When is Metric Temporal Logic Expressively Complete?*

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Abstract

A seminal result of Kamp is that over the reals Linear Temporal Logic (LTL) has the same expressive power as first-order logic with binary order relation $<$ and monadic predicates. A key question is whether there exists an analogue of Kamp's theorem for Metric Temporal Logic (MTL) – a generalization of LTL in which the Until and Since modalities are annotated with intervals that express metric constraints. Hirshfeld and Rabinovich gave a negative answer, showing that first-order logic with binary order relation $<$ and unary function $+1$ is strictly more expressive than MTL with integer constants. However, a recent result of Hunter, Ouaknine and Worrell shows that when rational timing constants are added to both languages, MTL has the same expressive power as first-order logic, giving a positive answer. In this paper we generalize these results by giving a precise characterization of those sets of constants for which MTL and first-order logic have the same expressive power. We also show that full first-order expressiveness can be recovered with the addition of counting modalities, strongly supporting the assertion of Hirshfeld and Rabinovich that Q2MLO is one of the most expressive decidable fragments of $\mathrm{FO}(<, +1)$.

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1 Introduction

One of the best-known and most widely studied logics in specification and verification is *Linear Temporal Logic (LTL)*: temporal logic with the modalities *Until* and *Since*. For discrete-time systems one considers interpretations of LTL over the integers $(\mathbb{Z}, <)$, and for continuous-time systems one considers interpretations over the reals $(\mathbb{R}, <)$. A celebrated result of Kamp [14] is that, over both $(\mathbb{Z}, <)$ and $(\mathbb{R}, <)$, LTL has the same expressiveness as the *Monadic Logic of Order ($\mathrm{FO}(<)$)*: first-order logic with binary order relation $<$ and uninterpreted monadic predicates. Thus we can benefit from the appealing variable-free syntax and elementary decision procedures of LTL, while retaining the expressiveness and canonicity of first-order logic.

Over the reals $\mathrm{FO}(<)$ cannot express quantitative properties such as "every request is followed by a response within one time unit". This motivates the introduction of *Monadic Logic of Order and Metric*, which augments $\mathrm{FO}(<)$ with a family of unary function symbols

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$+c$, for all $c \in \mathbb{R}$. We consider fragments of this logic by restricting the unary functions to some set $\mathcal{K} \subseteq \mathbb{R}$ of timing constants, and we denote this by $\mathrm{FO}_{\mathcal{K}}$. The traditional choice for $\mathcal{K}$ is $\mathbb{Z}$ (or, equivalently, $\{1\}$); however it seems more natural in the continuous time setting to have sets of timing constants that are not discrete. In [13] Hunter *et al.* considered rational timing constants, but sets of constants involving irrationals such as $\{1, \sqrt{2}\}$ or $\mathbb{R}$ have practical application: for example, in the specification of systems with two or more timing devices which are initially synchronized but have independent unit time length. In this paper we consider arbitrary subsets of $\mathbb{R}$ for $\mathcal{K}$; however we observe that with simple arithmetic any integer linear combination of elements in $\mathcal{K}$ can be derived as a unary function. Thus we restrict our attention to sets that are closed under integer linear combinations, that is, additive subgroups of $\mathbb{R}$.

There have been a variety of proposals for quantitative temporal logics, with modalities definable in $\mathrm{FO}_{\mathcal{K}}$ (see, e.g., [1, 2, 3, 7, 8, 9, 13]). Typically these temporal logics can be seen as quantitative extensions of LTL. However, until [13] there was no fully satisfactory counterpart to Kamp's theorem in the quantitative setting.

The best-known quantitative temporal logic is *Metric Temporal Logic (MTL)*, introduced over 20 years ago in [15]. MTL arises by annotating the temporal modalities of LTL with real intervals representing metric constraints. Again we consider fragments by restricting the endpoints of the intervals to some $\mathcal{K} \subseteq \mathbb{R}$, and as we are interested in various choices of $\mathcal{K}$ we denote this as $\mathrm{MTL}_{\mathcal{K}}$. Since the $\mathrm{MTL}_{\mathcal{K}}$ operators are definable in $\mathrm{FO}_{\mathcal{K}}$, it is immediate that one can translate $\mathrm{MTL}_{\mathcal{K}}$ into $\mathrm{FO}_{\mathcal{K}}$. The main question addressed by this paper is when does the converse apply?

Several previous results, illustrating that the question is non-trivial, can be succinctly specified with our notation:

- Kamp [14]: $\mathrm{MTL}_{\{0\}} = \mathrm{LTL} = \mathrm{FO}(<) = \mathrm{FO}_{\{0\}}$.
- Hirshfeld and Rabinovich [11]: $\mathrm{MTL}_{\mathbb{Z}} \neq \mathrm{FO}_{\{1\}} = \mathrm{FO}_{\mathbb{Z}}$.
- Hunter, Ouaknine and Worrell [13]: $\mathrm{MTL}_{\mathbb{Q}} = \mathrm{FO}_{\mathbb{Q}}$.

The first main result of this paper generalizes these results by giving a precise characterization of when $\mathrm{MTL}_{\mathcal{K}}$ is expressively complete.

▶ **Theorem 1.** *Let $\mathcal{K}$ be an additive subgroup of $\mathbb{R}$. Then $MTL_{\mathcal{K}} = FO_{\mathcal{K}}$ if and only if $\mathcal{K}$ is dense.*

Two consequences of this theorem are that $\mathrm{MTL}_{\mathbb{R}}$ is expressively complete (for the Monadic Logic of Order and Metric), and, in contrast to $\mathrm{MTL}_{\mathbb{Z}} \neq \mathrm{FO}_{\{1\}}$, MTL with interval endpoints taken from $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is able to express all of $\mathrm{FO}_{\{1,\sqrt{2}\}}$.

Our proof for the "if" part of this theorem is similar to the strategy used in [13]: we prove expressive completeness for bounded formulas and then use a generalization of the concept of *separation*, introduced by Gabbay [4], to extend this to all formulas. However our proof departs significantly from that result in a number of areas. In [13], the authors used scaling to reduce the problem from $\mathrm{FO}_{\mathbb{Q}}$ formulas to $\mathrm{FO}_{\mathbb{Z}}$ formulas. For general sets of constants however, this is not always possible: both in the scaling operation, which may no longer be reversible, and in the reduction to $\mathrm{FO}_{\mathbb{Z}}$. We overcome this by introducing a more general notion of separability in Lemma 13 which can account for unary functions other than $+1$. Secondly, the regularity of the integers was exploited to remove occurrences of the $+1$ function, resulting in $\mathrm{FO}_{\{0\}}$ formulas restricted to unit intervals. In our setting there is no obvious interval length that will achieve this. Instead we introduce a normal form for $\mathrm{FO}_{\mathcal{K}}$ formulas that enables us to remove the $+c$ functions and reduce the problem to $\mathrm{FO}_{\{0\}}$ formulas restricted to bounded intervals.

