Dynamics on Games: Simulation-Based Techniques and Applications to Routing

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Abstract
We consider multi-player games played on graphs, in which the players aim at fulfilling their own (not necessarily antagonistic) objectives. In the spirit of evolutionary game theory, we suppose that the players have the right to repeatedly update their respective strategies (for instance, to improve the outcome w.r.t. the current strategy profile). This generates a dynamics in the game which may eventually stabilise to an equilibrium. The objective of the present paper is twofold. First, we aim at drawing a general framework to reason about the termination of such dynamics. In particular, we identify preorders on games (inspired from the classical notion of simulation between transitions systems, and from the notion of graph minor) which preserve termination of dynamics. Second, we show the applicability of the previously developed framework to interdomain routing problems.

2012 ACM Subject Classification Theory of computation → Algorithmic game theory

Keywords and phrases games on graphs, dynamics, simulation, network

Introduction

Games are nowadays a well-established model to reason about several problems in computer science. In the game paradigm, several agents (called players) are assumed to be rational, and interact in order to reach a fixed objective. As such, games have found numerous applications, such as controller synthesis [15, 20] or network protocols [13]. In this paper, we are mainly concerned about multi-player games played on graphs, in which \( n \geq 2 \) players interact trying to fulfil their own objectives (which are not necessarily antagonistic to the others); and where the arena (defining the possible actions of the players) is given as a finite graph.

An example of such game is given in Figure 1, modelling an instance of an interdomain routing problem which is typical of the Internet. In this case, two service providers \( v_1 \) and \( v_2 \) want to route packets to a target node \( v_\bot \) through the links that are represented by the graph edges. For economical reasons, \( v_1 \) prefers to route the traffic to \( v_\bot \) through \( v_2 \) (using path \( c_1 s_2 \)) instead of sending them directly to \( v_\bot \), and symmetrically for \( v_2 \). (Assume for instance that both \( v_1 \) and \( v_2 \) are located in Europe, and that \( v_\bot \) is in America. Then, \( s_1 \) and \( s_2 \) are transatlantic links that incur a huge cost of operation for the origin nodes.) Then, assume that, initially, \( v_1 \) and \( v_2 \) route the packets through \( s_1 \) and \( s_2 \) respectively, and
broadcast this information through the network. When \( v_1 \) becomes aware of the choice of \( v_2 \), he could decide to rely on the \( c_1 \) link instead, trying to route his packets through \( v_2 \). However, due to the asynchronous nature of the network, \( v_2 \) could decide to route through \( c_2 \) before the new choice of \( v_1 \) reaches it. Hence, the packets get blocked in a cycle \( c_1 c_2 c_1 \cdots \) and do not reach \( v_1 \) anymore. Then, \( v_1 \) and \( v_2 \) could decide simultaneously to reverse to \( s_1 \) and \( s_2 \) respectively which brings the network in its initial state, where the same behaviour can start again... Clearly, such oscillations in the routing policies must be avoided.

This simple example illustrates the main notions we will consider in the paper. We study the notion of dynamics in games, which model the behaviour of the players when they update repeatedly their strategy (i.e. their choices of actions) in order to achieve a better outcome. Then, the main objectives of the paper are to draw a general framework to reason about the termination of such dynamics and to show its applicability to interdomain routing problems (as sketched above). We say that a dynamics terminates when the players converge to an equilibrium, i.e. a state in which they have no incentive to further update their respective strategies. Our framework is introduced in Section 3 and 4. It relies on notions of preorders, in particular the simulation preorder [12]. Simulations are usually defined on transition systems: intuitively, a system \( A \) simulates a system \( B \) if each step of \( B \) can be mimicked in \( A \).

We consider two kinds of preorders: preorders defined on game graphs, i.e. on the structure of the games; and simulation defined on the dynamics, which are useful to reason about termination (indeed, if a dynamics \( D_1 \) simulates a dynamics \( D_2 \), and if \( D_1 \) terminates, then \( D_2 \) terminates as well). We show how the existence of a relation between game graphs implies the existence of a simulation between the induced dynamics of those games (Theorem 4).

This technique allows us to check the termination of the dynamics using structural criteria about the game graph.

The motivation of this framework comes from the literature: we have found several unrelated examples of problems which are (sometimes implicitly) reduced to checking the termination of a dynamics in a multi-player game, and where sufficient criteria are proposed that can be expressed as the existence of a preorder between game graphs. We thus seek to unify these results, hoping that our framework will foster new applications of the game model. For instance, several sufficient conditions for termination in the network problem sketched above consist in checking that the game graph does not contain a forbidden pattern [7]. This containment can naturally be expressed as a preorder.

To this aim, we introduce, in Section 4 a preorder relation on game graphs, which is inspired from the classical notion of graph minor [11]. Intuitively, a game graph \( \mathcal{G}' \) is a minor of \( \mathcal{G} \) if \( \mathcal{G}' \) can be obtained by deleting edges and vertices from \( \mathcal{G} \) (under well-chosen conditions that are compatible with the game setting). Then, the relation ‘is a minor of’ forms a preorder relation on game graphs and allows one to reason on the termination of dynamics (see Theorem 8 and Theorem 9).

Finally, in Section 5, we achieve our second objective, by casting questions about Interdomain Routing into our framework. Interdomain Routing is the process of constructing routes across the networks that compose the Internet. The Border Gateway Protocol (BGP), is the de facto standard interdomain routing protocol. As sketched in the example above, it grows a routing tree towards every destination network in a distributed manner. The example also shows that the behaviour of the BGP is naturally modelled as a game, as already pointed out before (see [5, 18] for example). In particular, checking for so-called safety (does the protocol always converge to a stable state?) amounts to checking termination of some dynamics. In Section 5, we formally express BGP in our game model; revisit a classical result of Sami et al. that we re-prove within our framework; and finally obtain a new result regarding BGP: we provide a novel necessary and sufficient condition for convergence in the restricted (yet realistic) setting where the preferences of the nodes range on the next-hop in the route only.

