Computability of Data Word Functions Defined by Transducers

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Synthesis

- ➔ Derive an implementation from the specification of a behaviour

Example (Nondeterministic Finite Automata)

An NFA A *specifies* a language, or equivalently a program that takes as input a word w and outputs 0 or 1.

Nondeterminism does not exist in practice $\Rightarrow$ how to implement such program?

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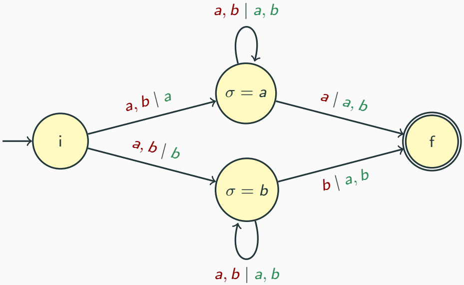
- Enumerate all possible runs of A over w and output 1 as soon as an accepting run is found (0 otherwise).
- There can be (exponentially) many runs $\Rightarrow$ we can do better
- NFA can always be determinised $\Rightarrow$ an equivalent DFA is a program which implements A and is guaranteed to take only a finite amount of memory.

➔ Derive an implementation from the specification of a behaviour

Example (Nondeterministic Finite Transducers)

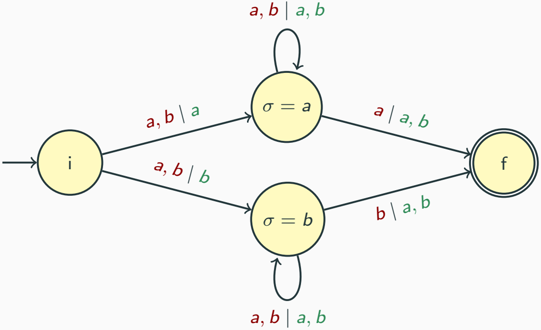
A transducer is an automaton with outputs. To every input word, it associates a set of acceptable outputs

$\Rightarrow$ An implementation chooses an acceptable output for each input.



A transducer recognising $S = \{(u\sigma, \sigma w) \mid \sigma \in \Sigma, u, w \in \Sigma^*, |u| = |w|\}$

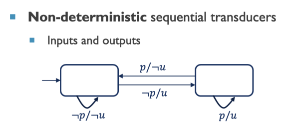
Synthesis



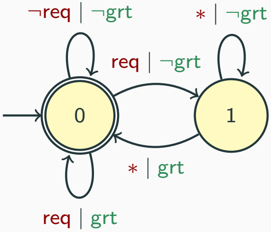
- The above specification can for instance be implemented by a program that computes $f : u\sigma \mapsto \sigma u$
- It cannot be implemented by any deterministic transducer
- Nor by any *synchronous* program, which outputs a letter as soon as it reads a letter

Synthesis: Nondeterministic ω-Transducers

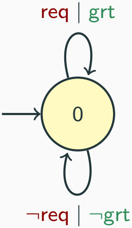
➔ Generalisation of transducers to infinite words, with a parity acceptance condition.



Infinite words do not exist in practice: we are specifying the behaviour of a non-terminating program *in the limit*.



An ω-transducer specifying that every request must eventually be granted

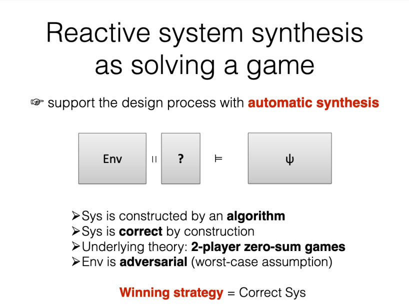


A program which immediately grants any request, represented as a deterministic transducer

What does it mean to be computable for non-terminating behaviours?

In the classical reactive synthesis setting

- An implementation is a *synchronous* program, i.e. a strategy in the parity game induced by the transducer.
- As parity games are positionally determined, we can restrict to finite-memory *synchronous* programs, also known as deterministic transducers.