It follows from our proof of Theorem 1 and the result of [11] that if $\mathrm{MTL}_K \neq \mathrm{FO}_K$ then even with a (possibly infinite) set of arbitrary additional modal operators of bounded quantifier depth the inequality remains. Examples of separating formula are $\mathbf{C}_n\ \varphi$ and $\overline{\mathbf{C}}_n\ \varphi$, for sufficiently large n, where $\mathbf{C}_n$ is the modal operator which asserts that a formula is satisfied at least n distinct times in the next time interval and $\overline{\mathbf{C}}_n$ is its temporal dual. Our second main result is to show that for expressive completeness it is sufficient to add the (infinite) set of these counting operators. That is, if we define $\mathrm{MTL}_{\mathbb{Z}}$+C as the logic of $\mathrm{MTL}_{\mathbb{Z}}$ with the additional operators $\{\mathbf{C}_n, \overline{\mathbf{C}}_n \colon n \in \mathbb{N}\}$, then

▶ **Theorem 2.** *$MTL_{\mathbb{Z}}$+C has the same expressive power as $FO_{\mathbb{Z}}$.*

In [9] Hirshfeld and Rabinovich considered the addition of counting modalities to MITL: Metric Temporal Logic without singleton (punctual) intervals. They showed the resulting logic had the same expressive power as Q2MLO, a decidable fragment of $\mathrm{FO}_{\{1\}}$. Our result supports their claim that Q2MLO is one of the most expressive decidable fragments of $\mathrm{FO}_{\{1\}}$: by adding the operators $\Diamond_{\{1\}}X$ (X occurs in exactly one time unit) and $\overleftarrow{\Diamond}_{\{1\}}X$ (X occurred exactly one time unit ago) the resulting logic has the full expressive power of $\mathrm{FO}_{\{1\}}$.

2 Preliminaries

In this section we define the concepts and notation used throughout the paper.

We say $\mathcal{K} \subseteq \mathbb{R}$ is an *additive subgroup of* $\mathbb{R}$ if it is non-empty and closed under addition and unary minus, and we say $\mathcal{K}$ is *dense* if for all $a < b \in \mathcal{K}$, there exists $c \in \mathcal{K}$ such that $a < c < b$. We write $\mathcal{K}_{\geq 0}$ for the set $\{c\ :\ c \in \mathcal{K} \text{ and } c \geq 0\}$.

First-order logic.

Formulas of *Monadic Logic of Order and Metric with constants $\mathcal{K}$ ($FO_{\mathcal{K}}$)* are first-order formulas over a signature with a binary relation symbol $<$, an infinite collection of unary predicate symbols $P_1, P_2, \ldots$, and a (possibly infinite) family of unary function symbols $+c$, $c \in \mathcal{K}$. Formally, the terms of $\mathrm{FO}_{\mathcal{K}}$ are generated by the grammar $t ::= x \mid t + c$, where x is a variable and $c \in \mathcal{K}$. Formulas of $\mathrm{FO}_{\mathcal{K}}$ are given by the following syntax:

$$\varphi ::= \mathbf{true} \mid P_i(t) \mid t = t \mid t < t \mid \varphi \wedge \varphi \mid \neg\varphi \mid \exists x\, \varphi\,,$$

where x denotes a variable and t a term. When $\mathcal{K}$ is an additive subgroup it suffices to consider only terms of the form $x + c$, $c \in \mathcal{K}$.

We consider interpretations of $\mathrm{FO}_{\mathcal{K}}$ over the real line, $\mathbb{R}$, with the natural interpretations of $<$ and $+c$. It follows that a structure for $\mathrm{FO}_{\mathcal{K}}$ is determined by an interpretation of the monadic predicates.

Given terms $t_1 = x_1 + c_1$ and $t_2 = x_2 + c_2$, we define $\mathrm{Bet}_{\mathcal{K}}(t_1, t_2)$ to consist of the $\mathrm{FO}_{\mathcal{K}\cup\{c_1,c_2\}}$ formulas in which

(i) each subformula $\exists z\, \psi$ has the form $\exists z\, ((t_1 < z < t_2) \wedge \chi)$, i.e., each quantifier is relativized to the open interval between t_1 and t_2;
(ii) in each atomic subformula $P(t)$ the term t is a bound occurrence of a variable;
(iii) For $i \in \{1, 2\}$, if $c_i \notin \mathcal{K}$ then $+c_i$ only occurs in t_i.

These conditions ensure that a formula in $\mathrm{Bet}_{\mathcal{K}}(t_1, t_2)$ is essentially a $\mathrm{FO}_{\mathcal{K}}$ formula that only refers to the values of monadic predicates on points in the open interval (t_1, t_2). Clause (iii) allows us to use endpoints not necessarily definable in $\mathrm{FO}_{\mathcal{K}}$. We say that a formula $\varphi(x)$ in $\mathrm{Bet}_{\mathcal{K}}(x - N, x + N)$ is *N-bounded*.

Metric Temporal Logic.

Given a set $\boldsymbol{P}$ of atomic propositions, the formulas of *Metric Temporal Logic with constants $\mathcal{K}$ (MTL$_\mathcal{K}$)* are built from $\boldsymbol{P}$ using boolean connectives and time-constrained versions of the *Until* and *Since* operators $\mathbf{U}$ and $\mathbf{S}$ as follows:

$$\varphi ::= \mathbf{true} \mid P \mid \varphi \wedge \varphi \mid \neg\varphi \mid \varphi\ \mathbf{U}_I\ \varphi \mid \varphi\ \mathbf{S}_I\ \varphi\,,$$

where $P \in \boldsymbol{P}$ and $I \subseteq (0, \infty)$ is an interval with endpoints in $\mathcal{K}_{\geq 0} \cup \{\infty\}$.

Intuitively, the meaning of $\varphi_1\ \mathbf{U}_I\ \varphi_2$ is that φ_2 will hold at some time in the interval I, and until then φ_1 holds. More precisely, the semantics of MTL$_\mathcal{K}$ are defined as follows. A *signal* is a function $f : \mathbb{R} \to 2^{\boldsymbol{P}}$. Given a signal f and $r \in \mathbb{R}$, we define the satisfaction relation $f, r \models \varphi$ by induction over φ as follows:

- $f, r \models p$ iff $p \in f(r)$,
- $f, r \models \neg\varphi$ iff $f, r \not\models \varphi$,
- $f, r \models \varphi_1 \wedge \varphi_2$ iff $f, r \models \varphi_1$ and $f, r \models \varphi_2$,
- $f, r \models \varphi_1\ \mathbf{U}_I\ \varphi_2$ iff there exists $t > r$ such that $t - r \in I$, $f, t \models \varphi_2$ and $f, u \models \varphi_1$ for all $u \in (r, t)$,
- $f, r \models \varphi_1\ \mathbf{S}_I\ \varphi_2$ iff there exists $t < r$ such that $r - t \in I$, $f, t \models \varphi_2$ and $f, u \models \varphi_1$ for all $u \in (t, r)$.

LTL can be seen as a restriction of MTL with only the interval $I = (0, \infty)$, so in particular LTL $=$ MTL$_{\{0\}}$. MITL is a restriction of MTL$_\mathbb{Z}$ where singleton intervals, that is intervals of the form $\{c\}$, do not occur in the $\mathbf{U}$ and $\mathbf{S}$ operators.

We say the $\mathbf{U}_I$ and $\mathbf{S}_I$ operators are *bounded* if I is bounded, otherwise we say that the operators are *unbounded*.

We introduce the derived connectives $\Diamond_I\varphi := \mathbf{true}\ \mathbf{U}_I\ \varphi$ (φ will be true at some point in interval I) and $\diamondminus_I\varphi := \mathbf{true}\ \mathbf{S}_I\ \varphi$ (φ was true at some point in interval I in the past). We also have the dual connectives $\Box_I\varphi := \neg\Diamond_I\neg\varphi$ (φ will hold at all times in interval I in the future) and $\boxminus_I := \neg\diamondminus_I\neg\varphi$ (φ was true at all times in interval I in the past).

Counting modalities.

The counting modalities $\mathbf{C}_n\ \varphi$ and $\overline{\mathbf{C}}_n\ \varphi$ are defined for all $n \in \mathbb{N}$ and are interpreted as φ will be true for at least n distinct occasions in the next/previous time unit. That is, for any signal f and $r \in \mathbb{R}$:

- $f, r \models \mathbf{C}_n\ \varphi$ iff there exists $r_1 < \cdots < r_n \in (r, r+1)$ with $f, r_i \models \varphi$ for all i.
- $f, r \models \overline{\mathbf{C}}_n\ \varphi$ iff there exists $r_1 < \cdots < r_n \in (r-1, r)$ with $f, r_i \models \varphi$ for all i.