Due to space constraints, full proofs and some examples are to be found in the appendix.
Figure 1 Left: a 2-player game $G^{DIS}$. Middle: $G^{DIS}(\overset{\rightarrow}{P_1})$. Right: $G^{DIS}(\overset{\rightarrow}{P_2})$

## 2 Preliminaries

### Graphs.
A (directed) graph is a pair $G = (V, E)$ where $V$ is a set of states, $E \subseteq V \times V$ is the set of edges. A labelled graph is a tuple $G = (V, E, L)$ where $(V, E)$ is a graph, and $L : E \to S$ is function associating, to each edge $e$, a label $L(e)$ from a set $S$ of labels. A (labelled) graph $G$ is finite iff $V$ is finite. A path in a (labelled) graph $G$ is a finite sequence $v_1 v_2 \cdots v_k$ or an infinite sequence $v_1 v_2 \cdots$ of states such that $(v_i, v_{i+1}) \in E$ for all $i$. We denote $v_1$, the first state of a path $\pi$, by first($\pi$). When $\pi = v_1 v_2 \cdots v_k$ is finite, we let last($\pi$) = $v_k$.

We let $V_\perp = \{ v \in V \mid \nexists v' : (v, v') \in E \}$ be the set of terminal states. We say that a path $\pi$ is maximal iff: either $\pi$ is infinite, or $\pi$ is finite and last($\pi$) $\in V_\perp$. Let $\pi_1 = v_1 \cdots v_k$ and $\pi_2 = u_1 u_2 \cdots$ be two paths such that $(v_k, u_1) \in E$. Then, we write $\pi_1 \prec \pi_2$ to denote the new path $v_1 \cdots v_k u_1 u_2 \cdots$, obtained by the concatenation of $\pi_1$ and $\pi_2$.

Following automata terminologies, a labelled graph $G$ is said to be complete deterministic if for all states $v$ and label $\ell$, there is exactly one edge $(v, \ell')$ s.t. $L(v, \ell') = \ell$.

### Games played on graphs.
An $n$-player game is a tuple $G = (V, E, (V_i)_{1 \leq i \leq n}, (\leq_i)_{1 \leq i \leq n})$ where players are denoted by $1, \ldots, n$ and: $(V, E)$ is a finite graph which forms the arena of the game, with $V_\perp$ the terminal states; $(V_i)_{1 \leq i \leq n}$ is a partition of $V \setminus V_\perp$ indicating which player owns each (non-terminal) state of the game ($v$ belongs to player $i$ iff $v \in V_i$); and $\leq_i$ describes the preference of player $i$ as a reflexive, transitive and total (i.e. for all $\pi, \pi'$, $\pi \leq_i \pi'$ or $\pi' \leq_i \pi$) binary relation defined on maximal paths which we call plays (the set of all plays being denoted by $Play$). Intuitively, player $i$ prefers play $\pi$ to play $\pi'$ iff $\pi' \leq_i \pi$. We can extract from $\leq_i$ a strict partial order relation by letting $\pi \prec_i \pi'$ if player $i$ strictly prefers play $\pi'$ to play $\pi$, i.e. if $\pi \leq_i \pi'$ and $\pi' \nleq_i \pi$. We also write $\pi \sim_i \pi'$ if $\pi \leq_i \pi'$ and $\pi' \leq_i \pi$, and say that $\pi$ and $\pi'$ are equivalent for player $i$. From now on, we describe preferences by mentioning plays of interest only (implicitly, all unmentioned plays are equivalent, and below in the preference order). We also abuse notations and identify a game with its arena: so, we can write, for instance, about the ‘paths of $G$’, meaning the paths of the underlying arena.

**Example 1.** Consider the example of [7]. In our context, it is modelled with the 2-player game $G^{DIS} = (V, E, (V_1, V_2), (\leq_1, \leq_2))$ depicted on the left of Figure 1. The state $v_\perp$ is terminal. Player 1 owns $V_1 = \{v_1\}$, and player 2 owns $V_2 = \{v_2\}$. Let $E = \{c_1, s_1, c_2, s_2\}$ be such that $s_1 = (v_1, v_0)$ and $c_1 = (v_1, v_2)$, $c_2 = (v_2, v_1)$. Edges $c_i$ stand for ‘continue’, and edges $s_i$ stand for ‘stop’. For player 1, we let the preferences be $(v_1 v_2)^\omega \prec_1 v_1 v_\perp \prec_1 v_1 v_2 v_\perp$, where $\pi^\omega$ denotes an infinite number of iterations of the cycle $\pi$. Symmetrically, player 2 has preferences $(v_2 v_1)^\omega \prec_2 v_2 v_\perp \prec_2 v_1 v_2 v_\perp$. In this case, all unmentioned plays are equally worst for both players, in particular the plays that do not start in the state owned by the player (this will always be the case in the routing application of Section 5).

### Strategies and strategy profiles.
The game is played by letting players move a token along the edges of the arena. Note that, in our games, there is no designated initial state, so the play can start in any state $v_1$. The choice of the initial state is not under the control of any
player. Then, the player who owns \( v_1 \) picks an edge \((v_1, v_2)\) and moves the token to \( v_2 \). It is then the turn of the player who owns \( v_2 \) to choose an edge \((v_2, v_3)\) and so forth. The game continues ad infinitum or until a terminal node has been reached, thereby forming a play. Of course, each player will act in order to yield a play that is best according to his preference order \( \prec_i \). Since no player controls the choice of the initial vertex, the players will seek to obtain the best path considering any possible initial vertex (see the formal definitions below).

This will be important for the application of Interdomain Routing in Section 5, where the games are networks and each state corresponds to a network node that wants to send a packet to one of the terminal states.

Formally, a non-maximal path is called a **history** in the following, and the set of all histories is denoted by Hist. We let \( \text{Hist}_i \) be the set of histories \( h \) such that last\((h)\) \( \in \Sigma_i \), i.e. \( h \) ends in a state that belongs to player \( i \). We further let player\((h) = i \) iff \( h \in \text{Hist}_i \).

The way players behave in the game is captured by the central notion of **strategy**, which is a mapping from a history \( h \) to a successor state in the graph, indicating how the player will play from \( h \). A **player \( i \) strategy** is thus a function \( \sigma_i : \text{Hist}_i \rightarrow \Sigma_i \) such that, for all \( h \in \text{Hist}_i \), \((\text{last}(h), \sigma_i(h)) \in E \). A strategy profile \( \sigma \) is a tuple \( (\sigma_i)_{1 \leq i \leq n} \) of strategies, one for each player \( i \). In the following, when we consider a strategy profile \( \sigma \), we always assume that \( \sigma_i \) is the corresponding strategy of player \( i \). We also abuse notations, and write \( \sigma(h) \) to denote the node obtained by playing the relevant strategy of \( \sigma \) from \( h \), i.e. \( \sigma(h) = \sigma_i(h) \) with \( i = \text{player}(h) \).

