When is Metric Temporal Logic Expressively Complete?

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Reasoning about time

Timed systems are everywhere:

- Hardware circuits
- Communication protocols
- Cell phones
- Plant controllers
- Aircraft navigation systems

Want to specify:

“If I press the brake pedal then the pads will be applied.”

Expressiveness vs Computability
Reasoning about time

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Expressiveness vs Computability
LTL has emerged as the definitive temporal logic.

- “Computable”
- As expressive as first order logic [Kamp 68]

LTL cannot express quantitative properties

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Reasoning about time

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LTL cannot express quantitative properties

“If I press the brake pedal then the pads will be applied between 0.5ms and 1ms.”
Metric Temporal Logic (MTL)

[Koymans; de Roever; Pnueli ~1990] is LTL with timing constraints added to the temporal modalities

Problem:
How expressive is MTL?
How expressive is MTL?

Depends on the timing constants used...

- With no constants: $\text{MTL} = \text{FO}$ [Kamp 68]
- With integer constants: $\text{MTL} \neq \text{FO}$ [Hirshfeld and Rabinovich 07]
- With rational constants: $\text{MTL} = \text{FO}$ [H., Ouaknine and Worrell 13]

Problem:
When is MTL expressively complete?
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- With rational constants: $\text{MTL} = \text{FO}$ [H., Ouaknine and Worrell 13]

**Problem:**
When is MTL expressively complete?
Temporal models

- A set $\mathbf{MP}$ of propositions: $P, Q, R, \ldots$
- Continuous time model: $\mathbb{R}$
Temporal models

- A set \( \mathbf{MP} \) of propositions: \( P, Q, R, \ldots \)
- Continuous time model: \( \mathbb{R} \)

\[
f : \mathbb{R} \rightarrow 2^{\mathbf{MP}} \quad \text{(flow or signal)}
\]
Classic temporal predicate logic

FO(<): First-order logic with < and monadic predicates for each proposition $P \in MP$:

$$\varphi ::= x < y \mid P(x) \mid \varphi_1 \land \varphi_2 \mid \varphi_1 \lor \varphi_2 \mid \neg \varphi \mid \forall x \varphi \mid \exists x \varphi$$

For example:

$$\forall x. \text{PEDAL}(x) \rightarrow \exists y. ((y > x) \land \text{BRAKE}(y))$$
Given a set $\mathcal{K} \subseteq \mathbb{R}$ of constants we add many unary functions $\{+c : c \in \mathcal{K}\}$ to $\text{FO}(\prec)$ to model moving $c$ time units into the future.

$$\forall x. \text{PEDAL}(x) \rightarrow \exists y. (x+5 < y < x+10) \land \text{BRAKE}(y),$$

a formula of $\text{FO}_{\{5,10\}}$. 
Temporal logic: LTL

LTL: Propositional logic with temporal modalities:

$$\theta ::= P \mid \theta_1 \land \theta_2 \mid \theta_1 \lor \theta_2 \mid \neg \theta$$

- **F \theta** (\theta occurs in the Future)
- **G \theta** (\theta occurs always (Globally))
- **\theta_1 U \theta_2** (\theta_1 holds Until \theta_2)
- **P \theta** (\theta occurred in the Past)
- **H \theta** (\theta has always occurred (Historically))
- **\theta_1 S \theta_2** (\theta_1 has held Since \theta_2)

For example,

$$G(PEDAL \rightarrow F BRAKE)$$
Metric Temporal Logic

${\text{MTL}}_{\mathcal{K}} = \text{LTL} + \text{timing constraints taken from } \mathcal{K} \text{ on operators:}$

\[
\theta ::= P \mid \theta_1 \land \theta_2 \mid \theta_1 \lor \theta_2 \mid \neg \theta \\
\mid F_I \theta \quad (\theta \text{ occurs in the Future in the interval } I) \\
\mid G_I \theta \quad (\theta \text{ occurs always (Globally) in the interval } I) \\
\mid \theta_1 U_I \theta_2 \quad (\theta_2 \text{ holds in } I \text{ and Until then } \theta_1 \text{ holds}) \\
\mid P_I \theta \quad (\theta \text{ occurred in the Past in the interval } I) \\
\mid H_I \theta \quad (\theta \text{ always occurred in the interval } I) \\
\mid \theta_1 S_I \theta_2 \quad (\theta_2 \text{ held in } I \text{ and } \theta_1 \text{ has held Since})
\]

where $I$ is an interval with end-points in $\mathcal{K}$.

\[
\mathbf{G} (\text{PEDAL} \rightarrow \mathbf{F}_{(5,10)} \text{ BRAKE})
\]
Adding time metrics to the models

What sets of constants $\mathcal{K}$?