Synthesis: to sum up

Definition (Real($\mathcal{S}, \mathcal{I}$))

- $\mathcal{S}$: class of specifications
- $\mathcal{I}$: class of implementations Given $S \in \mathcal{S}$, decide whether there exists $I \in \mathcal{I}$ such that I *implements* S, i.e. I and S have same domain and for all $x \in \text{dom}(S), (x, I(x)) \in S$

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- $\textsc{Real}(NFA, TM) = \textsc{Real}(NFA, DFA)$ and is always true
- $\textsc{Real}(NFT_{\mathsf{syn}}, TM)$ is always true, but not $\textsc{Real}(NFT_{\mathsf{syn}}, DFT_{\mathsf{syn}})$ (which is decidable)
- $\textsc{Real}(\omega NT_{\mathsf{syn}}, SP) = \textsc{Real}(\omega NT_{\mathsf{syn}}, \omega DT_{\mathsf{syn}})$ is decidable and equivalent with finding a winning strategy in a parity game

What does it mean to be computable for non-terminating behaviours?

Relax the synchronicity requirement

An implementation is a program which outputs longer and longer prefixes of an acceptable output as it reads longer and longer prefixes of the input.

Example (Guessing the last letter of a chunk)

Consider a specification that takes as input an ω-word of the form $u_1\sigma_1\#u_2\sigma_2\#u_3\sigma_3\ldots$ and accepts any output of the form $\sigma_1 w_1\#\sigma_2 w_2\#\sigma_3 w_3\ldots$

- It cannot be implemented by a synchronous program
- It can be implemented by a program computing $f_\# : u\sigma \mapsto \sigma u$ on each chunk

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Example (Guessing if the first letter appears again)

To an input σu, associate the output σ^ω if σ occurs in u, and σu otherwise.

- Such specification is definable by a transducer which initially guesses whether σ will appear again and checks such property
- It cannot be implemented by any program

What does it mean to be computable for non-terminating behaviours?

Relax the synchronicity requirement

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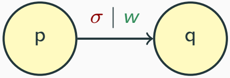
Asynchronous specifications

We now consider asynchronous transducers: on reading a letter $\sigma \in \Sigma$, a transducer can output a word $w \in \Sigma^*$.



Theorem ([Holtmann et al., 2012])

Deciding whether a specification defined by a transducer is realisable by a computable function is undecidable.

Computability

A function $f : \Sigma^\omega \to \Sigma^\omega$ is *computable* if

there exists a deterministic Turing machine M

which outputs longer and longer prefixes of the output

when reading longer and longer prefixes of the input

- Three tape deterministic Turing machine
  - Read-only one-way input tape
  - Two-way working tape
  - Write-only one-way output tape
- $M(x, k)$: the output written after having the k first input letters of x
- Since the output is write-only, $M(x, k)$ is nondecreasing

M computes f if

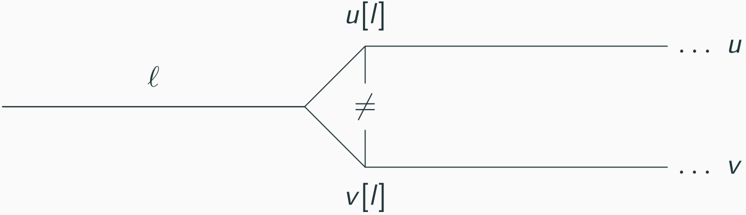
for all $x \in \mathrm{dom}(f)$, $M(x, k)$ converges towards $f(x)$

Continuity

Cantor distance

For $u, v \in \Sigma^\omega$, $d(u, v) = \begin{cases} 0 \text{ if } u = v \\ 2^{-\|u \wedge v\|} \text{otherwise} \end{cases}$

where $u \wedge v$ denotes the longest common prefix ℓ of u and v



Continuity

Continuous function

A function $f : \Sigma^\omega \to \Sigma^\omega$ is *continuous* at $x \in \mathrm{dom}(f)$ if:

(a) for all sequences of data words $(x_n)_{n\in\mathbb{N}}$ converging towards, we have that $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$. (where for all $i \in \mathbb{N}$, $x_i \in \mathrm{dom}(f)$), or equivalently:

(b) $\forall i \geq 0, \exists j \geq 0, \forall y \in \mathrm{dom}(f), \|x \wedge y\| \geq j \Rightarrow \|f(x) \wedge f(y)\| \geq i$.