We denote by \( \Sigma_i(\mathcal{G}) \) and \( \Sigma(\mathcal{G}) \) the sets of player \( i \) strategies and of strategy profiles respectively (if the game \( \mathcal{G} \) is clear from the context, we may drop it and write \( \Sigma \) and \( \Sigma_i \)). As usual, given a strategy profile \( \sigma = (\sigma_i)_{1 \leq i \leq n} \) and a strategy \( \sigma'_j \) for some player \( j \), we denote by \( (\sigma_{-j}, \sigma'_j) \) the strategy profile obtained from \( \sigma \) by replacing the player \( j \) strategy \( \sigma_j \) with \( \sigma'_j \). Fixing a history \( h \) (or, in particular, an initial node) and a profile of strategies \( \sigma \) is sufficient to define a unique play that is called the **outcome**: we let \( \text{Outcome}(\sigma, h) \) be the (unique) play \( hv_1v_2 \cdots \) such that for all \( i \geq 1: v_i = \sigma(hv_1 \cdots v_{i-1}) \).

Of particular interest are the **positional strategies** (sometimes called memoryless), i.e. the set of strategies such that the action of the player depends on the last state of the history only. That is, \( \sigma_i \) is positional iff for all pairs of histories \( h_1 \) and \( h_2 \) in \( \text{Hist}_i \) : \( \text{last}(h_1) = \text{last}(h_2) \) implies \( \sigma_i(h_1) = \sigma_i(h_2) \). For a positional strategy profile \( \sigma \), and a state \( v \in \Sigma \), we write \( \sigma(v) \) to denote the (unique) state \( \sigma(h) \) returned by \( \sigma \) for all \( h \) with \( \text{last}(h) = v \). We denote by \( \Sigma^P(\mathcal{G}) \) the set of strategy profiles composed of positional strategies only, and by \( \Sigma_i^P(\mathcal{G}) \) the set of player \( i \) positional strategies. From all states \( v \), applying a positional strategy profile builds a play such that the very same decision is always taken at a particular state: therefore, it either creates a finite path without cycles, or a lasso (infinite path that starts with a finite path without cycle and continues with an infinite simple cycle, disjoint from the finite path). We let \( \text{Play}^P \) be the set of all positional plays thus generated. In a game where we are only interested in positional strategies (as this will be the case in the application to routing, for instance), the preferences may only be defined on positional plays. Indeed, all other plays will never be obtained as an outcome, and can be assumed to be worse than any other positional play.

**Game Dynamics.** Let us now turn our attention to the central notion of **dynamics**. Intuitively, a dynamics consists in letting players update their strategies according to some criteria. For example, a player will want to update his strategy in order to yield a better outcome according to his preferences. Therefore, a dynamics can be understood as a graph whose states are the strategy profiles, and whose edges correspond to possible updates.

\[\text{Definition 2.} \ Let \ \mathcal{G} \ be a game. A dynamics for \ \mathcal{G} \ is a binary relation \( \rightarrow \subseteq \Sigma \times \Sigma \) over the strategy profiles of \ \mathcal{G} \. Its associated graph is \( \mathcal{G}(\rightarrow) = (\Sigma, \rightarrow) \), where \( \Sigma \) is the set of states. The terminal profiles \( \sigma \) of \( \mathcal{G}(\rightarrow) \) (without outgoing edges) are called the **equilibria** of \( \rightarrow \).\]
We will focus on five dynamics, modelling certain rational behaviours of the players:

- The **one-step** dynamics $\dynamics$. It corresponds to the minimal update that can occur, where only one player changes a single decision in order to improve the outcome from his point of view: $\sigma \xrightarrow{1} \sigma'$ if there is a player $i \in \{1, \ldots, n\}$ and a history $h \in \text{Hist}_i$ such that (i) $\sigma(h) \neq \sigma'(h)$; (ii) $\text{Outcome}(\sigma, h) \prec_i \text{Outcome}(\sigma', h)$; and (iii) $\sigma(h') = \sigma'(h')$ for all $h' \neq h$. Note that the equilibria of the one-step dynamics are exactly the so-called subgame perfect equilibria (SPE) introduced in \cite{19} (see also \cite{14}).

- The **positional one-step** dynamics $\dynamicsP$. It ranges over positional strategy profiles only, and corresponds to a single player updating his strategy from a single state. Formally, $\sigma \xrightarrow{\dynamicsP} \sigma'$ (with $\sigma, \sigma' \in \Sigma^P$) iff there is a player $i \in \{1, \ldots, n\}$ and a state $v \in V_i$ s.t. (i) $\sigma(v) \neq \sigma'(v)$; (ii) $\text{Outcome}(\sigma, \sigma', v)$; and (iii) $\sigma(v') = \sigma'(v')$ for all $v' \neq v$.

- The **best reply positional one-step** dynamics $\dynamicsBP$. We let $\sigma \xrightarrow{\dynamicsBP} \sigma'$ iff there exists a player $i \in \{1, \ldots, n\}$ and a state $v \in V_i$ such that the three properties of the positional one-step dynamics are satisfied, and, in addition, the following best-reply condition is satisfied: (iv) for all $\sigma'' \neq \sigma'$ such that $\sigma \xrightarrow{\dynamicsP} \sigma''$ if player $i$ is the one that has changed its strategy between $\sigma$ and $\sigma''$, then: $\text{Outcome}(\sigma'', v) \prec_i \text{Outcome}(\sigma', v)$.

- The **positional concurrent** dynamics $\dynamicsC$ and its best reply version $\dynamicsBP$. Several players can update their strategies at the same time (in a ‘one step’ fashion), but each individual update would yield a better play when performed independently (in some sense, each player performing an update ‘believes’ he will improve). Formally, for $\sigma, \sigma' \in \Sigma^P$, we let $\sigma \xrightarrow{\dynamicsC} \sigma'$ (respectively, $\sigma \xrightarrow{\dynamicsBP} \sigma'$) iff for all $i \in P(\sigma, \sigma')$, $\sigma \xrightarrow{\dynamicsP} (\sigma'_i, \sigma_{-i})$ (respectively, $\sigma \xrightarrow{\dynamicsBP} (\sigma'_i, \sigma_{-i})$).

Observe that other dynamics can be defined, corresponding to other behaviours of the players. We focus on these five dynamics as they fit the applications we target in Section 5.

We have already said that the equilibria of $\dynamics$ are SPEs, and we can also see from the definitions that the equilibria of the four other dynamics coincide.

