Traffic Light Queues and the Poisson Clumping Heuristic

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Abstract. In discrete time, ℓ-blocks of red lights are separated by ℓ-blocks of green lights. Cars arrive at random. We seek the distribution of maximum line length of idle cars, and justify conjectured probabilistic asymptotics for $2 \leq \ell \leq 3$.

Assorted expressions emerge for a certain traffic light problem [1]. Let $\ell \geq 1$ be an integer. Let $X_0 = 0$ and X_1, X_2, $\ldots$, X_n be a sequence of independent random variables satisfying

$$\mathbb{P}\{X_i = 1\} = p, \qquad \mathbb{P}\{X_i = 0\} = q \qquad \text{if } i \equiv 1, 2, \ldots, \ell \bmod 2\ell;$$

$$\mathbb{P}\{X_i = 0\} = p, \qquad \mathbb{P}\{X_i = -1\} = q \qquad \text{if } i \equiv \ell+1, \ell+2, \ldots, 2\ell \bmod 2\ell$$

for each $1 \leq i \leq n$. Define $S_0 = X_0$ and $S_j = \max\{S_{j-1} + X_j, 0\}$ for all $1 \leq j \leq n$. Thus cars arrive at a one-way intersection according to a Bernoulli(p) distribution; when the signal is red ($1 \leq i \leq \ell$), no cars may leave; when the signal is green ($\ell+1 \leq i \leq 2\ell$), a car must leave (if there is one). The quantity $M_n = \max_{0 \leq j \leq n} S_j$ is the worst-case traffic congestion (as opposed to the average-case often cited). Only the circumstance when $\ell = 1$ is amenable to rigorous treatment [2], as far as is known. We assume that $p < q$ throughout.

The Poisson clumping heuristic [3], while not a theorem, gives results identical to exact asymptotic expressions when such exist, and evidently provides excellent predictions otherwise. Consider an irreducible positive recurrent Markov chain with stationary distribution π. For sufficiently large k, the maximum of the chain satisfies

$$\mathbb{P}\{M_n < k\} \sim \exp\left(-\frac{\pi_k}{\mathbb{E}(C)} n\right)$$

as $n \to \infty$, where C is the sojourn time in k during a clump of nearby visits to k.

Here is a simple example: for an asymmetric random walk with weak reflection at the origin, we have

$$\pi_j = p\pi_{j-1} + q\pi_{j+1}, \qquad j \geq 1;$$

$$\pi_0 = q\pi_0 + q\pi_1$$

hence

$$\pi_1 = \frac{p}{q}\pi_0.$$

From

$$\pi_1 = p\pi_0 + q\pi_2$$

we deduce

$$q\pi_2 = \pi_1 - p\pi_0 = (1-q)\,\pi_1 = p\pi_1$$

hence

$$\pi_2 = \frac{p}{q}\pi_1 = \frac{p^2}{q^2}\pi_0.$$

Defining

$$\begin{aligned} F(z) = \sum_{j=2}^{\infty}\pi_j z^j &= pz\sum_{j=2}^{\infty}\pi_{j-1}z^{j-1} + \frac{q}{z}\sum_{j=2}^{\infty}\pi_{j+1}z^{j+1} \\ &= pz\left[F(z)+\pi_1 z\right] + \frac{q}{z}\left[F(z)-\pi_2 z^2\right], \end{aligned}$$

we have

$$\left[1 - pz - \frac{q}{z}\right]F(z) = p\pi_1 z^2 - q\pi_2 z$$

hence

$$\left[q - z + pz^2\right]F(z) = (q\pi_2 - p\pi_1 z)\,z$$

hence

$$(1-z)\,(q-pz)\,F(z) = \frac{p^2\,(1-z)\,z}{q}\pi_0$$

hence

$$F(z) = \frac{p^2 z}{q\,(q-pz)}\pi_0$$

hence

$$L = \lim_{z\to 1}F(z) = \frac{p^2}{q\,(q-p)}\pi_0.$$

Because

$$\pi_0 + \pi_1 + L = 1$$

it follows that

$$\left[1 + \frac{p}{q} + \frac{p^2}{q\,(q-p)}\right]\pi_0 = 1$$

therefore

$$\pi_0 = \frac{q-p}{q}$$

and thus

$$\pi_j = \frac{q-p}{q}\left(\frac{p}{q}\right)^j, \qquad j \geq 0.$$

Note that, if $k = \log_{q/p}(n) + h + 1$, we have

$$\left(\frac{q}{p}\right)^k = n\left(\frac{q}{p}\right)^{h+1}$$

thus

$$\pi_k n = \frac{q-p}{q}\left(\frac{p}{q}\right)^k n = \frac{q-p}{q}\left(\frac{q}{p}\right)^{-(h+1)} = \frac{p(q-p)}{q^2}\left(\frac{q}{p}\right)^{-h}.$$

By [3],

$$\mathbb{E}(C) = 1 + p\mathbb{E}(C) + q\left(\frac{p}{q}\right)\mathbb{E}(C)$$

equivalently

$$\mathbb{E}(C) = \frac{1}{q-p}$$

which implies

$$\begin{aligned}\mathbb{P}\left\{M_n \leq \log_{q/p}(n) + h\right\} &= P\left\{M_n < \log_{q/p}(n) + h + 1\right\} \\ &\sim \exp\left[-\frac{p(q-p)^2}{q^2}\left(\frac{q}{p}\right)^{-h}\right]\end{aligned}$$

as $n \to \infty$. This formula appears in [2]. A similar argument gives an analogous result for random walks with strong reflection at the origin.

1. TRAFFIC LIGHT: $\ell = 1$

We separate the walk into two subwalks: $i \equiv 0 \bmod 2$ and $i \equiv 1 \bmod 2$. Let

$$U = \begin{pmatrix} q & p & 0 & 0 & 0 & \cdots \\ 0 & q & p & 0 & 0 & \cdots \\ 0 & 0 & q & p & 0 & \cdots \\ 0 & 0 & 0 & q & p & \cdots \\ 0 & 0 & 0 & 0 & q & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \mathbb{P}\{j+1|\, j\} = p$$

denote the (infinite) transition matrix from 0 to 1, and

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ q & p & 0 & 0 & 0 & \cdots \\ 0 & q & p & 0 & 0 & \cdots \\ 0 & 0 & q & p & 0 & \cdots \\ 0 & 0 & 0 & q & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \mathbb{P}\{j-1|\,j\} = q$$

denote the transition matrix from 1 to 0.