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Functionality

A specification is *functional* if any input admits at most one acceptable output.

- Unless otherwise stated, specifications are now functional.
- Deciding if a transducer T is functional is doable in polynomial time.

Computability and Continuity

Computability

$f : \Sigma^\omega \to \Sigma^\omega$ is computable if there exists a deterministic Turing machine which outputs longer and longer prefixes of the output when reading longer and longer prefixes of the input.

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Computability $\Rightarrow$ Continuity

If $f : \Sigma^\omega \to \Sigma^\omega$ is computable, then it is continuous.

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Computability $\Rightarrow$ Continuity

If $f : \Sigma^\omega \to \Sigma^\omega$ is computable, then it is continuous.

$\to$ For $i \geq 0$ take $j \geq 0$ such that $\|M(x, j)\| \geq i$.

Since M is deterministic, any input y such that $\|x \wedge y\| \geq j$ satisfies $M(y, j) = M(x, j)$.

Thus, $M(x, j)$ is a common prefix of $f(x)$ and $f(y)$, so $\|f(x) \wedge f(y)\| \geq i$.

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$$\forall i \geq 0, \exists j \geq 0, \forall y \in \mathrm{dom}(f), \|x \wedge y\| \geq j \Rightarrow \|f(x) \wedge f(y)\| \geq i$$

Continuity $\Rightarrow$ Computability [Dave et al., 2019]

Let $f : \Sigma^\omega \to \Sigma^\omega$ be a function definable by a nondeterministic transducer. Then f is continuous iff it is computable.

Algorithm 1: Algorithm describing the machine M_f computing f.

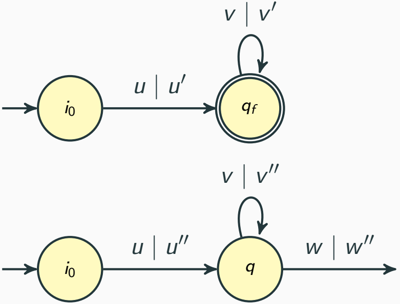
```
Data: x ∈ dom(f)
o := ε; // the output so far
for j = 0 to ∞ do
    for σ ∈ Σ do
        if o.σ ⪯ f̂(x[:j]) then // σ can be safely output
            o := o.σ;
            output σ;
        end
    end
end
```

Characterising continuity with a pattern

Theorem (Excluded pattern [Dave et al., 2019])

Let T be transducer defining a function f_T.

f_T is continuous iff T does not have the following pattern:



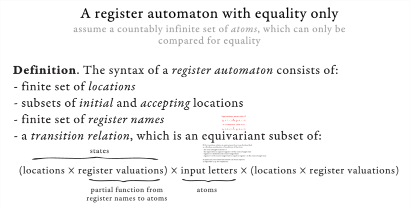
where $mismatch(u', u'') \vee (v'' = \varepsilon \wedge mismatch(u', u''w''))$

Our Contribution: extension to the infinite alphabet case

Until now

- Behaviour specified by functional asynchronous transducers
- Computability defined with deterministic Turing machines

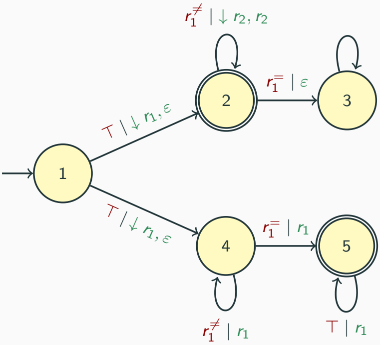
Extend to devices computing over (slightly) infinite sets