**Example 3.** Let $G^{\text{DIS}}$ be the game from Example 1. The graphs $G^{\text{DIS}}(\dynamicsP)$ and $G^{\text{DIS}}(\dynamicsC)$ are given in the middle and the right of Figure 1, where each strategy profile is represented by the choices of the players from $v_1$ and $v_2$. For example, $c_1c_2$ is the strategy profile s.t. $\sigma_1(v_1) = v_2$ and $\sigma_2(v_2) = v_1$. Note that, in this example, $\dynamicsP = \dynamicsBP$ and $\dynamicsC = \dynamicsBP$.

The main problem we study is whether a given dynamics terminates on a certain game: we say that a dynamics $\dynamics$ terminates on the game $G$ if there is no infinite path in the graph $G(\dynamics)$ of the dynamics. As illustrated in the introduction (Example 1), such infinite paths may be problematic in certain applications, like in the Interdomain Routing problem, where an infinite path in the dynamics means that the routing protocol does not stabilise. We are thus interested in techniques to check whether a dynamics terminates on a given game.

Sometimes, a dynamics does not terminate in general, but does when we restrict ourselves to **fair executions** where all players will eventually have the opportunity to update their strategies if they want to. Formally, given a dynamics $\dynamics$, an infinite path $\sigma^1 \rightarrow \sigma^2 \rightarrow \cdots$ of the graph $G(\dynamics)$ is not fair if there exists a player $i$, and a position $k$ such that for all $\ell \geq k$, player $i$ can switch his strategy in $\sigma^\ell$ (i.e. there is $\sigma^\ell \rightarrow \sigma'$ where $\sigma_i^\ell \neq \sigma_i'$), but for all $\ell \leq k$, player $i$ keeps the same strategy forever (i.e. $\sigma_i^\ell = \sigma_i^k$). We say that the dynamics $\dynamics$ **fairly terminates** for the game $G$ if there are no infinite fair paths in the graph $G(\dynamics)$: this is a weakening of the notion of termination seen before (Figure 5 in Appendix A shows an example of a dynamics that does not terminate but terminates fairly).
3 Simulations: preorders on the dynamics graphs

At this point of the paper, it is important to understand that a game is characterised by two graphs: the game graph which gives its structure (see for example, Figure 1, left); and the dynamics graph, which, given a fixed dynamics \( \rightarrow \), defines the semantics of the game as the long-term behaviour of the players (Figure 1, middle and right). In the present section, we study preorder relations on the dynamics graphs, relying on the classical notion of simulation [12]. They are the key ingredients to reason about the termination of dynamics.

The domain of a binary relation \( R \subseteq A \times B \) is the set of elements \( a \in A \) such that there exists \( b \in B \) with \((a, b) \in R\). The co-domain or \( R \) is the set of elements \( b \in B \) such that there exists \( a \in A \) with \((a, b) \in R\). We denote the domain of \( R \) by \( \text{dom}(R) \). The transitive closure \( R^+ \) of relation \( R \) is defined as \((a, b) \in R^+ \) if there are \( a_0 = a, a_1, a_2, \ldots, a_n = b \) such that for all \( i \in \{0, 1, \ldots, n - 1\} \), \((a_i, a_{i+1}) \in R\).

Partial simulations and simulations. We start with some weak version of the notion of simulation, called partial simulation \( \sqsubseteq \). Intuitively, we say that a state \( u \) partially simulates a state \( u' \) (noted \( u' \sqsubseteq u \)) if for all successor states \( v' \) of \( u' \), the following holds: if \( v' \) is in the domain of the simulation, then there must be some \( v \) simulating \( v' \) such that \( v \) is a successor of \( u \). Formally, if \( G = (V,E) \) and \( G' = (V',E') \) are two graphs, a binary relation \( \sqsubseteq \) contained in \( V' \times V \) is a partial simulation of \( G' \) by \( G \) if: for all \((u', v') \in E' \cap \text{dom}(\sqsubseteq)^2 \), for all \( u \in V \): \( u' \sqsubseteq u \) implies there is \( v \in V \) such that \( (u,v) \in E \) and \( v' \sqsubseteq v \). Then, a simulation \( \sqsubseteq \) of \( G' \) by \( G \) is a partial simulation of \( G' \) by \( G \) s.t. \( \text{dom}(\sqsubseteq) = V' \), i.e. all states of \( G' \) are simulated by some state of \( G \). When a (partial) simulation \( \sqsubseteq \) of \( G' \) by \( G \) exists, we say that \( G \) (partially) simulates \( G' \). The following example highlights the difference between partial simulations and simulations. Assume \( G \) with only one edge \( u \rightarrow v \) and \( G' \) with only two edges \( u' \rightarrow v_1' \) and \( u' \rightarrow v_2' \). Then, the relation \( \sqsubseteq \) s.t. \( u' \sqsubseteq u \) and \( v_1' \sqsubseteq v \) (but \( v_2' \not\sqsubseteq v \)) is a partial simulation (its domain is \( \{u', v_1'\} \) so it is not a problem that \( v_2' \) is not simulated) but is not a simulation relation.

Simulations between dynamics graphs help in showing termination properties, as shown by the following easy result:

**Theorem 4.** Let \( G_1 \) and \( G_2 \) be two games, \( \rightarrow_1 \) and \( \rightarrow_2 \) be two dynamics on \( G_1 \) and \( G_2 \) respectively. If \( G_1(\rightarrow_1) \) simulates \( G_2(\rightarrow_2) \) and the dynamics \( \rightarrow_1 \) terminates on \( G_1 \), then the dynamics \( \rightarrow_2 \) terminates on \( G_2 \).

Bisimulations and transitive closure. We can define other preorder relations on dynamics graphs. A bisimulation is a simulation \( \sqsubseteq \) such that the inverse relation \( \sqsubseteq^{-1} \) is also a simulation. We say that \( G = (V,E) \) and \( G' = (V',E') \) are bisimilar when there is a bisimulation between them. As a corollary of the previous theorem, if \( G_1(\rightarrow_1) \) and \( G_2(\rightarrow_2) \) are bisimilar, then \( \rightarrow_1 \) terminates on \( G_1 \) if and only if \( \rightarrow_2 \) terminates on \( G_2 \).

For termination purposes, it is also perfectly fine to simulate a single step of \( G' \) in several steps of \( G \) for instance. The following proposition stems from Theorem 4 and mixes the notions of transitive closures and partial simulations.

**Proposition 5.** Let \( G_1 \) and \( G_2 \) be two games, \( \rightarrow_1 \) and \( \rightarrow_2 \) be dynamics on \( G_1 \) and \( G_2 \) resp.