We define *MTL$_\mathcal{K}$ with counting (MTL$_\mathcal{K}$+C)* to be the extension of MTL$_\mathcal{K}$ by the operations $\{\mathbf{C}_n, \overline{\mathbf{C}}_n : n \in \mathbb{N}\}$.

Expressive Equivalence.

Given a set $\boldsymbol{P} = \{P_1, \ldots, P_m\}$ of monadic predicates, a signal $f : \mathbb{R} \to 2^{\boldsymbol{P}}$ defines an interpretation of each P_i, where $P_i(r)$ iff $P_i \in f(r)$. As observed earlier, this is sufficient to define the model-theoretic semantics of FO$_\mathcal{K}$, enabling us to relate the semantics of FO$_\mathcal{K}$ and MTL$_\mathcal{K}$.

Let $\varphi(x)$ be an FO$_\mathcal{K}$ formula with one free variable and ψ an MTL$_\mathcal{K}$ formula. We say φ and ψ are *equivalent* if for all signals f and $r \in \mathbb{R}$:

$$f \models \varphi[r] \iff f, r \models \psi.$$

We say $\mathrm{MTL}_{\mathcal{K}}$ and $\mathrm{FO}_{\mathcal{K}}$ have the same expressive power, written $\mathrm{MTL}_{\mathcal{K}} = \mathrm{FO}_{\mathcal{K}}$, if for all formulas with one free variable $\varphi(x) \in \mathrm{FO}_{\mathcal{K}}$ there is an equivalent formula $\varphi^{\dagger} \in \mathrm{MTL}_{\mathcal{K}}$ and vice versa.

3 Characterization of expressively complete MTL

The goal of this section is to prove:

▶ **Theorem 1.** *Let $\mathcal{K}$ be an additive subgroup of $\mathbb{R}$. Then $MTL_{\mathcal{K}} = FO_{\mathcal{K}}$ if and only if $\mathcal{K}$ is dense.*

First we consider the "only if" direction. Central to this is the following easily proven result:

▶ **Lemma 3.** *Let $\mathcal{K}$ be an additive subgroup of $\mathbb{R}$. If $\mathcal{K}$ is not dense then $\mathcal{K} = \epsilon\mathbb{Z}$ for some $\epsilon > 0$.*

It now follows by a simple scaling argument and the result $\mathrm{MTL}_{\mathbb{Z}} \neq \mathrm{FO}_{\mathbb{Z}}$ [11] that if $\mathcal{K}$ is not dense then $\mathrm{MTL}_{\mathcal{K}} \neq \mathrm{FO}_{\mathcal{K}}$. We refer the reader to the full version of the paper for details.

In fact [11] showed a much stronger result: even with (possibly infinite) additional arbitrary modal operators of bounded quantifier depth $\mathrm{MTL}_{\mathbb{Z}}$ cannot fully express $\mathrm{FO}_{\mathbb{Z}}$. This result clearly carries over to $\mathcal{K} = \epsilon\mathbb{Z}$, thus in the non-dense case $\mathrm{MTL}_{\mathcal{K}}$ is "quite far" from $\mathrm{FO}_{\mathcal{K}}$.

▶ **Corollary 4.** *Let $\mathcal{K}$ be a non-dense additive subgroup of $\mathbb{R}$. With additional arbitrary modal operators of bounded quantifier depth $MTL_{\mathcal{K}}$ cannot fully express $FO_{\mathcal{K}}$.*

Returning to the "if" direction in the proof of Theorem 1, we focus on the non-trivial case ($\mathcal{K}$ infinite), as the trivial case $\mathcal{K} = \{0\}$ is covered by Kamp's theorem [14]. As mentioned earlier, our strategy parallels the proof of expressive completeness of $\mathrm{MTL}_{\mathbb{Q}}$ in [13]: we first show expressive completeness for bounded formulas, and then, using a refinement of syntactic separation [4, 5, 13], extend this to all $\mathrm{FO}_{\mathcal{K}}$ formulas.

3.1 Expressive completeness for bounded formulas

To show that bounded $\mathrm{FO}_{\mathcal{K}}$ formulas can be expressed by $\mathrm{MTL}_{\mathcal{K}}$ we proceed in a similar manner to [13].

Step 1. We first remove any occurrence of a unary $+c$ function applied to a bound variable.

Step 2. Using a composition argument (see e.g. [6, 10]) we then reduce the problem to showing expressive completeness for formulas in $\mathrm{Bet}_{\{0\}}(x, x+c)$.

Step 3. Exploiting a normal form of [6] and the denseness of $\mathcal{K}$ we show how an $\mathrm{MTL}_{\mathcal{K}}$ formula can express any formula in $\mathrm{Bet}_{\{0\}}(x, x+c)$, and hence any bounded formula.

Our proof differs significantly to that of [13] notably at Steps 1 and 2. In [13] the authors were able to scale $\mathrm{FO}_{\mathbb{Q}}$ formulas to $\mathrm{FO}_{\{1\}}$ and then use the regularity of the integers to reduce the problem to formulas in $\mathrm{Bet}_{\{0\}}(x, x+1)$ (so-called *unit-formulas*). For more general $\mathcal{K}$ however neither of these steps are applicable so instead we introduce a normal form for $\mathrm{FO}_{\mathcal{K}}$ formulas which simplifies the removal of the unary functions.

Step 1. Removing unary functions.

Given an N-bounded $\mathrm{FO}_{\mathcal{K}}$ formula with one free variable x, we show that it is equivalent to a N'-bounded formula (over a possibly larger set of monadic predicates, suitably interpreted) in which the unary functions are only applied to x. We can remove occurrences of unary functions within the scope of monadic predicates by introducing new predicates. That is, we replace $P(y+c)$ with $P^c(y)$, the intended interpretation of P^c being $\{r : r+c \in P\}$. We will later replace $P^c(y)$ with $\Diamond_{\{c\}}P$ when completing the translation to $\mathrm{MTL}_{\mathcal{K}}$. Thus it suffices to demonstrate how to remove the unary functions from the scope of the $<$ relation. For this we introduce a normal form where all inequality constraints are replaced with interval inclusions and the intervals satisfy the following hierarchical condition: if y is quantified to $(x+c, z+c')$ then all intervals involving y and a variable that was free when y was quantified are affine translations of $(x+c, y)$ or $(y, z+c')$. We note that the results of this section apply for any additive subgroup $\mathcal{K} \subseteq \mathbb{R}$.

▶ **Definition 5.** An *interval-guarded formula* is a $\mathrm{FO}_{\mathcal{K}}$-formula such that all quantifiers are of the form $\exists x \in (y+c, y'+c')$ where y, y' are free variables and $c, c' \in \mathcal{K}$. A *Hierarchical Interval Formula (HIF)* is an interval-guarded $\mathrm{FO}_{\mathcal{K}}$-formula defined inductively as follows.

- Any $<$-free, quantifier-free $\mathrm{FO}_{\mathcal{K}}$-formula is a HIF;
- If φ_1, φ_2 are HIFs then so are $\neg\varphi_1$ and $\varphi_1 \vee \varphi_2$; and
- If $\varphi(\overline{x}, y)$ is a HIF and there exists $x_l, x_r \in \overline{x}$ and $c_l, c_r \in \mathcal{K}$ such that the only intervals in φ involving y and a free variable are of the form $(x_l + c_l + c, y + c)$ or $(y + c, x_r + c_r + c)$ for some $c \in \mathcal{K}$, then $\exists y \in (x_l + c_l, x_r + c_r).\varphi(\overline{x}, y)$ is a HIF.

