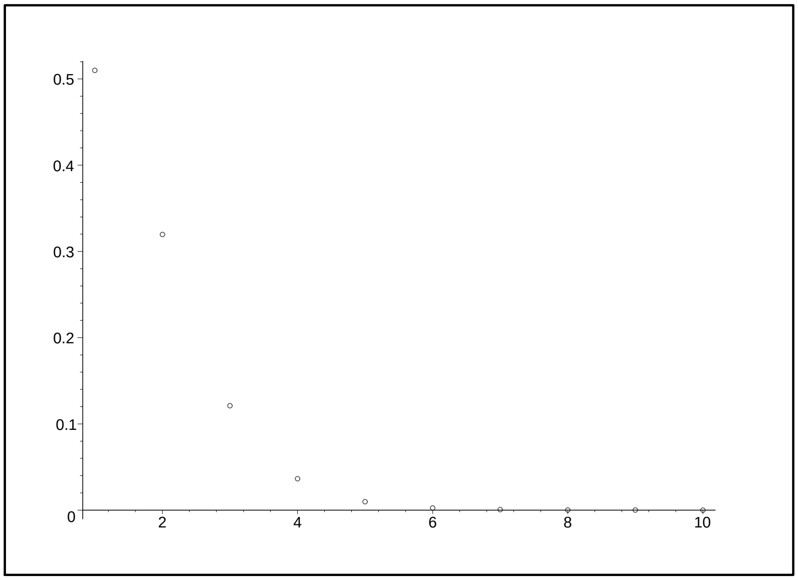

Figure 3: $P(\infty, j), j = 1..10$.

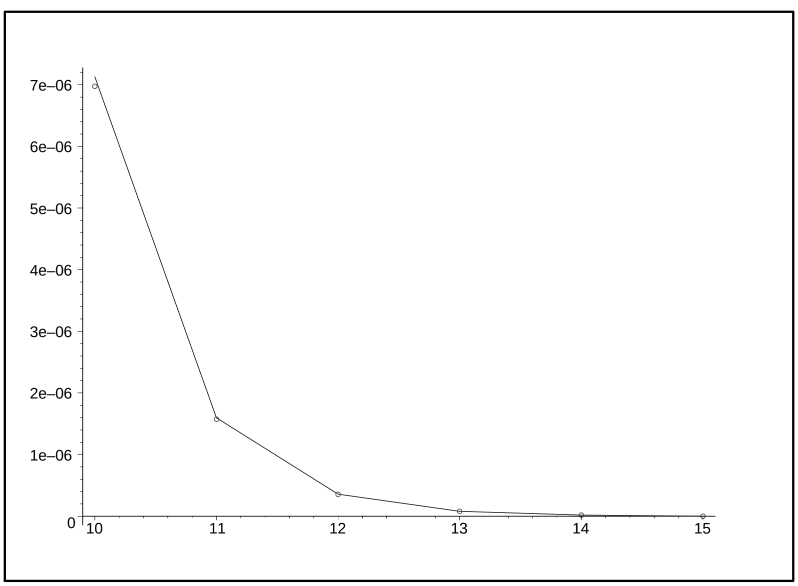

Figure 4: Comparison between $P(\infty, j), j = 1..20$ (circle) and the asymptotic (7) (line).

9 Probability of ending with 0 leader, Model 2

Let $P^*(n)$ be the probability that we end the game with 0 leader, starting with n players. We have

$$
\begin{aligned}
P^*(n) &= \frac{p}{1-q^{n+1}} + \sum_{l=2}^{n} \frac{pq^l}{1-q^{n+1}} P^*(l), \\
P^*(2) &= \frac{p}{1-q^3-pq^2}, \\
P^*(n)(1-q^{n+1}) &= p + \sum_{l=2}^{n} pq^l P^*(l), \\
P^*(n-1)(1-q^n) &= p + \sum_{l=2}^{n-1} pq^l P^*(l), \\
P^*(n) &= \prod_{i=2}^{n} P_i^*, \\
P_2^* &= P^*(2), \\
P_i^* &:= \frac{1-q^i}{1-q^{i+1}-pq^i}, \quad i > 2.
\end{aligned}
$$

We could of course similarly analyze the joint probability that the game ends at time j, with 0 leader, starting with n players. Let us summarize our results in the following theorem

Theorem 9.1 ***Model*** **2**
Asymptotically, mean, variance, distribution and probability of ending with 0 *leader are given by the following expressions*

$$
\begin{aligned}
M(n) &\sim C_2 - qq^n + \Theta(q^{2n}), \\
\mathbb{V}(X(n)) &= \frac{(q^2+q+1)q^2}{(q+1)^2} - \sum_{i=3}^{\infty} \frac{pq^i(q^{i+1}-1)}{(1-q^i)^2} - qq^n + \Theta(q^{2n}), \\
P(k,j) &\sim R(k) z^{*-j}, \quad j \to \infty, \\
P(\infty, j) &\sim \rho z^* z^{*-j}, \quad j \to \infty, \\
P^*(n) &= \prod_{i=2}^{n} P_i^*.
\end{aligned}
$$

10 Survivors, Model 2

10.1 Survivors near the beginning of the game

If only 1 or 0 players survive, the game is over, and the possible surviving player is not counted a a survivor. We have

$$S(n,l) = \frac{pq^l}{1-q^{n+1}}, \quad l = 0..n,$$

Set now

$$T(n) := S(n,0) + S(n,1) = \frac{p(1+q)}{1-q^{n+1}}.$$

This gives

$$Q(n,1,l) = S(n,l), \quad l = 2..n,$$
$$Q(n,i,l) = \sum_{u=2}^{n} Q(n,i-1,u)S(u,l) = S^i(n,l), \qquad \text{in matrix notation.}$$

10.2 Survivors near the end of the game

We have

$$\Pi(n,1,2,l) = S(n,l)T(l), \quad l = 2..n,$$
$$\Pi(n,1,j,l) = \sum_{u=l}^{n} S(n,u)\Pi(u,1,j-1,l),$$
$$\Pi(l,1,2,l) = S(l,l)T(l) = \frac{pq^l}{1-q^{l+1}}T(l).$$

We have

$$V(n,1,l) - \Pi(n,1,2,l) = \sum_{u=l}^{n} S(n,u)V(u,1,l)$$
$$= S(n,n)V(n,1,l) + \sum_{u=l}^{n-1} S(n,u)V(u,1,l), \quad n \geq l+1,$$
$$V(n,1,l)[1-q^{n+1}-pq^n] = \sum_{u=l}^{n-1} pq^u V(u,1,l) + pq^l T(l),$$
$$V(n-1,1,l)[1-q^n-pq^{n-1})] = \sum_{u=l}^{n-2} pq^u V(u,1,l) + pq^l T(l),$$
$$V(n,1,l) = V(n-1,1,l), \quad n \geq l+1.$$

Again, this should have a simple probabilistic explanation!

Now we must write down the boundary relations. Note carefully that here, we only need $V(l,1,l)$.

$$V(l+1,1,l) = V(l,1,l),$$
$$V(l,1,l)[1-q^{l+1}-pq^l)] = pq^l T(l),$$
$$V(l,1,l) = \frac{p^2q^l(q+1)}{(-1+q^{l+1})/(-1+q^{l+1}+pq^l)}.$$

The case of 2 steps before the end of the game i.e. $i = 2$ is analyzed as in **Model** 1, we omit the details.

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