1.1. Subwalk on Even Times. The subwalk for $i \equiv 0 \bmod 2$ has transition matrix

$$UV = \begin{pmatrix} (1+p)q & p^2 & 0 & 0 & 0 & \cdots \\ q^2 & 2pq & p^2 & 0 & 0 & \cdots \\ 0 & q^2 & 2pq & p^2 & 0 & \cdots \\ 0 & 0 & q^2 & 2pq & p^2 & \cdots \\ 0 & 0 & 0 & q^2 & 2pq & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

thus

$$\pi_j = p^2\pi_{j-1} + 2pq\pi_j + q^2\pi_{j+1}, \qquad j \geq 1;$$
$$\pi_0 = (1+p)q\pi_0 + q^2\pi_1$$

hence

$$q^2\pi_1 = [1-(1+p)q]\,\pi_0 = \left[1-\left(1-p^2\right)\right]\pi_0$$

hence

$$\pi_1 = \frac{p^2}{q^2}\pi_0.$$

From

$$\pi_1 = p^2\pi_0 + 2pq\pi_1 + q^2\pi_2$$

we deduce

$$q^2\pi_2 = (1-2pq)\pi_1 - p^2\pi_0 = \left(1-2pq-q^2\right)\pi_1 = p^2\pi_1$$

hence

$$\pi_2 = \frac{p^2}{q^2}\pi_1 = \frac{p^4}{q^4}\pi_0.$$

Defining

$$\begin{aligned} F(z) &= \sum_{j=2}^{\infty} \pi_j z^j \\ &= p^2 z \sum_{j=2}^{\infty} \pi_{j-1} z^{j-1} + 2pq \sum_{j=2}^{\infty} \pi_j z^j + \frac{q^2}{z} \sum_{j=2}^{\infty} \pi_{j+1} z^{j+1} \\ &= p^2 z \left[F(z) + \pi_1 z \right] + 2pq F(z) + \frac{q^2}{z} \left[F(z) - \pi_2 z^2 \right] \end{aligned}$$

we have

$$\left[1 - p^2 z - 2pq - \frac{q^2}{z} \right] F(z) = p^2 \pi_1 z^2 - q^2 \pi_2 z$$

hence

$$\left[q^2 - (1 - 2pq) z + p^2 z^2 \right] F(z) = \left(q^2 \pi_2 - p^2 \pi_1 z \right) z^2$$

hence

$$(1 - z) \left(q^2 - p^2 z \right) F(z) = \frac{p^4 (1 - z) z^2}{q^2} \pi_0$$

hence

$$F(z) = \frac{p^4 z^2}{q^2 (q^2 - p^2 z)} \pi_0$$

hence

$$L = \lim_{z \to 1} F(z) = \frac{p^4}{q^2 (q^2 - p^2)} \pi_0 = \frac{p^4}{q^2 (q - p)} \pi_0.$$

Because

$$\pi_0 + \pi_1 + L = 1$$

it follows that

$$\left[1 + \frac{p^2}{q^2} + \frac{p^4}{q^2 (q - p)} \right] \pi_0 = 1$$

therefore

$$\pi_0 = \frac{q - p}{q^2}$$

and thus

$$\pi_j = \frac{q - p}{q^2} \left(\frac{p^2}{q^2} \right)^j, \qquad j \geq 0.$$

Note that, if $k = \log_{q^2/p^2}(n) + h + 1$, we have

$$\left(\frac{q^2}{p^2} \right)^k = n \left(\frac{q^2}{p^2} \right)^{h+1}$$

thus

$$\pi_k \frac{n}{2} = \frac{q-p}{2q^2}\left(\frac{p^2}{q^2}\right)^k n = \frac{q-p}{2q^2}\left(\frac{q^2}{p^2}\right)^{-(h+1)} = \frac{p^2(q-p)}{2q^4}\left(\frac{q^2}{p^2}\right)^{-h}.$$

By [3],

$$\mathbb{E}(C) = \frac{1}{q^2-p^2} = \frac{1}{q-p}$$

as before, which implies

$$\begin{aligned}\mathbb{P}\left\{M_n \le \log_{q^2/p^2}(n) + h\right\} &= P\left\{M_n < \log_{q^2/p^2}(n) + h + 1\right\} \\ &\sim \exp\left[-\frac{p^2(q-p)^2}{2q^4}\left(\frac{q^2}{p^2}\right)^{-h}\right]\end{aligned}$$

as $n \to \infty$.

1.2. Subwalk on Odd Times. The subwalk for $i \equiv 1 \bmod 2$ has transition matrix

$$VU = \begin{pmatrix} q & p & 0 & 0 & 0 & \cdots \\ q^2 & 2pq & p^2 & 0 & 0 & \cdots \\ 0 & q^2 & 2pq & p^2 & 0 & \cdots \\ 0 & 0 & q^2 & 2pq & p^2 & \cdots \\ 0 & 0 & 0 & q^2 & 2pq & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

thus

$$\pi_j = p^2\pi_{j-1} + 2pq\pi_j + q^2\pi_{j+1}, \qquad j \ge 2;$$
$$\pi_1 = p\pi_0 + 2pq\pi_1 + q^2\pi_2;$$
$$\pi_0 = q\pi_0 + q^2\pi_1$$

hence

$$q^2\pi_1 = (1-q)\pi_0 = p\pi_0$$

hence

$$\pi_1 = \frac{p}{q^2}\pi_0.$$

Also

$$q^2\pi_2 = (1-2pq)\pi_1 - p\pi_0 = \left(1-2pq-q^2\right)\pi_1 = p^2\pi_1$$

hence

$$\pi_2 = \frac{p^2}{q^2}\pi_1 = \frac{p^3}{q^4}\pi_0.$$

As earlier, we have

$$\left[q^2 - (1-2pq)z + p^2 z^2\right] F(z) = \left(q^2 \pi_2 - p^2 \pi_1 z\right) z^2$$

hence

$$(1-z)\left(q^2 - p^2 z\right) F(z) = \frac{p^3 (1-z) z^2}{q^2} \pi_0$$

hence

$$F(z) = \frac{p^3 z^2}{q^2 (q^2 - p^2 z)} \pi_0$$

hence

$$L = \lim_{z \to 1} F(z) = \frac{p^3}{q^2 (q^2 - p^2)} \pi_0 = \frac{p^3}{q^2 (q-p)} \pi_0.$$

Because

$$\pi_0 + \pi_1 + L = 1$$

it follows that

$$\left[1 + \frac{p}{q^2} + \frac{p^3}{q^2 (q-p)}\right] \pi_0 = 1$$

therefore

$$\pi_0 = \frac{q-p}{q}$$

and thus

$$\pi_j = \frac{q-p}{pq} \left(\frac{p^2}{q^2}\right)^j, \qquad j \geq 1.$$

With k as before,

$$\pi_k \frac{n}{2} = \frac{q-p}{2pq} \left(\frac{p^2}{q^2}\right)^k n = \frac{q-p}{2pq} \left(\frac{q^2}{p^2}\right)^{-(h+1)} = \frac{p(q-p)}{2q^3} \left(\frac{q^2}{p^2}\right)^{-h}$$

and

$$\mathbb{P}\left\{M_n \leq \log_{q^2/p^2}(n) + h\right\} \sim \exp\left[-\frac{p(q-p)^2}{2q^3} \left(\frac{q^2}{p^2}\right)^{-h}\right].$$

This latter formula is the one we desire and also appears in [2]. Observe that the subwalk exponential coefficient ε_0 for $i \equiv 0 \bmod 2$ possesses an extra p/q factor compared to the coefficient ε_1 for $i \equiv 1 \bmod 2$, equivalently,[1]

$$\varepsilon_1 = \frac{1}{2} \cdot \frac{p^2}{q^2} \cdot \frac{q}{p} \cdot \frac{(q-p)^2}{q^2} = \frac{p(q-p)^2}{2q^3}.$$

Of course, the two maxima are not independent.