- Traditional approach: intervals over $\mathbb{Z}$
- Continuous but finitely presentable: intervals over $\mathbb{Q}$
- Intervals over an arbitrary additive subgroup of $\mathbb{R}$...
What sets of constants $\mathcal{K}$?

- Traditional approach: intervals over $\mathbb{Z}$
- Continuous but finitely presentable: intervals over $\mathbb{Q}$
- Intervals over an arbitrary additive subgroup of $\mathbb{R}$...
Additive subgroup?

- Can easily form integer linear combinations of timing constants.
- Integer linear combinations of $\mathcal{K} = \text{Subgroup of } (\mathbb{R}, +)$ generated by $\mathcal{K}$.

Motivation:
- Includes most general case ($\mathcal{K} = \mathbb{R}$)
- Generalizes previous cases ($\mathcal{K} = \mathbb{Z}, \mathbb{Q}$, or $\{0\}$)
- Can be used to model multiple independent asynchronous timing systems (e.g. $\mathbb{Z}[\sqrt{2}]$)
Main result

Theorem

$MTL_{\mathcal{K}} = FO_{\mathcal{K}}$ if and only if $\mathcal{K}$ is dense.
Proof: “Only if”

Lemma

If $\mathcal{K}$ is a non-dense additive subgroup of $\mathbb{R}$ then $\mathcal{K} = \epsilon \mathbb{Z}$ for some $\epsilon \in \mathbb{R}$. 
Proof: “If”

1. Use “metric separation” to reduce to bounded formulas.
2. Use a normal form for FO_\mathcal{K} formulas to remove \(+c\) functions.
3. Use denseness of \mathcal{K} to express LTL statements restricted to an interval.
Expressive completeness of LTL can be proven by separating formulas into *past*, *present*, and *future*.

Separation does not hold in the quantitative setting.

For example,

\[ G(BRAKE \rightarrow P_{(5,10)}PEDAL) \]
General quantitative separation

Given a constant $c > 0$, a metric temporal formula is:

- **pure $c$-distant past** if it is invariant on flows that agree on $(-\infty, -c)$
- **pure $c$-distant future** if it is invariant on flows that agree on $(c, \infty)$
- **bounded** if there is an $N$ such that it is invariant on all flows that agree on $(-N, N)$

A temporal logic with constants from $\mathcal{K}$ is **generally metrically separable** if every formula is equivalent, for some $c \in \mathcal{K}_{>0}$, to a boolean combination of pure $c$-distant past, pure $c$-distant future and bounded formulas.

**Lemma**

$MTL_{\mathcal{K}}$ is generally metrically separable for non-trivial $\mathcal{K}$. 
Given a constant \( c > 0 \), a metric temporal formula is:

- **pure c-distant past** if it is invariant on flows that agree on \((-\infty, -c)\)
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A temporal logic with constants from \( \mathcal{K} \) is generally metrically separable if every formula is equivalent, for some \( c \in \mathcal{K}_{>0} \), to a boolean combination of pure c-distant past, pure c-distant future and bounded formulas.

**Lemma**

\( MTL_\mathcal{K} \) is generally metrically separable for non-trivial \( \mathcal{K} \).
General quantitative separation

Given a constant $c > 0$, a metric temporal formula is:

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A temporal logic with constants from $\mathcal{K}$ is generally metrically separable if every formula is equivalent, for some $c \in \mathcal{K}_{>0}$, to a boolean combination of pure $c$-distant past, pure $c$-distant future and bounded formulas.

**Corollary**

$\text{MTL}_{\mathcal{K}} = \text{FO}_{\mathcal{K}}$ iff $\text{MTL}_{\mathcal{K}}$ can express all bounded $\text{FO}_{\mathcal{K}}$ formulas.
Removing the unary functions

First remove unary functions from monadic predicates by introducing new predicates: e.g. \( P(x + 5) = P_5(x) \)

\[
\phi(x) = \exists y \in (x, x + 1) \exists z \in (y, y + \sqrt{2}) \ldots
\]

\[
= \exists y \in (x, x + 1) (\exists z \in (y, x + 1) \ldots \lor \exists z \in (x + 1, y + \sqrt{2}) \ldots)
\]

\[
= \exists y \in (x, x + 1) (\exists z \in (y, x + 1) \ldots \lor \exists z' \in (x + 1 - \sqrt{2}, y) \ldots)
\]

\[
\phi(x_0, x_1, x_2) = \exists y \in (x_1, x_2) (\exists z \in (y, x_2) \ldots \lor \exists z' \in (x_0, y) \ldots)
\]