1. If \( G_1(\rightarrow_1^+) \) simulates \( G_2(\rightarrow_2) \) and the dynamics \( \rightarrow_1 \) terminates on \( G_1 \), then the dynamics \( \rightarrow_2 \) terminates on \( G_2 \).
2. If \( \sqsubseteq \) is a partial simulation of \( G_2(\rightarrow_2^+) \) by \( G_1(\rightarrow_1^+) \), and the dynamics \( \rightarrow_1 \) terminates on \( G_1 \), then there are no paths in \( G_2(\rightarrow_2) \) that visit infinitely often a state of \( \text{dom}(\sqsubseteq) \).
Let us now introduce notions of preorders on game graphs. We introduce a new notion of graph minor which consists in lifting the classical notion of graph minor to the context of $n$-player games. To the best of our knowledge, this has not been done previously. This new preorder on game graphs enables to use in a simple context the results of Section 3 to reason about termination of dynamics. Let us start with the formal definition. For that purpose, we start by defining two transformations on game graphs. Let $G = (V, E, (V_i), (\leq_i))$ be an $n$-player game. Then we can modify it by applying either of the following transformations that yields a game $G' = (V'', E'', (V'_i), (\leq'_i))$.

- Deleting an edge $(u, v) \in E$. Then, $V'' = V, E'' = E \setminus \{(u, v)\}, (V'_i) = (V_i)$, and $\leq'_i$ is s.t. $\pi_1 \leq'_i \pi_2$ iff $\pi_1 \leq_i \pi_2$ and $\pi_1, \pi_2$ are both paths of $G'$.

- Deleting a state $v \in V_j$ (for a certain player $j$). This can happen in two different ways:
  1. either when $v$ is isolated, i.e. when $(u, v) \notin E$ and $(v, u) \notin E$ for all $u \in V$. Then, $V'' = V \setminus \{v\}, E'' = E, V'_i = V_i$ for all $i \neq j, V'_j = V_j \setminus \{v\}$, and $(\leq'_i)_i = (\leq_i)_i$.
  2. or when $v$ has a unique outgoing edge $(v, v')$ and all predecessors $u$ of $v$ (i.e. $(u, v) \in E)$ do not have $v'$ as a successor (i.e. $(u, v') \notin E$). In this case, we have $V'' = V \setminus \{v\}, V'_i = V_i$ for all $i \neq j$ and $V'_j = V_j \setminus \{v\}, E'' = (E \cap (V' \times V')) \cup \{(u, v') | (u, v) \in E\}$, and $\pi'_1 \preceq'_i \pi'_2$ iff $\pi_1 \preceq_i \pi_2$ where $\pi_1$ and $\pi_2$ are the plays of $G$ obtained from $\pi'_1$ and $\pi'_2$ respectively, by replacing all occurrences of $(u, v')$ (for some $u$) by $(u, v), (v, v')$.

Definition 6. Let $G$ and $G'$ be two $n$-player games. Then, $G'$ is a minor of $G$ if $G'$ can be obtained from $G$ by applying a (finite) sequence of edges and states deletions.

Example 7. An example of minor is depicted in Figure 2. If the original preferences of the player owning state $v_i$ are $v_1v_4v_5v_\perp < v_1v_3v_\perp < v_1v_2v_4v_\perp < v_1v_2v_4v_\perp$ (other plays being equally worse for this player), then after the deletion of the edge $(v_4, v_\perp)$, his preferences become $v_1v_4v_5v_\perp < v_1v_3v_\perp < v_1v_2v_4v_\perp < v_1v_2v_4v_\perp$ (the path $v_1v_2v_4v_\perp$ does not exist in the new graph and has simply been removed from the preferences). Next, the deletion of $v_4$ is allowed because it is a single outgoing edge $v_5$, and neither $v_1$ nor $v_2$ nor $v_3$ have an edge to $v_5$. After this deletion, the preferences become $v_1v_5v_\perp < v_1v_3v_\perp < v_1v_2v_5v_\perp$. Finally, after the deletion of the edge $(v_1, v_5)$, the preferences become $v_1v_5v_\perp < v_1v_2v_5v_\perp$.

The deletion of a state therefore consists in squeezing each path of length 2 around it in a single edge. The condition $(u, v') \notin E$ makes sure that this squeezing is not perturbed by the presence of an incident edge $(u, v')$ that could have contradictory preferences. For instance, in the previous example, we cannot remove vertex $v_6$ in the minor obtained before having removed edge $(v_1, v_5)$; otherwise, we would obtain as preferences for the owner of $v_1$ the chain $v_1v_5v_\perp < v_1v_2v_5v_\perp$ which is not possible.

We can link dynamics on graph games with the presence of minors, in the various dynamics introduced before: if we manage to find a game minor where the dynamics does not terminate, then the original game does not terminate either.
Theorem 8. Let $\mathcal{G}$ be a game, and $\mathcal{G}'$ be a minor of $\mathcal{G}$. If $\rightarrow \in \{\overrightarrow{\beta P1}, \overrightarrow{\beta PC}\}$, then $\mathcal{G}'(\rightarrow)$ simulates $\mathcal{G}(\rightarrow)$. In particular, if the dynamics $\rightarrow$ terminates for $\mathcal{G}$, then it terminates for $\mathcal{G}'$ too.

Sketch of proof. We prove the result for $\overrightarrow{\beta P1}$, the two other cases being similar. Since simulations are transitive relations, it is sufficient to only consider that $\mathcal{G}'$ has been obtained from $\mathcal{G}$ either by deleting a single edge, or by deleting a single node. Let us briefly detail the case where $\mathcal{G}'$ is obtained by deleting the deletion of a state $v$. If $h \in \text{Hist}(\mathcal{G}) \setminus \{h \mid \text{last}(h) = v\}$, we can construct a corresponding play $f(h)$ of $\mathcal{G}'$ by replacing a sequence $uv'$ of $h$ by $u'v$. The conditions over the deletion of $v$ implies that that $f$ is indeed a bijection from $\text{Hist}(\mathcal{G}) \setminus \{h \mid \text{last}(h) = v\}$ to $\text{Hist}(\mathcal{G}')$. We then consider the following relation on strategy profiles: $\sigma' \sqsubseteq \sigma$ if for all histories $h \in \text{Hist}(\mathcal{G}) \setminus \{h \mid \text{last}(h) = v\}$, $\sigma'(f(h)) = \sigma(h)$ if $\sigma(h) \neq v$, and $\sigma'(f(h)) = v'$ otherwise; and show that $\sqsubseteq$ is a simulation (indeed a bisimulation).