1.3. Expected Sojourn Time. In our treatment of the subwalk represented by UV, we indicated that $\mathbb{E}(C) = 1/(q-p)$ without comment [4]. Let us elaborate on this point. Consider a random walk on the integers consisting of incremental steps satisfying

$$\begin{cases} -1 & \text{with probability } q^2, \\ 0 & \text{with probability } 2pq, \\ 1 & \text{with probability } p^2. \end{cases}$$

[2]For nonzero j, let ν_j denote the probability that, starting from $-j$, the walker eventually hits 0. Let ν_0 denote the probability that, starting from 0, the walker eventually returns to 0 (at some future time). We have two values for ν_0: when it is used in a recursion, it is equal to 1; when it corresponds to a return probability, it retains the symbol ν_0. Using

$$\nu_j = p^2\nu_{j-1} + 2pq\nu_j + q^2\nu_{j+1}, \qquad j \geq 1;$$

$$\nu_0 = p^2\nu_{-1} + 2pq + q^2\nu_1$$

define

$$\begin{aligned} \tilde{F}(z) &= \sum_{j=1}^{\infty}\nu_j z^j \\ &= p^2 z\sum_{j=1}^{\infty}\nu_{j-1}z^{j-1} + 2pq\sum_{j=1}^{\infty}\nu_j z^j + \frac{q^2}{z}\sum_{j=1}^{\infty}\nu_{j+1}z^{j+1} \\ &= p^2 z\left[\tilde{F}(z) + 1\right] + 2pq\tilde{F}(z) + \frac{q^2}{z}\left[\tilde{F}(z) - \nu_1 z\right] \end{aligned}$$

equivalently

$$\left[1 - p^2 z - 2pq - \frac{q^2}{z}\right]\tilde{F}(z) = p^2 z - q^2\nu_1$$

equivalently

$$\left[q^2 - (1-2pq)z + p^2z^2\right]\tilde{F}(z) = \left(q^2\nu_1 - p^2 z\right)z$$

equivalently

$$(1-z)\left(q^2 - p^2 z\right)\tilde{F}(z) = \left(\nu_0 - p^2\nu_{-1} - 2pq - p^2 z\right)z.$$

Only the first of the zeroes 1, q^2/p^2 is of interest (the second is > 1). Substituting $z = 1$ into the numerator of $\tilde{F}(z)$ gives an equation

$$Eq_1 : \nu_0 - p^2\nu_{-1} - 2pq - p^2 = 0.$$

Also, using

$$\nu_{-j} = p^2\nu_{-j-1} + 2pq\nu_{-j} + q^2\nu_{-j+1}, \qquad j \geq 1$$

we deduce that[3]

$$\nu_{-j} = \nu_j \left(\frac{q^2}{p^2}\right)^j$$

since multiplying both sides of

$$\nu_j \left(\frac{q^2}{p^2}\right)^j = p^2\nu_{j+1} \left(\frac{q^2}{p^2}\right)^{j+1} + 2pq\nu_j \left(\frac{q^2}{p^2}\right)^j + q^2\nu_{j-1} \left(\frac{q^2}{p^2}\right)^{j-1}$$

by p^{2j}/q^{2j} gives an identity. Replacing $q^2\nu_1$ by $p^2\nu_{-1}$ in our initial expression for ν_0 gives another equation

$$Eq_2 : \nu_0 = 2p^2\nu_{-1} + 2pq.$$

Solving Eq_1 and Eq_2 simultaneously yields $\nu_0 = 2p$ and $\nu_{-1} = 1$. More generally, $\nu_{-j} = 1$ for $j \geq 1$. Most importantly,

$$\mathbb{E}(C) = \frac{1}{1-\nu_0} = \frac{1}{q-p}$$

as was to be shown.

We will similarly study random walks

$$\begin{cases} -2 & \text{with probability } q^4, \\ -1 & \text{with probability } 4pq^3, \\ 0 & \text{with probability } 6p^2q^2, \\ 1 & \text{with probability } 4p^3q, \\ 2 & \text{with probability } p^4; \end{cases}$$

$$\begin{cases} -3 & \text{with probability } q^6, \\ -2 & \text{with probability } 6pq^5, \\ -1 & \text{with probability } 15p^2q^4 \\ 0 & \text{with probability } 20p^3q^3, \\ 1 & \text{with probability } 15p^4q^2, \\ 2 & \text{with probability } 6p^5q, \\ 3 & \text{with probability } p^6 \end{cases}$$

in Sections 2.2 and 3.2 respectively. The formulas for $\mathbb{E}(C)$, however, will be somewhat more complicated.

2. Traffic Light: $\ell = 2$

We have four subwalks. Let us consider the subwalk represented by U^2V^2:

$$\begin{pmatrix} (1+2p+3p^2)\,q^2 & 4p^3q & p^4 & 0 & 0 & 0 & \cdots \\ (1+3p)q^3 & 6p^2q^2 & 4p^3q & p^4 & 0 & 0 & \cdots \\ q^4 & 4pq^3 & 6p^2q^2 & 4p^3q & p^4 & 0 & \cdots \\ 0 & q^4 & 4pq^3 & 6p^2q^2 & 4p^3q & p^4 & \cdots \\ 0 & 0 & q^4 & 4pq^3 & 6p^2q^2 & 4p^3q & \cdots \\ 0 & 0 & 0 & q^4 & 4pq^3 & 6p^2q^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

2.1. Stationary Distribution.

For $j \geq 2$, we have

$$\pi_j = p^4\pi_{j-2} + 4p^3q\pi_{j-1} + 6p^2q^2\pi_j + 4pq^3\pi_{j+1} + q^4\pi_{j+2};$$

$$\pi_0 = \left(1+2p+3p^2\right)q^2\pi_0 + (1+3p)q^3\pi_1 + q^4\pi_2$$

hence

$$\pi_2 = \frac{\left(1-q^2-2pq^2-3p^2q^2\right)\pi_0 - (1+3p)q^3\pi_1}{q^4};$$

$$\pi_1 = 4p^3q\pi_0 + 6p^2q^2\pi_1 + 4pq^3\pi_2 + q^4\pi_3$$

hence

$$\pi_3 = \frac{-4p^3q\pi_0 + \left(1-6p^2q^2\right)\pi_1 - 4pq^3\pi_2}{q^4}.$$

Defining

$$\begin{aligned} F(z) &= \sum_{j=2}^{\infty}\pi_j z^j \\ &= p^4z^2\sum_{j=2}^{\infty}\pi_{j-2}z^{j-2} + 4p^3qz\sum_{j=2}^{\infty}\pi_{j-1}z^{j-1} + 6p^2q^2\sum_{j=2}^{\infty}\pi_j z^j \\ &+ \frac{4pq^3}{z}\sum_{j=2}^{\infty}\pi_{j+1}z^{j+1} + \frac{q^4}{z^2}\sum_{j=2}^{\infty}\pi_{j+2}z^{j+2} \\ &= p^4z^2\left[F(z)+\pi_0+\pi_1 z\right] + 4p^3qz\left[F(z)+\pi_1 z\right] + 6p^2q^2F(z) \\ &+ \frac{4pq^3}{z}\left[F(z)-\pi_2z^2\right] + \frac{q^4}{z^2}\left[F(z)-\pi_2z^2-\pi_3z^3\right] \end{aligned}$$

we deduce

$$\left[1 - p^4z^2 - 4p^3qz - 6p^2q^2 - \frac{4pq^3}{z} - \frac{q^4}{z^2}\right] F(z)$$
$$= p^4z^2\left(\pi_0 + \pi_1 z\right) + 4p^3qz\left(\pi_1 z\right) - \frac{4pq^3}{z}\left(\pi_2 z^2\right) - \frac{q^4}{z^2}\left(\pi_2 z^2 + \pi_3 z^3\right)$$

hence

$$\left[q^4 + 4pq^3z - \left(1 - 6p^2q^2\right)z^2 + 4p^3qz^3 + p^4z^4\right] F(z)$$
$$= -p^4z^4\left(\pi_0 + \pi_1 z\right) - 4p^3qz^3\left(\pi_1 z\right) + 4pq^3z\left(\pi_2 z^2\right) + q^4\left(\pi_2 z^2 + \pi_3 z^3\right).$$