Note that if $\varphi(\overline{x}, y)$ is a HIF then so is $\varphi(\overline{x}, u)$ for any term u involving variables from $\overline{x}$. As an example, consider the following interval-guarded $\mathrm{FO}_{\{1\}}$-formula:

$$\varphi(x) = \exists y \in (x, x+1).\exists z \in (y, y+1).\psi(x, y, z).$$

This is not a HIF as neither endpoint of the interval $(y, y+1)$ corresponds to an endpoint of $(x, x+1)$, the interval defining y. However, it is easy to see that $\varphi(x)$ is equivalent to:

$$\varphi'(x) = \exists y \in (x, x+1).\big(\exists z \in (y, x+1).\psi(x, y, z) \vee \psi(x, y, x+1) \vee \exists z \in (x+1, y+1).\psi(x, y, z)\big)$$

which is a HIF. Indeed, HIFs are a normal form for N-bounded $\mathrm{FO}_{\mathcal{K}}$ formulas with one free variable:

▶ **Lemma 6.** *Every N-bounded $FO_{\mathcal{K}}$ formula with one free variable is equivalent to a HIF.*

To prove this, the following property of HIFs that allows us to break up intervals will be useful:

▶ **Lemma 7.** *Let $\varphi(\overline{x}) = \exists y \in (s, t).\psi(\overline{x}, y)$ be a HIF. Then for any term u involving variables from $\overline{x}$, the following equivalence holds:*

$$u \in (s, t) \wedge \varphi(\overline{x}) \qquad \longleftrightarrow \qquad u \in (s, t) \wedge \big(\theta^{<}(\overline{x}) \vee \theta^{=}(\overline{x}) \vee \theta^{>}(\overline{x})\big)$$

where $\theta^{<}(\overline{x}) = \exists y \in (s, u).\psi^{<}(\overline{x}, y)$ and $\theta^{>}(\overline{x}) = \exists y \in (u, t).\psi^{>}(\overline{x}, y)$ are HIFs and $\theta^{=}(\overline{x})$ is a HIF with strictly smaller quantifier depth than φ.

Proof. Clearly $u \in (s, t) \wedge \varphi(\overline{x})$ is equivalent to

$$u \in (s, t) \wedge \big(\exists y \in (s, u).\psi(\overline{x}, y) \vee \psi(\overline{x}, u) \vee \exists y \in (u, t).\psi(\overline{x}, y)\big).$$

As ψ has strictly smaller quantifier depth than φ, defining $\theta^{=}(\overline{x}) = \psi(\overline{x}, u)$ suffices. We focus on the first disjunct to define $\theta^{<}$, the definition of $\theta^{>}$ from the third disjunct is analogous. We proceed by induction on the quantifier depth of ψ. If ψ is quantifier-free then it is a HIF so set $\theta^{<} = \exists y \in (s, u).\psi$. As φ is a HIF, the only inductive case that is not straightforward is if $\psi = \exists z \in (y+c, t+c).\chi(\overline{x}, y, z)$ for some $c \in \mathcal{K}$ (the case $\psi = \forall z \in (y+c, t+c).\chi(\overline{x}, y, z)$ is also handled similarly). Applying the induction hypothesis yields:

$$\begin{aligned}
u \in (s,t) \wedge y \in (s,u) \wedge \psi \quad &\equiv \quad u \in (s,t) \wedge u + c \in (y+c, t+c) \wedge s < y \wedge \psi \\
&\equiv \quad u \in (s,t) \wedge u + c \in (y+c, t+c) \wedge s < y \wedge \\
&\qquad \big(\exists z \in (y+c, u+c).\eta^{<} \vee \eta^{=} \vee \exists z \in (u+c, t+c).\eta^{>}\big) \\
&\equiv \quad u \in (s,t) \wedge y \in (s,u) \wedge \\
&\qquad \big(\exists z \in (y+c, u+c)\eta^{<} \vee \eta^{=} \vee \exists z \in (u+c, t+c)\eta^{>}\big)
\end{aligned}$$

where $\eta^{<}$, $\eta^{=}$ and $\eta^{>}$ are HIFs. It follows that $\theta^{<} = \exists y \in (s, u).\psi$ is equivalent to a HIF. ◀

We now show that every N-bounded $\mathrm{FO}_{\mathcal{K}}$ formula with one free variable is equivalent to a HIF. As with the example above, the idea is to successively break up bad intervals from the innermost quantified variables, using Lemma 7 to ensure that such breaking up does not introduce more bad intervals on already processed subformulas. To simplify the procedure, we start with a more general statement.

▶ **Lemma 8.** *Every $\mathrm{FO}_{\mathcal{K}}$ formula $\psi(\overline{x}) \in \mathrm{Bet}_{\mathcal{K}}(x_0 - N, x_0 + N)$ is equivalent to a disjunction $\bigvee_i \big(\kappa_i(\overline{x}) \wedge \varphi_i(\overline{x})\big)$ where each κ_i is a conjunction of constraints of the form $x_j + c < x_k + c'$ and each φ_i is a HIF.*

Proof. We prove this by induction on the quantifier depth of ψ. We can remove the equality predicate by substitution (and induction on the number of variables), so for simplicity we assume that all inequalities are strict and occur within the scope of an even number of negations. In particular, we see that if the result holds for ψ then it also holds for $\neg\psi$ as negations of inequality constraints are also inequality constraints and negations of HIFs are also HIFs. Now if ψ is quantifier-free the result follows by taking a disjunctive normal form of ψ. So suppose $\psi = \exists y \varphi(\overline{x}, y)$. By the induction hypothesis we have $\varphi(\overline{x}, y)$ is equivalent to $\bigvee_i(\kappa_i(\overline{x}, y) \wedge \varphi_i(\overline{x}, y))$, so ψ is equivalent to

$$\bigvee_i (\kappa'_i(\overline{x}) \wedge \exists y \bigwedge_{j=0}^{n} y < x_j + c_j \wedge \bigwedge_{j=0}^{n} y > x_j + c'_j \wedge \varphi_i(\overline{x}, y)).$$

For technical reasons that will become clear shortly, we need to remove from each φ_i intervals of the form $(y + c, y + c')$. To do this, we observe that, by the pigeon-hole principle, $x_0 + n(c' - c) \in (y + c, y + c')$ for some $n \in \mathbb{Z}$. As $\psi \in \mathrm{Bet}_{\mathcal{K}}(x_0 - N, x_0 + N)$ we have $c - N < n(c' - c) < c' + N$, so there are a finite number of possibilities for n, and as $c, c' \in \mathcal{K}$, $n(c' - c) \in \mathcal{K}$. Thus for each interval $I = (y + c, y + c')$ occurring in φ_i we take a disjunction over all integers n in $(\frac{c-N}{c'-c}, \frac{c'+N}{c'-c})$, add the constraints $y + c < x_0 + n(c - c') < y + c'$, and use Lemma 7 to remove I. We also assume that all constraints amongst $\overline{x}$ and y implicitly defined[1] by φ_i are included in the conjunction of inequalities κ'_i.

The idea is to now take a disjunction over all possible choices for the greatest lower bound, $x_l + c_l$, and the least upper bound, $x_r + c'_r$, for y. This adds some additional constraints (e.g.