Corollary

\( MTL_{\kappa} = FO_{\kappa} \iff MTL_{\kappa} \) can express all bounded \( FO_{\{0\}} \) formulas.
Removing the unary functions

Move the remaining unary functions to the free variable

\[ \varphi(x) = \exists y \in (x, x + 1) \exists z \in (y, y + \sqrt{2}) \ldots \]
\[ = \exists y \in (x, x + 1) (\exists z \in (y, x + 1) \ldots \lor \exists z \in (x + 1, y + \sqrt{2}) \ldots) \]
\[ = \exists y \in (x, x + 1) (\exists z \in (y, x + 1) \ldots \lor \exists z' \in (x + 1 - \sqrt{2}, y) \ldots) \]
\[ \varphi(x_0, x_1, x_2) = \exists y \in (x_1, x_2) (\exists z \in (y, x_2) \ldots \lor \exists z' \in (x_0, y) \ldots) \]

Corollary

\( \text{MTL}_K = \text{FO}_K \) iff \( \text{MTL}_K \) can express all bounded \( \text{FO}_{\{0\}} \) formulas.
Removing the unary functions

Move the remaining unary functions to the free variable

\[ \varphi(x) = \exists y \in (x, x + 1) \exists z \in (y, y + \sqrt{2}) \ldots \]

\[ = \exists y \in (x, x + 1) (\exists z \in (y, x + 1) \ldots \lor \exists z \in (x + 1, y + \sqrt{2}) \ldots) \]

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\[ \varphi(x_0, x_1, x_2) = \exists y \in (x_1, x_2) (\exists z \in (y, x_2) \ldots \lor \exists z' \in (x_0, y) \ldots) \]

Corollary

\[ \text{MTL}_K = \text{FO}_K \text{ iff } \text{MTL}_K \text{ can express all bounded } \text{FO}_\{0\} \text{ formulas.} \]
Removing the unary functions

Move the remaining unary functions to the free variable

\[ \varphi(x) = \exists y \in (x, x+1) \exists z \in (y, y + \sqrt{2}) \ldots \]

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\[ \varphi(x_0, x_1, x_2) = \exists y \in (x_1, x_2) (\exists z \in (y, x_2) \ldots \lor \exists z' \in (x_0, y) \ldots) \]

Corollary

MTL\(_K\) = FO\(_K\) iff MTL\(_K\) can express all bounded FO\(_\{0\}\) formulas.
Removing the unary functions

Move the remaining unary functions to the free variable

\[ \varphi(x) = \exists y \in (x, x + 1) \exists z \in (y, y + \sqrt{2}) \ldots \]

\[ = \exists y \in (x, x + 1) (\exists z \in (y, x + 1) \ldots \lor \exists z \in (x + 1, y + \sqrt{2}) \ldots) \]

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\[ \varphi(x_0, x_1, x_2) = \exists y \in (x_1, x_2) (\exists z \in (y, x_2) \ldots \lor \exists z' \in (x_0, y) \ldots) \]

Corollary

\[ MTL_K = FO_K \iff MTL_K \text{ can express all bounded } FO_{\{0\}} \text{ formulas.} \]
Removing the unary functions

Replace the “milestones” \( \{x + 1 - \sqrt{2}, x, x + 1\} \) with new variables to obtain a \( \text{FO}(\prec) \) formula.

\[
\varphi(x) = \exists y \in (x, x + 1) \exists z \in (y, y + \sqrt{2}) \ldots \\
= \exists y \in (x, x + 1) (\exists z \in (y, x + 1) \ldots \\
\quad \lor \exists z \in (x + 1, y + \sqrt{2}) \ldots ) \\
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\]

Corollary

\( \text{MTL}_K = \text{FO}_K \) iff \( \text{MTL}_K \) can express all bounded \( \text{FO}_{\{0\}} \) formulas.
Removing the unary functions

Use a model-theoretic argument to break this into formulas on the intervals \( \{x_0\}, (x_0, x_1), \{x_1\}, \ldots \)

\[
\varphi(x) = \exists y \in (x, x + 1) \exists z \in (y, y + \sqrt{2}) \ldots \\
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\( MTL_K = FO_K \) iff \( MTL_K \) can express all bounded \( FO_{\{0\}} \) formulas.
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\varphi(x_0, x_1, x_2) = \exists y \in (x_1, x_2) (\exists z \in (y, x_2) \ldots \lor \exists z' \in (x_0, y) \ldots )
\]

Corollary

\(\text{MTL}_K = \text{FO}_K\) iff \(\text{MTL}_K\) can express all bounded \(\text{FO}_{\{0\}}\) formulas.
Failure of Kamp’s theorem

MTL_\mathbb{Z} is unable to express:

“P occurs twice in the next time interval.”