Replacing π_2 and π_3 by expressions in π_0 and π_1, then cancelling the common factor $1 - z$ between numerator and denominator, yields

$$F(z) = \frac{\left\{p^3(4 - 3p + pz)\pi_0 + \left[-1 + 6p^2 - 8p^3 + 3p^4 + (4 - 3p)p^3z + p^4z^2\right]\pi_1\right\} z^2}{(q^2 - p^2z)\left[q^2 + (1 + 2pq)z + p^2z^2\right]}$$

hence

$$L = \lim_{z\to 1} F(z) = \frac{2p^3(1 + q)\pi_0 - \left[2(q - p) - q^4\right]\pi_1}{2(q - p)}.$$

We observe three zeroes in the denominator $D(z)$ of $F(z)$. The first zero, of smallest modulus < 1, is negative and given by

$$z_1 = \frac{-1 - 2pq + \theta}{2p^2}$$

where $\theta = \sqrt{1 + 4pq}$. The second zero, of intermediate modulus, is positive and given by

$$z_2 = \frac{q^2}{p^2} > 1.$$

The third zero, of largest modulus > 1, is negative and given by

$$z_3 = \frac{-1 - 2pq - \theta}{2p^2}.$$

Finding the unknowns π_0 and π_1 is achieved by solving two simultaneous equations:

$$Eq_1 : subst\left(z = z_1, N\right) = 0$$

(substituting z_1 for z in the numerator $N(z)$ for $F(z)$ and setting this equal to zero)

$$Eq_2 : \pi_0 + \pi_1 + L = 1$$

which yields

$$\pi_0 = \frac{(q-p)\,(3-2p-\theta)}{2q^4},$$
$$\pi_1 = \frac{(q-p)\,[-1-p-2pq+(1+p)\theta]}{q^5}.$$

Thus we have a complete description of the stationary distribution. An exact expression for π_j is infeasible; therefore asymptotics as $j \to \infty$ are necessary. The second zero z_2 leads, by classical singularity analysis, to [5]

$$A(p) = -\frac{N(z_2)}{z_2 D'(z_2)} = \frac{(q-p)\,[1+(q-p)\theta]}{4q^4},$$
$$\pi_j \sim A(p)\left(\frac{p^2}{q^2}\right)^j.$$

This is the expression that we shall use in the clumping heuristic.

2.2. Clump Rate. Using

$$\nu_j = p^4\nu_{j-2} + 4p^3q\nu_{j-1} + 6p^2q^2\nu_j + 4pq^3\nu_{j+1} + q^4\nu_{j+2}, \qquad j \geq 1;$$
$$\nu_0 = p^4\nu_{-2} + 4p^3q\nu_{-1} + 6p^2q^2 + 4pq^3\nu_1 + q^4\nu_2$$

define

$$\begin{aligned}
\tilde{F}(z) &= \sum_{j=1}^{\infty}\nu_j z^j \\
&= p^4z^2\sum_{j=1}^{\infty}\nu_{j-2}z^{j-2} + 4p^3qz\sum_{j=1}^{\infty}\nu_{j-1}z^{j-1} + 6p^2q^2\sum_{j=1}^{\infty}\nu_j z^j \\
&\quad + \frac{4pq^3}{z}\sum_{j=1}^{\infty}\nu_{j+1}z^{j+1} + \frac{q^4}{z^2}\sum_{j=1}^{\infty}\nu_{j+2}z^{j+2} \\
&= p^4z^2\left[\tilde{F}(z) + \nu_{-1}z^{-1} + 1\right] + 4p^3qz\left[\tilde{F}(z) + 1\right] + 6p^2q^2\tilde{F}(z) \\
&\quad + \frac{4pq^3}{z}\left[\tilde{F}(z) - \nu_1 z\right] + \frac{q^4}{z^2}\left[\tilde{F}(z) - \nu_1 z - \nu_2 z^2\right]
\end{aligned}$$

equivalently

$$\begin{aligned}
&\left[1 - p^4z^2 - 4p^3qz - 6p^2q^2 - \frac{4pq^3}{z} - \frac{q^4}{z^2}\right]\tilde{F}(z) \\
&= p^4z^2\left(\nu_{-1}z^{-1} + 1\right) + 4p^3qz - \frac{4pq^3}{z}\left(\nu_1 z\right) - \frac{q^4}{z^2}\left(\nu_1 z + \nu_2 z^2\right)
\end{aligned}$$

equivalently

$$\left[q^4 + 4pq^3 z - \left(1 - 6p^2q^2\right) z^2 + 4p^3 q z^3 + p^4 z^4\right] \tilde{F}(z)$$
$$= -p^4 z^4 \left(\nu_{-1} z^{-1} + 1\right) - 4p^3 q z^3 + 4pq^3 z \left(\nu_1 z\right) + q^4 \left(\nu_1 z + \nu_2 z^2\right)$$

equivalently

$$(1-z)(q^2 - p^2 z)\left[q^2 + (1+2pq)z + p^2 z^2\right] \tilde{F}(z)$$
$$= -p^4 z^3 \nu_{-1} - p^4 z^4 - 4p^3 q z^3 + 4pq^3 z^2 \nu_1 + q^4 z \nu_1$$
$$+ z^2 \left(\nu_0 - p^4 \nu_{-2} - 4p^3 q \nu_{-1} - 6p^2 q^2 - 4pq^3 \nu_1\right)$$
$$= z^2 \nu_0 + q^4 z \nu_1 - p^4 z^3 \nu_{-1} - 4p^3 q z^2 \nu_{-1} - p^4 z^2 \nu_{-2} - 6p^2 q^2 z^2 - 4p^3 q z^3 - p^4 z^4.$$

Only the first two of the four zeroes z_1, 1, z_2, z_3 are of interest. Let $\tilde{N}(z)$ denote the numerator for $\tilde{F}(z)$. We have

$$\tilde{E}q_1 : subst\left(z = z_1, \tilde{N}\right) = 0,$$

$$\tilde{E}q_2 : subst\left(z = 1, \tilde{N}\right) = 0.$$

Replacing $q^4 \nu_2$ by $p^4 \nu_{-2}$ in our initial expression for ν_0 gives

$$\tilde{E}q_3 : \nu_0 = 2p^4 \nu_{-2} + 4p^3 q \nu_{-1} + 6p^2 q^2 + 4pq^3 \nu_1.$$

Also, replacing $q^2 \nu_1$ by $p^2 \nu_{-1}$ throughout $\tilde{E}q_1$, $\tilde{E}q_2$ and $\tilde{E}q_3$ reduces the number of variables to three. The simultaneous solution is

$$\nu_0 = \frac{-1 + 2p + 8p^2 - 8p^3 + (q-p)^2\theta}{4pq},$$

$$\nu_{-1} = \frac{1 - 8p^2 + 16p^3 - 8p^4 - (q-p)\theta}{8p^3 q},$$

$$\nu_{-2} = \frac{-1 - 2p + 12p^2 - 24p^4 + 24p^5 - 8p^6 + (q-p)(1+2p-4p^2)\theta}{8p^5 q}$$

yielding

$$\nu_1 = \frac{1 - 8p^2 + 16p^3 - 8p^4 - (q-p)\theta}{8pq^3}$$

in particular.