[1] For example $\exists z \in (x, y)$ implicitly implies $x < y$

$x_l + c_l > x_j + c_j$ for all $j \neq l$) which we add to κ'_i in each disjunct. Now ψ is equivalent to

$$\bigvee_{i'} (\kappa''_{i'}(\overline{x}) \wedge \exists y \in (x_l + c_l, x_r + c'_r)\, \varphi_i(\overline{x}, y)).$$

We next apply Lemma 7 to transform $\exists y \in (x_l + c_l, x_r + c'_r)\, \varphi_i(\overline{x}, y)$ into a HIF. Technically we apply it several times, once for each interval defined by free variables bounded above by $y + c$ and not bounded below by $x_l + c_l + c$ and once for each interval defined by free variables bounded below by $y + c$ and not bounded above by $x_r + c'_r$. The assumptions that there is no interval of the form $(y + c, y + c')$ and that all constraints implicitly defined by φ_i are included in κ_i together with the additional constraints imposed by the choice of x_l and x_r guarantee that $x_l + c_l + c$ is an element of any interval bounded above by $y + c$ and $x_r + c'_r + c$ is an element of any interval bounded below by $y + c$. Thus Lemma 7 guarantees that in the resulting HIF, φ'_i, all intervals involving y and some free variable are either of the form $(x_l + c_l + c, y + c)$ or $(y + c, x_r + c'_r + c)$. Thus $\exists y \in (x_l + c_l, x_r + c'_r)\, \varphi'_i(\overline{x}, y)$ is a HIF. ◀

Lemma 6 now follows as a corollary as inequality constraints over one variable can be trivially resolved.

The final stage of this step is to remove the application of unary functions to all bound variables.

▶ **Lemma 9.** *Let $\mathcal{K}$ be an additive subgroup of $\mathbb{R}$ and $\varphi(x)$ be an N-bounded $FO_{\mathcal{K}}$ formula with one free variable. Then $\varphi(x)$ is equivalent to an N'-bounded $FO_{\mathcal{K}}$ formula $\varphi'(x)$ in which the unary functions are only applied to x.*

Proof. Let us say there is a *violation* if a unary function is applied to a variable other than x. Following Lemma 6 and the comments at the start of the section it suffices to consider HIFs and remove all violations from intervals. We proceed from any maximal subformula of $\varphi(x)$, $\psi(x, \overline{y}) = \exists z \in (s, t).\theta(x, \overline{y}, z)$ where there is a violation, say $t = y_j + c$ (the case for $s = y_j + c$ being similar). Consider $\psi' = \exists z' \in (s - c, t - c).\theta(x, \overline{y}, z' + c)$. ψ' is clearly equivalent to ψ and is $(N + c)$-bounded. It suffices to show that $s - c$ is not a violation as this implies all violations in ψ' occur in proper subformulas and the result then follows by induction. The critical case is if $s = y_k + c'$. Then, as φ is a HIF and y_j and y_k are bound in φ, it follows that $j \neq k$. Suppose $j < k$. Then $y_j + c - c'$ must have been an endpoint on the interval constraining y_k at the point where y_k was quantified. As ψ is maximal, it follows that $c = c'$. Likewise if $k < j$. Therefore $s - c$ is not a violation. ◀

Step 2. Reduction to $\mathrm{Bet}_{\{0\}}(x, x + c)$ formulas.

Suppose now $\varphi(x)$ is an N-bounded $\mathrm{FO}_{\mathcal{K}}$ formula in which the unary functions are only applied to x. Let $c_0 < c_1 < \ldots < c_n$ be the constants in $\mathcal{K}$ (including 0) corresponding to the unary functions that are applied to x. Let $\varphi'(\overline{z})$ be the formula resulting from replacing each term $x + c_i$ with a new variable z_i. Then $\varphi(x)$ is equivalent to $\exists \overline{z}.(z_0 < \cdots < z_n) \wedge \varphi'(\overline{z}) \wedge \bigwedge(z_i = x + c_i)$. Moreover, φ' does not contain any unary functions and is thus a formula of $\mathrm{FO}_{\{0\}}$. A standard model-theoretic argument (see [14, 6, 10]) shows that $(z_0 < \cdots < z_n) \wedge \varphi'(\overline{z})$ can be written as a finite disjunction of formulas of the form $\bigwedge_{i=0}^{n} \psi_i(z_i) \wedge \bigwedge_{i=0}^{n-1} \chi_i(z_i, z_{i+1})$ where each ψ_i is a boolean combination of monadic predicates and each $\chi_i \in \mathrm{Bet}_{\{0\}}(z_i, z_{i+1})$. Thus $\varphi(x)$ can be written as a finite disjunction of formulas of the form

$$\bigwedge_{i=0}^{n} \psi_i(x + c_i) \wedge \bigwedge_{i=0}^{n-1} \chi_i(x + c_i, x + c_{i+1}).$$

Now $\psi_i(x+c_i)$ is clearly expressible by the $\mathrm{MTL}_{\mathcal{K}}$ formula $\Diamond_{\{c_i\}}\psi_i^\dagger$, where $\psi_i^\dagger$ is the obvious translation of $\psi_i(x)$ to $\mathrm{MTL}_{\mathcal{K}}$. Likewise, if $\chi_i^\dagger$ were an $\mathrm{MTL}_{\mathcal{K}}$ formula expressing $\chi_i(x, x+c_{i+1}-c_i)$ then $\Diamond_{\{c_i\}}\chi_i^\dagger$ would be an $\mathrm{MTL}_{\mathcal{K}}$ formula expressing $\chi_i(x+c_i, x+c_{i+1})$. Thus we have reduced the problem of expressing N-bounded $\mathrm{FO}_{\mathcal{K}}$ formulas to expressing every formula in $\mathrm{Bet}_{\{0\}}(x, x+c)$.

Step 3. Expressive completeness for bounded formulas.

Critical to this step is the following definition and lemma from [6].

A *decomposition formula* $\delta(x,y)$ is any formula of the form

$$\begin{aligned}
&x < y \wedge \exists z_0 \ldots \exists z_n\, (x = z_0 < \cdots < z_n = y)\\
&\qquad \wedge \bigwedge \{\varphi_i(z_i) : 0 < i < n\}\\
&\qquad \wedge \bigwedge \{\forall u\, ((z_{i-1} < u < z_i) \to \psi_i(u)) : 0 < i \leq n\}
\end{aligned}$$

where φ_i and ψ_i are LTL formulas regarded as unary predicates.

▶ **Lemma 10** ([6])**.** *Over any domain with a complete linear order, every formula $\psi(x,y)$ in* $\mathrm{Bet}_{\{0\}}(x,y)$ *is equivalent to a boolean combination of decomposition formulas $\delta(x,y)$.*

It follows that it suffices to show $\mathrm{MTL}_{\mathcal{K}}$ is able to express a decomposition formula. The proof of this result very closely follows the proof in [13], so we only outline the ideas and refer the reader to the full version of the paper for the details.

▶ **Lemma 11.** *Any decomposition formula $\delta(x, x+c)$ is equivalent to an $\mathrm{MTL}_{\mathcal{K}}$ formula.*

Proof (sketch). The proof is by induction on n, the number of existential quantifiers in $\delta(x, x+c)$. We divide the interval $(x, x+c)$ into small intervals of width $\nu \in \mathcal{K}$ where $0 < \nu \leq \frac{c}{2n}$. The fact that $\mathcal{K}$ is non-trivial and dense guarantees that ν exists. We then consider three cases depending on where the witnesses for the existential quantifiers of δ lie (taking a disjunction to cover all cases). If all witnesses lie in a single interval in the first half of $(x, x+c)$ then we can assert in $\mathrm{MTL}_{\mathcal{K}}$: ψ_1 holds until some point in the interval, then subsequent witness points occur within ν time units of the previous one. If instead all witnesses lie in a single interval in the second half of $(x, x+c)$ we assert: In c time units ψ_n would have held since a point in the interval, and each witness point was preceded within ν time units by another. Finally, if there is some k such that $x + k\nu$ separates the witnesses, we divide $\delta(x, x+c)$ into a $\mathrm{Bet}_{\{0\}}(x, x+k\nu)$ formula and a $\mathrm{Bet}_{\{0\}}(x+k\nu, x+c)$ formula and apply the inductive hypothesis. ◀

Combining Kamp's Theorem [14] and the results of this section yields:

▶ **Lemma 12.** *Let $\mathcal{K}$ be a dense additive subgroup of $\mathbb{R}$. Any N-bounded $FO_{\mathcal{K}}$ formula with one free variable is equivalent to an $MTL_{\mathcal{K}}$ formula.*

3.2 Syntactic separation of $\mathrm{MTL}_{\mathcal{K}}$

Having established that $\mathrm{MTL}_{\mathcal{K}}$ can express N-bounded $\mathrm{FO}_{\mathcal{K}}$ formulas when $\mathcal{K}$ is dense we now turn to extending the result to all $\mathrm{FO}_{\mathcal{K}}$. Our results for this section hold for all non-trivial additive subgroups $\mathcal{K}$.