In FO_\mathbb{Z}:

\varphi(z) = \exists x. \exists y. (z < x < z +) \land (z < y < z +) \land P(x) \land P(y).

In MTL_\mathbb{Z}:

(F_{(0,1)}P \land F_{(1,2)}P) \lor F_{=2}(P_{(0,1)}(P \land P_{(0,1)}P)).
Failure of Kamp’s theorem

MTL$_\mathbb{Z}$ is unable to express:

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In FO$_\mathbb{Z}$:

\[ \varphi(z) = \exists x. \exists y. (z < x < z + 1) \land (z < y < z + 1) \land P(x) \land P(y). \]

In MTL$_\mathbb{Z}$:

\[ \text{???)} \]
\[ (F_{(0,1)} P \land F_{(1,2)} P) \lor F_{=2} \left( P_{(0,1)} (P \land P_{(0,1)} P) \right) \]
MTL\(_\mathbb{Z}\) is able to express:

“\(P\) occurs twice in the next two time intervals.”

In FO\(_\mathbb{Z}\):

\[
\varphi(z) = \exists x. \exists y. (z < x < z + 2) \land (z < y < z + 2) \land P(x) \land P(y).
\]

In MTL\(_\mathbb{Z}\):

\[
\varphi = F_{(0,1)}(P \land F_{(0,1)}P) \lor (F_{(0,1)}P \land F_{(1,2)}P) \lor F_{=2}(P_{(0,1)}(P \land P_{(0,1)}P))
\]
MTL\(\mathbb{Z}\) is able to express:

“This \(P\) occurs twice in the next two time intervals.”

In FO\(\mathbb{Z}\):

\[\varphi(z) = \exists x. \exists y. (z < x < z + 2) \land (z < y < z + 2) \land P(x) \land P(y).\]

In MTL\(\mathbb{Z}\):

\[\varphi = F_{(0,1)}(P \land F_{(0,1)}P) \lor (F_{(0,1)}P \land F_{(1,2)}P) \lor F_{=2}(P_{(0,1)}(P \land P_{(0,1)}P)).\]
Failure of Kamp’s theorem

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“\(P\) occurs twice in the next two time intervals.”

In FO\(_\mathbb{Z}\):

\[
\varphi(z) = \exists x. \exists y. (z < x < z + 2) \land (z < y < z + 2) \land P(x) \land P(y).
\]

In MTL\(_\mathbb{Z}\):

\[
\varphi = F_{(0,1)}(P \land F_{(0,1)}P) \lor (F_{(0,1)}P \land F_{(1,2)}P) \lor F_{=2}\left(P_{(0,1)}\left(P \land P_{(0,1)}P\right)\right)
\]
Adding granularity

\[ \varphi = F_{(0,1)} \left( P \land F_{(0,1)} P \right) \lor \]
\[ \left( F_{(0,1)} P \land F_{(1,2)} P \right) \lor \]
\[ F_{=2} \left( P_{(0,1)} \left( P \land P_{(0,1)} P \right) \right) \]

**Corollary**

“\( P \) occurs twice in the next time interval” is expressible in \( \text{MTL}_Q \).
Adding granularity

\[ \varphi = F_{(0, \frac{1}{2})} (P \land F_{(0, \frac{1}{2})} P) \lor (F_{(0, \frac{1}{2})} P \land F_{(\frac{1}{2}, 1)} P) \lor F_{=1} \left( P_{(0, \frac{1}{2})} (P \land P_{(0, \frac{1}{2})} P) \right) \]

Corollary

“\(P\) occurs twice in the next time interval” is expressible in \(MTL_\mathbb{Q}\).
Theorem

$MTL_\mathbb{Z}$ with counting modalities has the same expressive power as $FO_\mathbb{Z}$.

Corollary

$MTL_K$ can express any bounded LTL formula if $K$ is dense and non-trivial.
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$MTL_\mathbb{Z}$ with counting modalities has the same expressive power as $FO_\mathbb{Z}$.

Corollary

$MTL_\mathcal{K}$ can express any bounded LTL formula if $\mathcal{K}$ is dense and non-trivial.
A true extension of Kamp’s theorem

Theorem

$MTL_{\mathcal{K}} = FO_{\mathcal{K}}$ if and only if $\mathcal{K}$ is dense.