[4]Readers might be tempted to use $\{0\}$ as the absorbing set Ω, imitating what we did in Section 1.3. But, starting from a negative integer, the walker could stray into

the positive integers without ever touching 0 due to the transition p^4. We must take $\Omega = \{0, -1\}$: no walk can venture above 0 without touching at least one of 0 or -1.

An idea of Aldous [3] now comes crucially into play. The rate λ of clumps of visits to Ω is equal to $\lambda_0 + \lambda_{-1}$ where parameters λ_0 and λ_{-1} are solutions of the system

$$\lambda_0 + \lambda_{-1}\nu_{-1} = (1-\nu_0)\,\pi_j,$$
$$\lambda_0\nu_1 + \lambda_{-1} = (1-\nu_0)\,\pi_{j+1} \sim \frac{p^2}{q^2}\,(1-\nu_0)\,\pi_j.$$

In words, for nonzero j, the ratio $\nu_j/\,(1-\nu_0)$ is the expected sojourn time in $\{0\}$, given that the walk started at $-j$. The total clump rate is consequently

$$\lambda \sim \frac{(q-p)\,[1+(q-p)\theta]}{2q^2}\pi_j$$

and thus the exponential coefficient is

$$\varepsilon_1 = \frac{1}{4}\cdot\frac{p^2}{q^2}\cdot\frac{q^2}{p^2}\cdot\frac{\lambda}{\pi_j}\cdot A(p) \sim \frac{(q-p)^2\,[1+(q-p)\theta]^2}{32q^6} = \frac{\chi_2(p)}{4}$$

where

$$\chi_2(p) = \frac{(q-p)^2}{4q^6}\left[\left(1-8p^2+16p^3-8p^4\right)+(q-p)\theta\right]$$

was conjectured in [1]. What was missed previously, however, is the expression for ε_0 as a perfect square. This fact is a corollary of the hidden relation

$$\frac{\lambda}{\pi_j} = 2q^2A(p)$$

("hidden" in the sense that our experimental methods in [1] overlooked this intriguing formula).

3. TRAFFIC LIGHT: $\ell = 3$

We have six subwalks. Let us consider the subwalk represented by U^3V^3:

$$\begin{pmatrix}
\left(1+3p+6p^2+10p^3\right)q^3 & 15p^4q^2 & 6p^5q & p^6 & 0 & 0 & 0 & 0 & \cdots\\
\left(1+4p+10p^2\right)q^4 & 20p^3q^3 & 15p^4q^2 & 6p^5q & p^6 & 0 & 0 & 0 & \cdots\\
(1+5p)q^5 & 15p^2q^4 & 20p^3q^3 & 15p^4q^2 & 6p^5q & p^6 & 0 & 0 & \cdots\\
q^6 & 6pq^5 & 15p^2q^4 & 20p^3q^3 & 15p^4q^2 & 6p^5q & p^6 & 0 & \cdots\\
0 & q^6 & 6pq^5 & 15p^2q^4 & 20p^3q^3 & 15p^4q^2 & 6p^5q & p^6 & \cdots\\
0 & 0 & q^6 & 6pq^5 & 15p^2q^4 & 20p^3q^3 & 15p^4q^2 & 6p^5q & \cdots\\
0 & 0 & 0 & q^6 & 6pq^5 & 15p^2q^4 & 20p^3q^3 & 15p^4q^2 & \cdots\\
0 & 0 & 0 & 0 & q^6 & 6pq^5 & 15p^2q^4 & 20p^3q^3 & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

3.1. Stationary Distribution. For $j \geq 3$, we have

$$\pi_j = p^6\pi_{j-3} + 6p^5q\pi_{j-2} + 15p^4q^2\pi_{j-1} + 20p^3q^3\pi_j + 15p^2q^4\pi_{j+1} + 6pq^5\pi_{j+2} + q^6\pi_{j+3};$$

$$\pi_0 = \left(1 + 3p + 6p^2 + 10p^3\right)q^3\pi_0 + \left(1 + 4p + 10p^2\right)q^4\pi_1 + (1 + 5p)q^5\pi_2 + q^6\pi_3$$

hence

$$\pi_3 = \frac{\left(1 - q^3 - 3pq^3 - 6p^2q^3 - 10p^3q^3\right)\pi_0 - (1 + 4p + 10p^2)q^4\pi_1 - (1 + 5p)q^5\pi_2}{q^6};$$

$$\pi_1 = 15p^4q^2\pi_0 + 20p^3q^3\pi_1 + 15p^2q^4\pi_2 + 6pq^5\pi_3 + q^6\pi_4$$

hence

$$\pi_4 = \frac{-15p^4q^2\pi_0 + \left(1 - 20p^3q^3\right)\pi_1 - 15p^2q^4\pi_2 - 6pq^5\pi_3}{q^6};$$

$$\pi_2 = 6p^5q\pi_0 + 15p^4q^2\pi_1 + 20p^3q^3\pi_2 + 15p^2q^4\pi_3 + 6pq^5\pi_4 + q^6\pi_5$$

hence

$$\pi_5 = \frac{-6p^5q\pi_0 - 15p^4q^2\pi_1 + \left(1 - 20p^3q^3\right)\pi_2 - 15p^2q^4\pi_3 - 6pq^5\pi_4}{q^6}.$$

Defining

$$\begin{aligned}
F(z) &= \sum_{j=3}^{\infty}\pi_j z^j \\
&= p^6z^3\sum_{j=3}^{\infty}\pi_{j-3}z^{j-3} + 6p^5qz^2\sum_{j=3}^{\infty}\pi_{j-2}z^{j-2} \\
&+ 15p^4q^2z\sum_{j=3}^{\infty}\pi_{j-1}z^{j-1} + 20p^3q^3\sum_{j=3}^{\infty}\pi_j z^j + \frac{15p^2q^4}{z}\sum_{j=3}^{\infty}\pi_{j+1}z^{j+1} \\
&+ \frac{6pq^5}{z^2}\sum_{j=3}^{\infty}\pi_{j+2}z^{j+2} + \frac{q^6}{z^3}\sum_{j=3}^{\infty}\pi_{j+3}z^{j+3} \\
&= p^6z^3\left[F(z) + \pi_0 + \pi_1 z + \pi_2 z^2\right] + 6p^5qz^2\left[F(z) + \pi_1 z + \pi_2 z^2\right] \\
&+ 15p^4q^2z\left[F(z) + \pi_2 z^2\right] + 20p^3q^3F(z) + \frac{15p^2q^4}{z}\left[F(z) - \pi_3 z^3\right] \\
&+ \frac{6pq^5}{z^2}\left[F(z) - \pi_3 z^3 - \pi_4 z^4\right] + \frac{q^6}{z^3}\left[F(z) - \pi_3 z^3 - \pi_4 z^4 - \pi_5 z^5\right]
\end{aligned}$$

we deduce

$$\left[1-p^6z^3-6p^5qz^2-15p^4q^2z-20p^3q^3-\frac{15p^2q^4}{z}-\frac{6pq^5}{z^2}-\frac{q^6}{z^3}\right]F(z)$$
$$=p^6z^3\left(\pi_0+\pi_1z+\pi_2z^2\right)+6p^5qz^2\left(\pi_1z+\pi_2z^2\right)+15p^4q^2z\left(\pi_2z^2\right)$$
$$-\frac{15p^2q^4}{z}\left(\pi_3z^3\right)-\frac{6pq^5}{z^2}\left(\pi_3z^3+\pi_4z^4\right)-\frac{q^6}{z^3}\left(\pi_3z^3+\pi_4z^4+\pi_5z^5\right)$$

hence

$$\left[q^6+6pq^5z+15p^2q^4z^2-\left(1-20p^3q^3\right)z^3+15p^4q^2z^4+6p^5qz^5+p^6z^6\right]F(z)$$
$$=-p^6z^6\left(\pi_0+\pi_1z+\pi_2z^2\right)-6p^5qz^5\left(\pi_1z+\pi_2z^2\right)-15p^4q^2z^4\left(\pi_2z^2\right)$$
$$+15p^2q^4z^2\left(\pi_3z^3\right)+6pq^5z\left(\pi_3z^3+\pi_4z^4\right)+q^6\left(\pi_3z^3+\pi_4z^4+\pi_5z^5\right).$$