Notice that Theorem 8 suffers from three weaknesses. First, it does not hold for the best reply dynamics $\overrightarrow{\beta PC}$ and $\overrightarrow{\beta P1}$, as shown by the following example. Consider again the game $\mathcal{G}^{DIS}$ from Example 1. Further, consider the game $\mathcal{G}$ (shown in Figure 4, Appendix A) obtained from $\mathcal{G}^{DIS}$ by adding a third player, who owns a single node $v_3$, s.t. the only edges to and from $v_3$ are $(v_1, v_3)$ and $(v_3, v_1)$, and where the preferences of player 1 are now $v_1v_1 \prec_1 v_1v_2v_1 \prec_1 v_1v_3v_1$ (observe that now, he prefers a path that traverses the new node $v_3$ above all other paths). Clearly, $\mathcal{G}^{DIS}$ is a minor of $\mathcal{G}'$. Using Theorem 8, and since we know that $\mathcal{G}^{DIS}(\overrightarrow{\beta P1})$ does not terminate, we deduce that $\mathcal{G}(\overrightarrow{\beta P1})$ does not terminate either. Moreover, in this example, $\mathcal{G}^{DIS}(\overrightarrow{\beta PC}) = \mathcal{G}(\overrightarrow{\beta P1})$, so even with the best-response property, the dynamics does not terminate in the minor. However, one can check that $\mathcal{G}(\overrightarrow{\beta P1})$ terminates thanks to the best-response property: Player 1 will not try to obtain path $v_1v_2v_1$ (which leads to a cycle in $\mathcal{G}^{DIS}(\overrightarrow{\beta PC})$), but will choose a strategy going to $v_3$ (see Figure 4). So, $\mathcal{G}^{DIS}$ is a minor of $\mathcal{G}$, s.t. $\overrightarrow{\beta P1}$ terminates for $\mathcal{G}^{DIS}$ but not in $\mathcal{G}$.

The example can be adapted to $\overrightarrow{\beta P1}$.

A second weakness is that Theorem 8 does not apply to fair termination: the dynamics $\rightarrow$ could fairly terminate for $\mathcal{G}$, but not for $\mathcal{G}'$. This could be the case if we remove every choice (except one) for a certain player in the minor $\mathcal{G}'$ creating a fair cycle in $\mathcal{G}'$ that would not be present in $\mathcal{G}$. See Figure 5 in Appendix A for an example. Finally, the reciprocal of Theorem 8 does not hold: all dynamics terminate on the trivial graph with a single state, but it is also minor of all games, including those where the dynamics does not terminate.

This motivates the introduction of a stronger notion of graph minor, where we allow to remove only the so-called dominated edges. Formally, let $\mathcal{G}$ be a game, let $v \in V \setminus v_i$ be a state, and let $e_1 = (v, v_1)$ and $e_2 = (v, v_2)$ be two outgoing edges of $v$. We say that $e_1$ is dominated by $e_2$ if for all positional strategies $\sigma \in \Sigma^p$, $\text{Outcome}(\sigma_1, v) \prec_i \text{Outcome}(\sigma_2, v)$, where $\sigma_1$ and $\sigma_2$ coincide with $\sigma$ except that $\sigma_1(v) = v_1$ and $\sigma_2(v) = v_2$. Intuitively, this means that the player always prefers $e_2$ to $e_1$. Then, a game $\mathcal{G}'$ is said to be a dominant minor of $\mathcal{G}$ if it can be obtained from $\mathcal{G}$ by deleting states as before, but only deleting dominated edges.

Equipped with this notion, we overcome the three limitations of Theorem 8 we had identified:

Theorem 9. Let $\mathcal{G}$ be a game and $\mathcal{G}'$ be a dominant minor of $\mathcal{G}$. If $\rightarrow \in \{\overrightarrow{\beta P1}, \overrightarrow{\beta PC}\}$, then we can build a simulation $\sqsubseteq$ of $\mathcal{G}'(\rightarrow)$ by $\mathcal{G}(\rightarrow)$ such that: (i) $\sqsubseteq^{-1}$ is a partial simulation of $\mathcal{G}(\rightarrow)$ by $\mathcal{G}'(\rightarrow)$; and (ii) if there is a fair cycle in $\mathcal{G}$ then there is a fair cycle in $\mathcal{G}'$. In particular, the dynamics $\rightarrow$ fairly terminates for $\mathcal{G}$ if and only if it does for $\mathcal{G}'$.

---

1 We restrict our definition to the context of positional strategies, for the sake of brevity, but it can be extended to a more general setting.
Now, Theorem 9 has some limitations too. We can show that it does not hold for
the ‘non-best-reply dynamics’ $P_1 \rightarrow$ and $P_C \rightarrow$. Moreover, even when we consider best-reply
dynamics, the fairness condition remains crucial: we can exhibit a case where there is a
(non-fair) cycle in $G$ but no cycle in $G'$. Those examples are given in Appendix A.

5 Applications to interdomain routing convergence

As explained in the introduction, the Border Gateway Protocol (BGP) is the de facto
interdomain routing protocol. Its role is to establish routes to all the networks that compose
the Internet. BGP does this by growing in a distributed manner a routing tree towards
every destination network, as follows. In the initial state, only the router in the destination
network has a route towards itself that it advertises to its neighbours. Each time a router
receives an advertisement, it selects among the neighbour routes the one it considers best
and then advertises it to its neighbours. The process repeats until no router wants to change
its best route. To select its best route, a router first filters the received routes to retain only
permitted ones and ranks them according to its preference. Both the filtering and ranking of
routes by a router are decided based on the network’s routing policy. For example, a route
can be preferred over another because it offers better performance or costs less and it can be
filtered out because it is not economically viable.

As shown in the introductory example, the routing approach at the heart of BGP
has known convergence issues. It could fail to reach an equilibrium, entering a persistent
oscillatory behaviour or it could have no equilibrium at all. This is a well-studied problem
that has led to considerable work [7, 6, 5, 18, 3, 4, 8, 13]. In their seminal work [7], Griffin et
al. analysed the BGP convergence properties using a simplified model named the Stable Path
Problem (SPP). The main questions they ask are the following: (1) whether an SPP instance
is Solvable, i.e., whether it admits a stable state; (2) whether the stable state is Unique;
and (3) whether the systems is Safe, i.e. it always converges to a stable state. They also give
a sufficient condition for an SPP instance to be safe: the absence of a substructure named a
Dispute Wheel. Later, Sami et al. [18] have shown that the existence of multiple stable
states is a sufficient condition to prevent safety (i.e. Safe $\Rightarrow$ Unique). These results have
later been refined by Cittadini et al. [3]. While the works just cited focus on the definition of
sufficient conditions for safety, another approach by Gao and Rexford [6] achieves convergence
by enforcing only local conditions on route preferences.