The notion of *separation* was introduced by Gabbay in [4] where he showed that every LTL formula can be equivalently rewritten as a boolean combination of formulas, each of which depends only on the past, present or future. This was later extended to LTL over the reals in [5]. Hunter, Ouaknine and Worrell [13] extended this idea for the metric setting, showing that each $\mathrm{MTL}_{\mathbb{Q}}$ formula can be equivalently rewritten as a boolean combination of formulas, each of which depends only on the distant past, bounded present, or distant future.

Here we use a similar approach; however we need to refine the definition of distant past and distant future in order to use the separation property in Section 3.3. This refinement is, however, simple enough that the proof of separability of $\mathrm{MTL}_{\mathbb{Q}}$ in [13] can largely be used and we need only indicate the two places where adjustments need to be made to account for our more general setting. The complete proof can be found in the full version of the paper.

Recall from [13] the inductive definitions of *future-reach* $fr : \mathrm{MTL}_{\mathcal{K}} \to \mathcal{K} \cup \{\infty\}$ and *past-reach* $pr : \mathrm{MTL}_{\mathcal{K}} \to \mathcal{K} \cup \{\infty\}$

- $fr(p) = pr(p) = 0$ for all propositions p,
- $fr(\mathbf{true}) = pr(\mathbf{true}) = 0$,
- $fr(\neg\varphi) = fr(\varphi)$, $pr(\neg\varphi) = pr(\varphi)$,
- $fr(\varphi \wedge \psi) = \max\{fr(\varphi), fr(\psi)\}$,
- $pr(\varphi \wedge \psi) = \max\{pr(\varphi), pr(\psi)\}$,
- If $n = \inf(I)$ and $m = \sup(I)$:
  - $fr(\varphi\, \mathbf{U}_I\, \psi) = m + \max\{fr(\varphi), fr(\psi)\}$,
  - $pr(\varphi\, \mathbf{S}_I\, \psi) = m + \max\{pr(\varphi), pr(\psi)\}$,
  - $fr(\varphi\, \mathbf{S}_I\, \psi) = \max\{fr(\varphi), fr(\psi) - n\}$,
  - $pr(\varphi\, \mathbf{U}_I\, \psi) = \max\{pr(\varphi), pr(\psi) - n\}$.

Our separation result is then:

▶ **Lemma 13.** *Let $\mathcal{K}$ be a non-trivial additive subgroup of $\mathbb{R}$. For any $c \in \mathcal{K}_{\geq 0}$, every $MTL_{\mathcal{K}}$ formula is equivalent to a boolean combination of:*

- $\Diamond_{\{N\}}\varphi$ *where $pr(\varphi) < N - c$ for some $N \in \mathcal{K}$,*
- $\overset{\leftarrow}{\Diamond}_{\{N\}}\varphi$ *where $fr(\varphi) < N - c$ for some $N \in \mathcal{K}$, and*
- φ *where all intervals occurring in the temporal operators are bounded.*

Proof (sketch). The proof follows directly from the proof of the separability of $\mathrm{MTL}_{\mathbb{Q}}$ in [13] as only few assumptions were made about the underlying set of constants, which we now address.

- For the equivalence defining K^+ and K^- as bounded formulas, we instead need to use: $K^+(\varphi) \leftrightarrow \neg(\neg\varphi\, \mathbf{U}_{<\nu}\, \mathbf{true})$ and $K^-(\varphi) \leftrightarrow \neg(\neg\varphi\, \mathbf{S}_{<\nu}\, \mathbf{true})$, where $\nu \in \mathcal{K}$ is such that $\nu > 0$. Note that as $\mathcal{K}$ is non-trivial such a ν exists.
- In Step 3 (*Completing the separation*) N was chosen so that $N > pr(\theta) + 1$. Now we choose $N \in \mathcal{K}$ such that $N > pr(\theta) + c$. Note that again as $\mathcal{K}$ is non-trivial such a choice is always possible.

◀

3.3 Expressive completeness for $\mathrm{FO}_{\mathcal{K}}$

We now use Lemmas 12 and 13 to complete the proof of Theorem 1. Our argument is similar to other expressive completeness results based on separation – we refer the reader to [5, 12, 13]. Let $\varphi(x)$ be a $\mathrm{FO}_{\mathcal{K}}$ formula. We prove by induction on the quantifier depth of $\varphi(x)$ that it is equivalent to an $\mathrm{MTL}_{\mathcal{K}}$ formula.

Base case.

All atoms are of the form $P_i(x)$, $x = x$, $x < x$, $x + c = x$. We replace these by P_i, **true**, **false**, **false** respectively and obtain an $\mathrm{MTL}_{\mathcal{K}}$ formula which is clearly equivalent to φ.

Inductive case.

Without loss of generality we may assume $\varphi = \exists y.\psi(x,y)$. We would like to remove x from ψ. To this end we take a disjunction over all possible choices for $\gamma : \{P_1(x), \ldots P_m(x)\} \to \{\mathbf{true}, \mathbf{false}\}$, and use γ to determine the value of $P_i(x)$ in each disjunct via the formula $\theta_\gamma := \bigwedge_{i=1}^m (P_i(x) \leftrightarrow \gamma(P_i))$. Thus we can equivalently write φ in the form $\bigvee_\gamma \big(\theta_\gamma(x) \wedge \exists y.\psi_\gamma(x,y)\big)$, where the propositions $P_i(x)$ do not appear in the ψ_γ.

Now in each ψ_γ, we may assume, after some arithmetic, x appears only in atoms of the form $x = z$, $x < z$, $x > z$ and $x + c = z$ for some variable z. We next introduce new monadic propositions $P_=$, $P_<$, $P_>$, and F_c for all c such that there is an atom $x + c = z$, and replace each of the atoms containing x in ψ_γ with the corresponding proposition. That is, $x = z$ becomes $P_=(z)$, $x < z$ becomes $P_<(z)$ and so on. This yields a formula $\psi'_\gamma(y)$ in which x does not occur, such that $\psi'_\gamma(y)$ has the same truth value as $\psi_\gamma(x,y)$ for suitable interpretations of the new propositions.

By the induction hypothesis, for each γ there is an $\mathrm{MTL}_\mathcal{K}$ formula $\theta^\dagger_\gamma$ equivalent to $\theta_\gamma(x)$, and an $\mathrm{MTL}_\mathcal{K}$ formula $\psi^\dagger_\gamma$ equivalent to $\psi'_\gamma(y)$. Then our original formula φ has the same truth value at each point x as

$$\varphi' := \bigvee_\gamma \big(\theta^\dagger_\gamma \wedge (\diamondminus\psi^\dagger_\gamma \vee \psi^\dagger_\gamma \vee \Diamond\psi^\dagger_\gamma)\big)$$

for suitable interpretations of $P_=$, $P_<$, $P_>$ and the F_c.

Let $c_{\max} \in \mathcal{K}$ be the largest, in absolute value, element of $\mathcal{K}$ for which the propositional variable F_c was introduced. By Lemma 13, φ' is equivalent to a boolean combination of formulas

(I) $\Diamond_{\{N\}}\theta$ where $pr(\theta) < N - |c_{\max}|$,
(II) $\diamondminus_{\{N\}}\theta$ where $fr(\theta) < N - |c_{\max}|$, and
(III) θ where all intervals occurring in the temporal operators are bounded.