Replacing π_3, π_4 and π_5 by expressions in π_0, π_1 and π_2, then cancelling the common factor $1-z$ between numerator and denominator, yields $F(z)$ to be

$$\frac{\left\{p^4\left[b+apz+p^2z^2\right]\pi_0+\left[c+bp^4z+ap^5z^2+p^6z^3\right]\pi_1+\left[d+cz+bp^4z^2+ap^5z^3+p^6z^4\right]\pi_2\right\}z^3}{(q^2-p^2z)\left[q^4+q^2(1+4pq)z+(1+2pq+6p^2q^2)z^2+p^2(1+4pq)z^3+p^4z^4\right]}$$

where

$$a=6-5p,\qquad b=15-24p+10p^2,$$
$$c=-\left(1-20p^3+45p^4-36p^5+10p^6\right),\qquad d=-\left(1-15p^2+40p^3-45p^4+24p^5-5p^6\right)$$

hence

$$L=\lim_{z\to1}F(z)=\frac{3p^4\left(1+2q+2q^2\right)\pi_0-\left[3(q-p)-2(1+2p)q^5\right]\pi_1-\left[3(q-p)-q^6\right]\pi_2}{3(q-p)}.$$

We observe five zeroes in the denominator $D(z)$ of $F(z)$. Two (complex conjugate) zeroes have modulus <1:

$$z_1=\frac{-1-i\sqrt{3}-4pq+\sqrt{-2+2i\sqrt{3}+8\left(1+i\sqrt{3}\right)pq}}{4p^2},$$

$$z_2=\frac{-1+i\sqrt{3}-4pq+\sqrt{-2-2i\sqrt{3}+8\left(1-i\sqrt{3}\right)pq}}{4p^2};$$

two zeroes have modulus >1:

$$z_4=\frac{-1-i\sqrt{3}-4pq-\sqrt{-2+2i\sqrt{3}+8\left(1+i\sqrt{3}\right)pq}}{4p^2},$$

$$z_5 = \frac{-1 + i\sqrt{3} - 4pq - \sqrt{-2 - 2i\sqrt{3} + 8\left(1 - i\sqrt{3}\right) pq}}{4p^2};$$

and the remaining (real) zero is

$$z_3 = \frac{q^2}{p^2} > 1.$$

Finding the unknowns π_0, π_1 and π_2 is achieved by solving three simultaneous equations (two involving the numerator $N(z)$ of $F(z)$):

$$Eq_1 : subst\,(z = z_1, N) = 0$$

$$Eq_2 : subst\,(z = z_2, N) = 0$$

$$Eq_3 : \pi_0 + \pi_1 + \pi_2 + L = 1$$

which yields

$$\pi_0 = \frac{(q-p)\left[7-10p+4p^2+\theta-\sqrt{2}\sqrt{1+28p-60p^2+40p^3-8p^4+(7-10p+4p^2)\theta}\right]}{4q^6},$$

$$\pi_1 = \frac{(q-p)\left[-3\left(1+7p-10p^2+4p^3\right)-3(1+p)\theta+\sqrt{6} \cdot\sqrt{-1+30p+71p^2-84p^3-100p^4+120p^5-24p^6+(7+16p-5p^2-18p^3+12p^4)\theta}\right]}{4q^7},$$

$$\pi_2 = \frac{(q-p)\left[3\left(-1+6p+14p^2-20p^3+8p^4\right)+3\left(1+4p+2p^2\right)\theta-\sqrt{6} \cdot\sqrt{-1-16p+64p^2+656p^3+52p^4-1072p^5+80p^6+480p^7-96p^8+(1+14p+120p^2+80p^3-92p^4-24p^5+48p^6)\theta}\right]}{4q^8}$$

where $\theta = \sqrt{1 + 4pq + 16p^2q^2}$. Thus we have a complete description of the stationary distribution. An exact expression for π_j is again infeasible. The third zero z_3 leads to

$$A(p) = -\frac{N(z_3)}{z_3 D'(z_3)} = \frac{(q-p)u + (q-p)^3\theta + \sqrt{2}(q-p)^2\sqrt{v + u\theta}}{12q^6},$$

$$\pi_j \sim A(p)\left(\frac{p^2}{q^2}\right)^j$$

where

$$u = 1 - 2p + 6p^2 - 8p^3 + 4p^4, \qquad v = 1 + 6p^2 - 28p^3 + 54p^4 - 48p^5 + 16p^6.$$

This is the expression that we shall use in the clumping heuristic.

3.2. Clump Rate. Using

$$\nu_j = p^6\nu_{j-3} + 6p^5q\nu_{j-2} + 15p^4q^2\nu_{j-1} + 20p^3q^3\nu_j + 15p^2q^4\nu_{j+1} + 6pq^5\nu_{j+2} + q^6\nu_{j+3}, \qquad j \geq 1;$$

$$\nu_0 = p^6\nu_{-3} + 6p^5q\nu_{-2} + 15p^4q^2\nu_{-1} + 20p^3q^3 + 15p^2q^4\nu_1 + 6pq^5\nu_2 + q^6\nu_3$$

define

$$\begin{aligned}\tilde{F}(z) &= \sum_{j=1}^{\infty}\nu_j z^j \\ &= p^6z^3\sum_{j=1}^{\infty}\nu_{j-3}z^{j-3} + 6p^5qz^2\sum_{j=1}^{\infty}\nu_{j-2}z^{j-2} \\ &+ 15p^4q^2z\sum_{j=1}^{\infty}\nu_{j-1}z^{j-1} + 20p^3q^3\sum_{j=1}^{\infty}\nu_j z^j + \frac{15p^2q^4}{z}\sum_{j=1}^{\infty}\nu_{j+1}z^{j+1} \\ &+ \frac{6pq^5}{z^2}\sum_{j=1}^{\infty}\nu_{j+2}z^{j+2} + \frac{q^6}{z^3}\sum_{j=1}^{\infty}\nu_{j+3}z^{j+3} \\ &= p^6z^3\left[\tilde{F}(z) + \nu_{-2}z^{-2} + \nu_{-1}z^{-1} + 1\right] + 6p^5qz^2\left[\tilde{F}(z) + \nu_{-1}z^{-1} + 1\right] \\ &+ 15p^4q^2z\left[\tilde{F}(z) + 1\right] + 20p^3q^3\tilde{F}(z) + \frac{15p^2q^4}{z}\left[\tilde{F}(z) - \nu_1 z\right] \\ &+ \frac{6pq^5}{z^2}\left[\tilde{F}(z) - \nu_1 z - \nu_2 z^2\right] + \frac{q^6}{z^3}\left[\tilde{F}(z) - \nu_1 z - \nu_2 z^2 - \nu_3 z^3\right]\end{aligned}$$