- Behaviour is defined using register transducers
- Computability is defined by allowing Turing machines to work over an infinite alphabet

Register Transducers

- $\mathcal{D}$ is a countably infinite set whose elements can be compared for equality only
- Equip a transducer with a finite set of registers
- Recognise relations S over data words, i.e. $S \subseteq (\Sigma \times \mathcal{D}) \times (\Sigma \times \mathcal{D})$



A register transducer taking as input dw and outputting w if d does not appear in w, d^ω otherwise (finite labels are irrelevant and not depicted)

Indistinguishability property [Kaminski and Francez, 1994]

As register machines only have k registers, any run over some data word w can be renamed into a run over some data word w' with at most $k+1$ data.

Corollary

Let A be a nondeterministic register automaton with k registers. If $L(A) \neq \varnothing$, then, for any $X \subseteq \mathcal{D}$ of size $|X| \geq k+1$ $L(A) \cap (\Sigma \times X)^\omega \neq \varnothing$.

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Theorem (Functionality)

Deciding whether a register transducer T is functional is PSPACE*-complete*

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Theorem (Functionality)

Deciding whether a register transducer T is functional is PSPACE*-complete*

$\rightarrow$ *Thanks to the indistinguishability property, we can show that T is functional if and only if it is functional over $(\Sigma \times X)^\omega$, where X is a finite subset of $\mathcal{D}$ of size $2k+1$.*

Continuity and computability

For functions defined by register transducers, computability and continuity again coincide.

Computability ⇒ Continuity is proved as before.

Continuity ⇒ Computability: requires to decide $o\sigma \preceq \hat{f}(x[:j])$

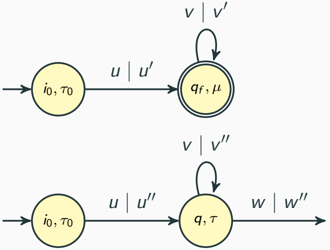
Algorithm 1: Algorithm describing the machine M_f computing f.

Data: $x \in \mathrm{dom}(f)$

```
o := ε ;
for j = 0 to ∞ do
    for (σ, d) ∈ Σ × (dt(x[:j]) ∪ {d_0}) do
        if o.(σ, d) ⪯ f̂(x[:j]) then // such test is decidable
            o := o.(σ, d);
            output (σ, d);
        end
    end
end
```

Continuity: extend the pattern characterisation

Theorem (Excluded pattern)



where:

$mismatch(u', u'') \vee$
$v'' = \varepsilon \wedge mismatch(u', u''w'')$

Moreover, such pattern is present iff it is present for data words with at most $2k + 1$ data.

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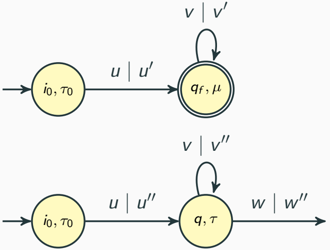

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f_T is continuous iff it is continuous over $(\Sigma \times X)^\omega$.

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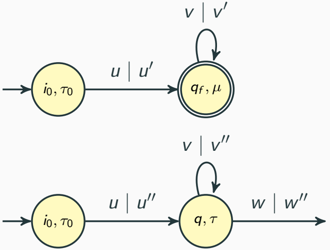

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Corollary

f_T is continuous iff it is continuous over $(\Sigma \times X)^\omega$.

This yields a PSPACE algorithm to decide whether a function f_T defined by a register transducer is computable.

Conclusion

- For functions defined by register transducers, continuity and computability coincide, and are decidable
- Such class is moreover closed under composition, and decidable
- Those problems are decidable in polynomial time for a subclass of functions, namely those recognised by test-free register-transducers

Future work

- Can we allow the devices to guess a data and put it in its registers?
- Extension to the 2-way case

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