Now in formulas of type (I) above, we know the intended value of each of the propositional variables $P_=, P_<, P_>$ and F_c: they are all **false** except $P_<$, which is **true**. So we can replace these propositional atoms by **true** and **false** as appropriate and obtain an equivalent $\mathrm{MTL}_\mathcal{K}$ formula which does not mention the new variables. Likewise we know the value of each of propositional variables in formulas of type (II): all are **false** except $P_>$, which is **true**; so we can again obtain an equivalent $\mathrm{MTL}_\mathcal{K}$ formula which does not mention the new variables. It remains to deal with each of the bounded formulas, θ. As $\mathrm{MTL}_\mathcal{K}$ is definable in $\mathrm{FO}_\mathcal{K}$, there exists a formula $\theta^*(x) \in \mathrm{FO}_\mathcal{K}$, with predicates from $\{P_=, P_<, P_>, F_c\}$, equivalent to θ. It is clear that as θ is bounded, there is an N such that θ^* is N-bounded. We now unsubstitute each of the introduced propositional variables. That is, replace in $\theta^*(x)$ all occurrences of $P_=(z)$ with $z = x$, all occurrences of $P_<(z)$ with $x < z$ etc. The result is an equivalent formula $\theta^+(x) \in \mathrm{FO}_\mathcal{K}$, which is still N-bounded as we have not removed any constraints on the variables of θ^*. From Lemma 12, it follows that there exists an $\mathrm{MTL}_\mathcal{K}$ formula δ that is equivalent to θ^+, i.e., equivalent to θ.

4 Expressive completeness of $\mathrm{MTL}_\mathbb{Z}$ with counting

In this section we show

▶ **Theorem 2.** *$\mathrm{MTL}_\mathbb{Z}$+C has the same expressive power as $\mathrm{FO}_\mathbb{Z}$.*

In fact we show a slightly stronger result involving an extension of Q2MLO (see [10]) by punctuality quantifiers.

▶ **Definition 14.** *Q2MLO with punctuality (PQ2MLO)* is an extension of $\mathrm{FO}_{\{0\}}$ (and a restriction of $\mathrm{FO}_{\{1\}}$) defined by the following syntax:

$$\varphi ::= \mathbf{true} \mid P_i(x) \mid x < y \mid \varphi \wedge \varphi \mid \neg\varphi \mid \exists x\, \varphi \mid \exists_x^{x+1} y\, \psi \mid \exists_{x-1}^{x} y\, \psi \mid \Diamond_1^x y.\, \chi \mid \diamondminus_1^x y.\, \chi\,,$$

where x and y denote variables, ψ denotes a PQ2MLO formula with two free variables x and y, and χ denotes a PQ2MLO formula with one free variable, y. *Q2MLO* is the restriction of PQ2MLO to formulas that do not contain the punctual quantifiers $\Diamond_1^x$ and $\diamondminus_1^x$.

The quantifiers $\exists_x^{x+1} y$, $\exists_{x-1}^{x} y$, $\Diamond_1^x y$ and $\diamondminus_1^x y$ are interpreted as $\exists y \in (x, x+1)$, $\exists y \in (x-1, x)$, $\exists y.\, (y = x+1)$ and $\exists y.\, (y = x-1)$ respectively.

▶ **Theorem 15.** *$\mathrm{FO}_{\mathbb{Z}}$, PQ2MLO and $\mathrm{MTL}_{\mathbb{Z}}$+C all have the same expressive power.*

It is clear that $\mathrm{FO}_{\mathbb{Z}}$ is at least as expressive as the other two. To show the equivalence of PQ2MLO and $\mathrm{MTL}_{\mathbb{Z}}$+C we use the following result of [10].

▶ **Theorem 16** ([10])**.** *MITL with counting has the same expressive power as Q2MLO.*

We also observe that if $\varphi(y)$ is a formula of PQ2MLO that is equivalent to $\varphi' \in \mathrm{MTL}_{\mathbb{Z}}$+C then $\Diamond_1^x y\varphi(y)$ is equivalent to $\Diamond_{\{1\}}\varphi'$ and $\diamondminus_1^x y\varphi(y)$ is equivalent to $\diamondminus_{\{1\}}\varphi'$. The result then follows by induction on the nesting depth of the punctual operators ($\Diamond_1^x y$ / $\diamondminus_1^x y$ and $\Diamond_{\{1\}}$ / $\diamondminus_{\{1\}}$) and Theorem 16.

It remains to show any formula in $\mathrm{FO}_{\mathbb{Z}}$ has an equivalent PQ2MLO formula. Using similar arguments to the previous section, it is sufficient to derive analogues of Lemma 11 for $\mathrm{FO}_{\{1\}}$ formulas and Lemma 13 for $\mathrm{MTL}_{\mathbb{Z}}$+C.

4.1 Expressive equivalence of bounded formulas

In order to show every bounded $\mathrm{FO}_{\mathbb{Z}}$ formula can be expressed by a PQ2MLO formula, the results of Section 3.1 imply that we need only consider $\mathrm{FO}_{\{1\}}$ formulas of the form $\delta(x) = \delta(x, x+1)$ where:

$$\begin{aligned}\delta(x,y) = \exists z_0 \ldots \exists z_n\, (x = z_0 < \cdots < z_n = y)& \\ \wedge \bigwedge\{\varphi_i(z_i) : 0 < i < n\}& \\ \wedge \bigwedge\{\forall u\, ((z_{i-1} < u < z_i) \rightarrow \psi_i(u)) : 0 < i \leq n\}&.\end{aligned}$$

Now for $1 \leq j \leq 2n-1$ let

$$\begin{aligned}\delta_j(x,y) = \exists z_0 \ldots \exists z_k\, (x = z_0 < \cdots < z_k = y)& \\ \wedge \bigwedge\{\varphi_i(z_i) : 0 < i \leq \lfloor \tfrac{j}{2} \rfloor\}& \\ \wedge \bigwedge\{\forall u\, ((z_{i-1} < u < z_i) \rightarrow \psi_i(u)) : 0 < i \leq k\}&,\end{aligned}$$

where $k = \lceil \frac{j}{2} \rceil$. That is, $\delta_j(x, y)$ is the formula obtained by restricting $\delta(x)$ to the first j formulas of $\psi_1, \varphi_1, \psi_2, \varphi_2, \ldots, \psi_n$. Now consider the PQ2MLO formula:

$$\delta'(x) \quad = \quad \forall_x^{x+1}\, u.\, \bigvee_{i=1}^{2n-1} \delta_i(x,u) \;\wedge\; \Diamond_1^x y.\exists_{y-1}^{y} u.\delta(u,y).$$

The following result provides the suitable analogue of Lemma 11.

▶ **Lemma 17.** *$\delta(x)$ is equivalent to $\delta'(x)$.*

Proof. $\delta(x) \Rightarrow \delta'(x)$. Let $x_0, \ldots, x_n \in [x, x+1]$ be witnesses for the existential quantifiers in δ. From the definition of δ_i, if $u \in (x_i, x_{i+1})$ (for $0 \leq i < n$) then $\delta_{2i+1}(x, u)$ holds. Further, if $u = x_i$ (for $1 \leq i < n$) then $\delta_{2i}(x, u)$ holds. Thus the first conjunct of δ' is satisfied for all $u \in (x, x+1)$. Any $u \in (x, x_1)$ is a witness for $\exists_x^{x+1} u.\delta(u, x+1)$, and as $x_1 \leq x+1$, $u \in (y-1, y)$ where $y = x+1$. Thus the second conjunct holds and $\delta'(z)$ is satisfied.