In this section, we show how SPP can be expressed in our $n$-player game model, therefore
Safety reduces to checking for termination of the game dynamics. We revisit the result of
Sami et al. by providing a new proof that relies on our framework. Then, we further exploit
this framework to obtain a new result about SPP: we provide a necessary and sufficient
condition for safety in a setting which is more restricted (yet still realistic) than Griffin’s.

One target games We first translate the SPP, as a combination of: (1) a reachability game
that models the network topology and routing policies; and (2) the best-reply positional
concurrent dynamics that models the asynchronous behaviour of the routing protocol. Using
this approach, the routing safety problem translates to a dynamics termination problem.

We rely on a particular class of games, that we call one target games (OTG for short):
they have a unique target, the destination network, that all players want to reach. Each
player corresponds to a network in the Internet and as such owns a single state. The routing
policies of networks are modelled by the preference relations and by the distinction between
permitted and forbidden paths. The preferences are only over positional strategies (paths),
meaning that each network picks its next-hop independently of its predecessors. Permitted
and forbidden paths model the fact that only some paths are allowed by the networks routing
policies. Forbidden paths are also used to take into account additional restrictions that cannot be directly modelled. In SPP, the paths are simple (no loops); and non-simple paths are forbidden, for obvious reasons of efficiency. Moreover, in SPP, if at some point a network reaches a forbidden path, he will inform his neighbours that he is not able to reach the target. To model this, we impose that if a path is permitted, all its suffixes are also permitted.

Formally, let \( G = (V, E, (V_i)_{1 \leq i \leq n}, (\leq_v)_{i \in P}) \) be an \( n \)-player game. For all \( 1 \leq i \leq n \), we assume that \( P_i \) is the set of permitted paths of player \( i \). All these paths are finite paths of the form \( v_1 \cdots v_k \in \text{Play}^b \). We denote by \( P_i^c \) the set of forbidden paths, i.e., all the positional plays starting in \( v_i \) that are not in \( P_i \) (in particular, all infinite paths are forbidden). We let \( P = \bigcup_{1 \leq i \leq n} P_i \) and \( P^c = \bigcup_{1 \leq i \leq n} P_i^c \). Then, \( G \) is a one target game (1TG) if:

- \( V_\bot = \{ v_\bot \} \), and, for all players \( i \): \( V_i = \{ v_i \} \);
- for all \( v_1 \in P_i^c \), there are \( v_1 \succ v_2 \) (permitted better than forbidden);
- for all \( v_1, v_2 \in P_i^c \): if \( v_1 \sim v_2 \) (all forbidden paths are equivalent);
- for all \( v_1, v_2 \in P_i^c \): \( v_1 \sim v_2 \) implies that then there are \( v \in V \) and \( \pi_1, \pi_2 \) s.t. \( v = a_1 a_2 \cdots \pi_1 \pi_2 \).

Our running example (Figure 1) is a 1TG. Since, in such a game, each player owns one and only one state, we will abuse notation by confusing each state \( v \in V \) with its player. For example, for \( v \in V_i \), we will write \( \prec_i \) instead of \( \prec_i \).

**Sami et al:** Termination implies a unique equilibrium. Equipped with this definition, we start by revisiting a result of Sami et al. saying that when an instance of SPP is safe, the solution is unique. In our setting, this translates as follows:

**Theorem 10.** Let \( G \) be a 1TG. If \( \rightarrow_{\text{bPC}} \) fairly terminates for \( G \) (i.e. the corresponding instance of SPP is safe), then it has exactly one equilibrium.

We (re-)prove this result in our setting. We rely on the notion of L-fair path that we define now. For a labelled graph \( G = (V, E, L) \), we write \( v_1 \rightarrow_a v_2 \) iff \( L(v_1, v_2) = a \) (for \( v_1, v_2 \in V \). We further write \( v_1 \rightarrow A v_2 \) with \( A = a_1 \cdots a_n \) iff \( v_1 \rightarrow a_1 \cdots \rightarrow a_n v_2 \). Then, a path \( \pi = v_1 v_2 \cdots \) is L-fair if all labels eventually occur in this path, i.e. for all \( e \in L \), there is \( k \geq 1 \) s.t. \( L(v_k, v_{k+1}) = a \). Then, we can show the following technical lemma:

**Lemma 11.** Let \( G = (V, E, L) \) be a finite complete deterministic labelled graph satisfying:

- for all \( v \in V \), for all \( a, b \in L \), there are \( A, B \in L^* \) and \( \bar{v} \in V \) such that \( v \rightarrow A \bar{v} \) and \( v \rightarrow B a b \bar{v} \).

If there exists a state from which we can reach two different terminal states, then \( G \) has an infinite L-fair path.

Thanks to this result, we can establish Theorem 10. We prove the contrapositive, as follows (full proof in Appendix B). We assume that \( G^{(\rightarrow_{\text{bPC}})} \) has more than one equilibrium. We introduce a new dynamics \( \rightarrow \) (taking into account the beliefs of the players about the other players’ strategies) and we use Lemma 11, to show that \( G^{(\rightarrow)} \) has an L-fair cycle. Then, we define a partial simulation \( \subseteq \) of \( G^{(\rightarrow)} \) by \( G^{(\rightarrow_{\text{bPC}})} \) and use Proposition 5 to conclude that \( G^{(\rightarrow_{\text{bPC}})} \) has a cycle, which is fair. Hence, \( \rightarrow_{\text{bPC}} \) does not fairly terminate.

**Griffin et al:** Dispute wheels. Another classical notion in the BGP literature is that of dispute wheel (DW for short), defined by Griffin et al. [7] as a “circular set of conflicting rankings between nodes”. They have shown that the absence of a DW is a sufficient condition for safety, which is of course of major practical interest to prove that BGP will converge in a given network. Moreover, a DW is an instance of a forbidden pattern in a game, and we will thus apply the results from Section 4.
We start by formally defining a DW. Let $G = (V, E, (V_i)_{1 \leq i \leq n}, (\preceq_i)_{1 \leq i \leq n})$ be a 1TG with $P_i$ the set of permitted paths of $v_i$. A triple $D = (U, P, H)$ is a DW of $G$ if:

- $U = (u_1, \ldots, u_k) \in V^k$ is a tuple of states;
- $P = (\pi_1, \ldots, \pi_k)$ is a tuple of permitted paths such that for all $1 \leq i \leq k$: $\pi_i \in P_{u_i}$, i.e., $\pi_i$ is a permitted path starting in $u_i$;
- $H = (h_1, \ldots, h_k)$ is a tuple of non-maximal paths such that for all $1 \leq i \leq k$:
  - $h_i\pi_{(i \mod k)+1} \in P_{u_i}$; and
  - $h_i$ for all $1 \leq i \leq k$: $\pi_i \not\sim u_i h_i\pi_{i+1}$.