equivalently

$$\begin{aligned}&\left[1 - p^6z^3 - 6p^5qz^2 - 15p^4q^2z - 20p^3q^3 - \frac{15p^2q^4}{z} - \frac{6pq^5}{z^2} - \frac{q^6}{z^3}\right]\tilde{F}(z) \\ &= p^6z^3\left(\nu_{-2}z^{-2} + \nu_{-1}z^{-1} + 1\right) + 6p^5qz^2\left(\nu_{-1}z^{-1} + 1\right) + 15p^4q^2z \\ &- \frac{15p^2q^4}{z}\left(\nu_1 z\right) - \frac{6pq^5}{z^2}\left(\nu_1 z + \nu_2 z^2\right) - \frac{q^6}{z^3}\left(\nu_1 z + \nu_2 z^2 + \nu_3 z^3\right)\end{aligned}$$

equivalently

$$\begin{aligned}&\left[q^6 + 6pq^5z + 15p^2q^4z^2 - \left(1 - 20p^3q^3\right)z^3 + 15p^4q^2z^4 + 6p^5qz^5 + p^6z^6\right]\tilde{F}(z) \\ &= -p^6z^6\left(\nu_{-2}z^{-2} + \nu_{-1}z^{-1} + 1\right) - 6p^5qz^5\left(\nu_{-1}z^{-1} + 1\right) - 15p^4q^2z^4 \\ &+ 15p^2q^4z^2\left(\nu_1 z\right) + 6pq^5z\left(\nu_1 z + \nu_2 z^2\right) + q^6\left(\nu_1 z + \nu_2 z^2 + \nu_3 z^3\right)\end{aligned}$$

equivalently

$$\begin{aligned}
&(1-z)(q^2-p^2z)\left[q^4+q^2(1+4pq)z+(1+2pq+6p^2q^2)z^2+p^2(1+4pq)z^3+p^4z^4\right]\tilde{F}(z) \\
&= -p^6z^4\nu_{-2}-p^6z^5\nu_{-1}-p^6z^6-6p^5qz^4\nu_{-1}-6p^5qz^5-15p^4q^2z^4 \\
&+15p^2q^4z^3\nu_1+6pq^5z^2\nu_1+6pq^5z^3\nu_2+q^6z\nu_1+q^6z^2\nu_2 \\
&+z^3\left(\nu_0-p^6\nu_{-3}-6p^5q\nu_{-2}-15p^4q^2\nu_{-1}-20p^3q^3-15p^2q^4\nu_1-6pq^5\nu_2\right) \\
&= z^3\nu_0+q^6z\nu_1+6pq^5z^2\nu_1+q^6z^2\nu_2-15p^4q^2z^3\nu_{-1}-6p^5qz^4\nu_{-1}-p^6z^5\nu_{-1} \\
&-6p^5qz^3\nu_{-2}-p^6z^4\nu_{-2}-p^6z^3\nu_{-3}-20p^3q^3z^3-15p^4q^2z^4-6p^5qz^5-p^6z^6.
\end{aligned}$$

Only the first three of the six zeroes z_1, z_2, 1, z_3, z_4, z_5 are of interest. Let $\tilde{N}(z)$ denote the numerator for $\tilde{F}(z)$. We have

$$\tilde{E}q_1 : subst\left(z=z_1,\tilde{N}\right)=0,$$

$$\tilde{E}q_2 : subst\left(z=z_2,\tilde{N}\right)=0,$$

$$\tilde{E}q_3 : subst\left(z=1,\tilde{N}\right)=0.$$

Replacing $q^6\nu_3$ by $p^6\nu_{-3}$ in our initial expression for ν_0 gives

$$\tilde{E}q_4 : \nu_0 = 2p^6\nu_{-3}+6p^5q\nu_{-2}+15p^4q^2\nu_{-1}+20p^3q^3+15p^2q^4\nu_1+6pq^5\nu_2.$$

Also, replacing $q^2\nu_1$ by $p^2\nu_{-1}$ and $q^4\nu_2$ by $p^4\nu_{-2}$ throughout $\tilde{E}q_1$, $\tilde{E}q_2$, $\tilde{E}q_3$ and $\tilde{E}q_4$ reduces the number of variables to four. The simultaneous solution is

$$\nu_0 = \frac{\begin{aligned}&-2p\left(-3-18p+92p^2-120p^3+816p^4-2816p^5+3840p^6-2304p^7+512p^8\right)\\&+2(q-p)^3\left(1+4p+12p^2-32p^3+16p^4\right)\theta-(q-p)^2\theta\sqrt{2}\\&\cdot\Big[\left(-1-8p+8p^2\right)\left(-1-2p-46p^2-32p^3+848p^4-2432p^5+3200p^6-2048p^7+512p^8\right)\\&+\left(1+8p+88p^2-448p^3+1888p^4-4864p^5+6400p^6-4096p^7+1024p^8\right)\theta\Big]^{1/2}\end{aligned}}{1-256p^3+768p^4-768p^5+256p^6},$$

$$\nu_{-1} = \frac{\begin{aligned}&-6pq-(q-p)^2\left(1+32p^2-64p^3+32p^4\right)\theta+(q-p)\theta\\&\cdot\Big[(q-p)^2\left(-1-8p+8p^2\right)\left(1-8p-56p^2-128p^3+704p^4-768p^5+256p^6\right)\\&+2\left(1-32p^2+64p^3+992p^4-4096p^5+6144p^6-4096p^7+1024p^8\right)\theta\Big]^{1/2}\end{aligned}}{2p^2(1-256p^3+768p^4-768p^5+256p^6)},$$

$$\nu_{-2} = \frac{\begin{gathered}-6p^2q^2\left(5-256p^3+768p^4-768p^5+256p^6\right)-(q-p)^4\left(1+8p+24p^2-64p^3+32p^4\right)\theta+(q-p)\theta \\ \cdot\Big[\left(-1+8p+8p^2+288p^3-944p^4-3136p^5+3776p^6-73728p^7+712704p^8-2445312p^9\right. \\ \left.+4345856p^{10}-4489216p^{11}+2736128p^{12}-917504p^{13}+131072p^{14}\right) \\ +2\left(1+4p+12p^2-32p^3+16p^4\right)\left(1-8p+40p^2-320p^3+1824p^4-4864p^5+6400p^6-4096p^7+1024p^8\right)\theta\Big]^{1/2}\end{gathered}}{2p^4(1-256p^3+768p^4-768p^5+256p^6)},$$

$$\nu_{-3} = \frac{\begin{gathered}-2p\left(-3-18p-33p^2+255p^3+441p^4-643p^5-8448p^6+28416p^7-40448p^8+30720p^9-12288p^{10}+2048p^{11}\right) \\ -(q-p)^2\left(-2-10p-41p^2+190p^3+89p^4-1088p^5+1728p^6-1152p^7+288p^8\right)\theta-(q-p)\theta \\ \cdot\Big[\left(2+24p+186p^2+272p^3-1239p^4-8796p^5+43998p^6-51924p^7-581577p^8+3019604p^9-4358340p^{10}-6991872p^{11}\right. \\ \left.+38416256p^{12}-72366336p^{13}+79163136p^{14}-54779904p^{15}+23804928p^{16}-5971968p^{17}+663552p^{18}\right) \\ +2\left(1+10p+69p^2-18p^3-330p^4-3840p^5+8160p^6+62940p^7-250095p^8+56320p^9+1459360p^{10}\right. \\ \left.-4044096p^{11}+5577696p^{12}-4608000p^{13}+2322432p^{14}-663552p^{15}+82944p^{16}\right)\theta\Big]^{1/2}\end{gathered}}{2p^6(1-256p^3+768p^4-768p^5+256p^6)},$$