$\delta'(x) \Rightarrow \delta(x)$. From the second conjunct of δ', $\delta(u, x+1)$ is satisfied for some $u \in (x, x+1)$. Let $x_0, \ldots, x_n \in [x, x+1]$ be the witnesses for $\delta(u, x+1)$. Now take any $v \in (x_{n-1}, x_n)$. From the first conjunct of δ', there is some $r \leq 2n-1$ such that $\delta_r(x, v)$ is satisfied. Note that if $r = 2n-1$ then as $\psi_n(x)$ holds for all $y \in (x_{n-1}, x_n)$ and $v \in (x_{n-1}, x_n)$, $\delta_{2n-1}(x, v)$ can be extended to the whole interval $[x, x+1)$, and thus $\delta(x)$ holds. So assume $r < 2n-1$, and let $x'_0, \ldots, x'_{r'} \in [x, v]$ be the witnesses for $\delta_r(x, v)$ where $r' = \lceil \frac{r}{2} \rceil < n$. Let m be the smallest index such that $x_m < x'_m$. As $x_{n-1} < v = x'_{r'}$ such an index must exist. Then we claim that $x, x'_1, \ldots, x'_{m-1}, x_m, x_{m+1}, \ldots, x_{n-1}, x+1$ are witnesses for $\delta(x)$. Every interval I defined by these witnesses, except (x'_{m-1}, x_m), is either an interval defined by witnesses of $\delta_r(x, x+1)$ or an interval defined by witnesses of $\delta(u, x+1)$, so all points in I satisfy ψ_i or φ_i as required. For the remaining interval, we observe that $(x'_{m-1}, x_m) \subseteq (x'_{m-1}, x'_m)$, thus all points satisfy ψ_{m-1} as required. Thus $\delta(x)$ is satisfied. ◀

4.2 Syntactic separation of $\mathrm{MTL}_{\mathbb{Z}}$+C

We now derive an extension of Lemma 13 for $\mathrm{MTL}_{\mathbb{Z}}$ with counting. We first extend the past-reach and future-reach functions as follows:

- $fr(\mathbf{C}_n\ \varphi) = 1 + fr(\varphi)$ and $fr(\overline{\mathbf{C}}_n\ \varphi) = fr(\varphi)$ for all n.
- $pr(\mathbf{C}_n\ \varphi) = pr(\varphi)$ and $pr(\overline{\mathbf{C}}_n\ \varphi) = 1 + pr(\varphi)$ for all n.

Our separation result for $\mathrm{MTL}_{\mathcal{K}}$ with counting is a generalization of Lemma 13 for any set $\mathcal{K}$ such that $1 \in \mathcal{K}$ (so that fr and pr are correctly defined).

▶ **Lemma 18.** *Let $\mathcal{K}$ be an additive subgroup of $\mathbb{R}$ such that $1 \in \mathcal{K}$. For any $c \in \mathcal{K}$, every $MTL_{\mathcal{K}}$+C formula is equivalent to a boolean combination of:*

- *$\Diamond_{\{N\}}\varphi$ where $pr(\varphi) < N - c$,*
- *$\overline{\Diamond}_{\{N\}}\varphi$ where $fr(\varphi) < N - c$, and*
- *φ where all intervals occurring in the temporal operators are bounded.*

The proof of Lemma 18 proceeds in the same way as the proof of Lemma 13. In that proof, the unbounded temporal operators were removed from the scope of bounded temporal operators, then the separation result of Gabbay [4] is applied treating the formulas in the scope of bounded temporal operators as atomic propositions. Here we also include formulas in the scope of the counting modalities as atomic propositions, thus it suffices to show how unbounded until and since operators can be removed from the scope of the counting modalities. This follows by induction from the following observation:

▶ **Lemma 19.** *For all $n \in \mathbb{N}$, the following equivalences and their temporal duals hold over all signals.*

(i) $\quad \mathbf{C}_1\ \varphi \quad \longleftrightarrow \quad \Diamond_1\varphi$

$$(ii) \qquad \mathbf{C}_n\left(\chi \wedge (\varphi\, \mathbf{U}\, \psi)\right) \quad\longleftrightarrow\quad \begin{array}{c} \mathbf{C}_n\left(\chi \wedge (\varphi\, \mathbf{U}_{<1}\, \psi)\right) \\ \vee \\ \Diamond_{\{1\}}\left(\varphi \wedge (\varphi\, \mathbf{U}\, \psi)\right) \wedge \\ \bigvee_{k=0}^{n-1}\left[\, \mathbf{C}_k\left(\chi \wedge (\varphi\, \mathbf{U}_{<1}\, \psi)\right)\right. \\ \left. \wedge\, \mathbf{C}_{n-k}\left(\chi \wedge \Box_{<1}(\varphi \wedge \neg\psi)\right)\right] \end{array}$$

$$(iii) \qquad \mathbf{C}_n\left(\chi \wedge \Box\varphi\right) \quad\longleftrightarrow\quad \mathbf{C}_n\left(\chi \wedge \Box_{<1}\varphi\right) \wedge \Box_{[1,\infty)}\varphi$$

$$(iv) \qquad \mathbf{C}_n\left(\chi \wedge (\varphi\, \mathbf{S}\, \psi)\right) \quad\longleftrightarrow\quad \begin{array}{c} \mathbf{C}_n\left(\chi \wedge (\varphi\, \mathbf{S}_{<1}\, \psi)\right) \\ \vee \\ \left(\varphi \wedge (\varphi\, \mathbf{S}\, \psi)\right) \wedge \\ \bigvee_{k=0}^{n-1}\left[\, \mathbf{C}_k\left(\chi \wedge (\varphi\, \mathbf{S}_{<1}\, \psi)\right)\right. \\ \left. \wedge\, \mathbf{C}_{n-k}\left(\chi \wedge \boxminus_{<1}(\varphi \wedge \neg\psi)\right)\right] \end{array}$$

$$(v) \qquad \mathbf{C}_n\left(\chi \wedge \boxminus\varphi\right) \quad\longleftrightarrow\quad \mathbf{C}_n\left(\chi \wedge \boxminus_{<1}\varphi\right) \wedge \varphi \wedge \boxminus\varphi$$

4.3 Equivalence of $\mathrm{MTL}_{\mathbb{Z}}{+}\mathrm{C}$, PQ2MLO and $\mathrm{FO}_{\mathbb{Z}}$

To complete the proof of Theorem 15 and hence Theorem 2 we apply the arguments of Section 3.3 together with Lemmas 17 and 18. We need only observe that $\mathrm{MTL}_{\mathbb{Z}}$+C formulas of type (III), that is where all intervals occurring in the temporal operators are bounded, are themselves bounded when translated to $\mathrm{FO}_{\mathbb{Z}}$. However, this follows directly from the definition of the counting modalities as they are defined by bounded formulas.

5 Conclusion and further work

We have given a precise characterization of the sets of timing constants $\mathcal{K}$ for which $\mathrm{MTL}_{\mathcal{K}}$ and $\mathrm{FO}_{\mathcal{K}}$ have the same expressive power. We have also shown that adding counting modalities to $\mathrm{MTL}_{\mathbb{Z}}$ (and punctuality modalities to Q2MLO) yields the full expressive power of $\mathrm{FO}_{\mathbb{Z}}$. This result can also be extended to other $\mathrm{MTL}_{\mathcal{K}}$ that are not as expressive as their first-order counterparts by adding the ability to count in the smallest definable non-zero interval (which, from the characterization, is known to exist).

Whilst most logics considered here have undecidable satisfaction tests, the cost of the translation in terms of formula size would still be an interesting area of exploration. Another area of ongoing work is an exploration of the notion of metric separation, and whether a suitable analogue of Gabbay's Theorem [4] can be derived.

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