yielding

$$\nu_{1} = \frac{\begin{gathered}-6pq-(q-p)^2\left(1+32p^2-64p^3+32p^4\right)\theta+(q-p)\theta \\ \cdot\Big[(q-p)^2\left(-1-8p+8p^2\right)\left(1-8p-56p^2-128p^3+704p^4-768p^5+256p^6\right) \\ +2\left(1-32p^2+64p^3+992p^4-4096p^5+6144p^6-4096p^7+1024p^8\right)\theta\Big]^{1/2}\end{gathered}}{2(1-256p^3+768p^4-768p^5+256p^6)q^2},$$

$$\nu_{2} = \frac{\begin{gathered}-6p^2q^2\left(5-256p^3+768p^4-768p^5+256p^6\right)-(q-p)^4\left(1+8p+24p^2-64p^3+32p^4\right)\theta+(q-p)\theta \\ \cdot\Big[\left(-1+8p+8p^2+288p^3-944p^4-3136p^5+3776p^6-73728p^7+712704p^8-2445312p^9\right. \\ \left.+4345856p^{10}-4489216p^{11}+2736128p^{12}-917504p^{13}+131072p^{14}\right) \\ +2\left(1+4p+12p^2-32p^3+16p^4\right)\left(1-8p+40p^2-320p^3+1824p^4-4864p^5+6400p^6-4096p^7+1024p^8\right)\theta\Big]^{1/2}\end{gathered}}{2(1-256p^3+768p^4-768p^5+256p^6)q^4},$$

in particular.

We must take $\Omega = \{0, -1, -2\}$ as the absorbing set. The rate λ of clumps of visits to Ω is equal to $\lambda_0 + \lambda_{-1} + \lambda_{-2}$ where parameters λ_0, λ_{-1} and λ_{-2} are solutions of the system

$$\lambda_0 + \lambda_{-1}\nu_{-1} + \lambda_{-2}\nu_{-2} = (1-\nu_0)\,\pi_j,$$

$$\lambda_0\nu_1 + \lambda_{-1} + \lambda_{-2}\nu_{-1} = (1-\nu_0)\,\pi_{j+1} \sim \frac{p^2}{q^2}\,(1-\nu_0)\,\pi_j,$$

$$\lambda_0\nu_2 + \lambda_{-1}\nu_1 + \lambda_{-2} = (1-\nu_0)\,\pi_{j+2} \sim \frac{p^4}{q^4}\,(1-\nu_0)\,\pi_j.$$

The total clump rate is consequently

$$\lambda \sim \frac{(q-p)u + (q-p)^3\theta + \sqrt{2}(q-p)^2\sqrt{v+u\theta}}{4q^4}\pi_j$$

where u and v appear at the end of Section 3.1. Thus the exponential coefficient is

$$\varepsilon_1 = \frac{1}{6}\cdot\frac{p^2}{q^2}\cdot\frac{q^3}{p^3}\cdot\frac{\lambda}{\pi_j}\cdot A(p) \sim \frac{\left[(q-p)u + (q-p)^3\theta + \sqrt{2}(q-p)^2\sqrt{v+u\theta}\right]^2}{288pq^9} = \frac{\chi_3(p)}{6}$$

where

$$\chi_3(p) = \frac{(q-p)^2}{12pq^9}\left[\alpha + (q-p)^2\beta\theta + (q-p)\sqrt{2}\sqrt{\gamma+\alpha\beta\theta}\right],$$

$$\alpha = 1 - 4p + 10p^2 - 52p^3 + 226p^4 - 520p^5 + 640p^6 - 400p^7 + 100p^8,$$

$$\beta = 1 - 2p + 6p^2 - 8p^3 + 4p^4,$$

$$\gamma = 1 - 4p + 16p^2 - 104p^3 + 506p^4 - 1808p^5 + 5604p^6 - 15576p^7 + 35574p^8 - 61160p^9 + 75152p^{10} - 63440p^{11} + 34840p^{12} - 11200p^{13} + 1600p^{14}$$

was conjectured in [1]. Again, what was missed previously is the expression for ε_0 as a perfect square, a corollary of the hidden relation

$$\frac{\lambda}{\pi_j} = 3q^2A(p).$$

We conjecture, for arbitrary $\ell \geq 2$, that

$$\frac{\lambda}{\pi_j} = \ell q^2A(p)$$

among many possible unsolved problems.[56]

Acknowledgement Our reliance in Sections 2 and 3 on the computer algebra systems Mathematica and Maple is obvious; we thank all people who created such software.

Notes

[1]Looking ahead, in Section 2, ε_1 will correspond to the subwalk for $i \equiv 2 \bmod 4$ and ε_0 will correspond to the subwalk for $i \equiv 0 \bmod 4$, that is, to V^2U^2 and U^2V^2 respectively. It can be shown that ε_0 will possess an extra p^2/q^2 factor compared to ε_1. In Section 3, ε_1 will correspond to the subwalk for $i \equiv 3 \bmod 6$ and ε_0 will correspond to the subwalk for $i \equiv 0 \bmod 6$, that is, to V^3U^3 and U^3V^3 respectively. It can be shown that ε_0 will possess an extra p^3/q^3 factor compared to ε_1.

[2]Defining ν_j is best done as follows. I am at a large level, say, J. I place the origin at J and I wish to find the probability ν_j of returning to J starting from $J-j$, equivalently, to 0 now. I reverse the walk direction. Now $J-j$ is j and $J+j$ is $-j$. The trend is now towards the positive integers rather than the negative integers.

[3]The same identity connecting ν_{-j} and ν_j will be true in Sections 2 and 3 as well, by the same reasoning.

[4]Employing the original coordinate axis may aid understanding. Readers might be tempted to use the level j as the absorbing set S. But the maximum could be above j without ever touching level j because of the transition p^4. So we must use as S the levels j and $j+1$: no maximum can be above $j+1$ without touching at least one of the levels j or $j+1$. In the revised notation, this leads to the absorbing set $\Omega = \{0, -1\}$.

[5]In this paper, we imagined a single line of traffic flowing from east to west. In the following, imagine instead two independent Bernoulli(p) queues with no ℓ parameter at all. Think of one queue for east-to-west traffic (EW) and one queue for north-to-south (NS) traffic. The intersection of the two lines of traffic is governed by one stoplight. An (old) green light for EW traffic turns red precisely when then are no EW cars left, the (new) green light for NS traffic turns red precisely when then are no NS cars left, and so on. What is the stationary distribution for this scenario? This may be a difficult question to answer. From [3], it seems that, if π can be calculated, then the distribution of the maximum line length of idle EW cars (say) can be readily found via the clumping heuristic.

[6]Another conceptualization of the $\ell = 2$ scenario involves patients randomly arriving at a hospital emergency room. One doctor treats a new patient starting at times $\equiv 3 \bmod 4$; the other likewise at times $\equiv 4 \bmod 4$. Services lengths are constant. Allowing these to be nondeterministic complicates the analysis.

References

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