Intuitively, in a DW, all players $u_i$ (for $i = 1, \ldots, k$) can choose between two paths to $v_\perp$:
- either a ‘direct’ path $\pi_i$, or an ‘indirect’ path $h_i\pi_{(i \mod k)+1}$, which traverses $u_{(i \mod k)+1}$
- and where the latter is always preferred. So $u_1$ prefers to reach through $v_2$, $v_3$ through $u_3$, and so on until $u_k$ who prefers to reach through $u_1$. Such a conflict clearly yields loops where the target is never reached. The game in Figure 1 is a typical example of game that has a DW, if we let $U = (v_1, v_2)$, $P = (v_1v_\perp, v_2v_\perp)$ and $H = (v_1, v_2)$. Indeed, $v_1v_\perp \sim v_1v_2v_\perp$ and $v_2v_\perp \not\sim v_2v_1v_\perp$. Then, in our setting the sufficient condition of Griffin et al. [7] becomes:

> **Theorem 12** ([7]). Let $G$ be a 1TG. If $G$ has no DW then $bPC\xrightarrow{\text{fairly}}$ fairly terminates for $G$.

**New result: strong dispute wheels for a necessary condition** It is well-known, however, that the absence of a DW is not necessary (see for example Figure 4 in Appendix A for a game that has a DW but where $bPC\xrightarrow{\text{terminates}}$). As far as we know, finding a unique and necessary condition for the fair termination of $bPC\xrightarrow{\text{in 1TGs}}$ is still an open problem.

Relying on our framework, we manage to obtain such a necessary and sufficient condition in a restricted setting. We first strengthen the definition of DW by introducing the notion of strong dispute wheel (SDW for short). We then obtain two original (as far as we know) results regarding SDW. First, the absence of SDW is a necessary condition for the termination of $PC\xrightarrow{\text{in 1TGs}}$ (i.e., we drop the best-reply and the fairness hypothesis). Second, the absence of an SDW is also a sufficient condition in the restricted setting where the preferences of the players are not only on their next-hop. This means for example that $u_1$ prefers to reach the target through $u_2$ rather than through $u_3$, but does not mind the route $u_2$ uses (as long as $v_\perp$ is reached). While this is a restriction, we believe that it is still meaningful in practice, since networks usually have little control about the routes chosen by their neighbours.

We first define the notion of SDW. Let $G$ be a 1TG and $D = (U, P, H)$ be a DW of $G$. Then, $D$ is a strong dispute wheel (SDW) of $G$ if:

1. for all $1 \leq i \leq k$: all states $u_i \in U$ occur only in $\pi_i$, $h_i$ and $h_{i-1}$ (we identify $h_0$ with $h_k$) and not in the other paths of $P$ and $H$; and
2. for all $\pi_i, \pi_j \in P$, for all $h_k, h_\ell \in H$ with $k \neq \ell$: $\pi_i, h_k$ and $h_\ell$ share no states of $V \setminus U$;
   and if $\pi_i$ and $\pi_j$ share a state $v$ of $V \setminus U$ then $\pi_i$ and $\pi_j$ have the same suffix after $v$.

An important property of this definition is that, whenever a game $G$ contains an SDW $D = (U, P, H)$, we can extract a minor $G'$ which is essentially an SDW restricted to the states of $U$ (formally, $G'$ contains an SDW $D' = (U', P', H')$ where $U' = U$ is the set of states of $G'$). We do so by first deleting from $G$ all edges that do not occur in $P$ and $H$; then all $v \not\in U$ (which have at most one outgoing edge at this point), using the procedure described in Section 4. Note that the two extra conditions in the definition of an SDW guarantee that the deletion of all the states $v \not\in U$ can occur.

> **Theorem 13.** Let $G$ be a 1TG. If $PC\xrightarrow{\text{terminates}}$ for $G$, then $G$ has no SDW.

**Proof.** By Theorem 8, it is sufficient to prove that the dynamics $PC\xrightarrow{\text{does not terminate}}$ in the minor game $G'$ extracted from the SDW (see above). We let, for all $1 \leq i \leq k$,
Thus, the absence of an SDW is a necessary condition for the termination of \( \sigma \). We can further show that this condition is sufficient in the restricted case where any two (permitted) paths that have the same next-hop are equivalent. Formally, let \( G \) be a NITG. We say that it is a neighbour one target game (NITG for short) if for all players \( i \), for all permitted paths \( \pi_1, \pi_2 \in \mathcal{P}_i \) of player \( i \): \( \pi_1 = v_i v_{i+1}^\pi \) and \( \pi_2 = v_i v_{i+2}^\pi \) implies that \( \pi_1 \sim \pi_2 \). Then, we can show the following, relying on Theorem 13 (and thus, also on Theorem 8):

\[
\text{Lemma 14. Let } G \text{ be a NITG. Then, } G \text{ has a DW+ if and only if } G \text{ has an SDW.}
\]

\textbf{Sketch of proof.} In [7], Griffin et al. prove a stronger result than Theorem 12, showing that if \( G(\sigma) \) has a fair cycle, then \( G \) has a DW satisfying the following additional properties: 1. for all \( u_i \in U \): \( j \neq i \) implies \( u_i \notin \pi_j \); 2. for all \( v \notin U \), for all \( i, j \): \( v \notin \pi_i \cap \pi_j \); and 3. for all \( v \in \pi_i \cap \pi_j \): \( \pi_i(v) = \pi_j(v) \). We call DW+ such DW. Then, the general schema of our proof is summarised in Figure 3: first, we show that the existence of a DW+ implies an SDW by showing the required additional properties. By Theorem 13, this implies \( G(\sigma) \) has a cycle. Then, we conclude by showing that this implies the existence of a fair cycle.

\textbf{Finding an SDW in practice} Because of the intricate definition of SDW, finding an SDW in a real network may be challenging in practice. However, we have:

\[
\text{Proposition 15. Let } G \text{ be an NITG. Then } G \text{ has an SDW if and only if } G^{DIS} \text{ is a minor of } G.
\]

\textbf{Future Works} We envision multiple directions of future work. First, we could consider games with imperfect information. In the application to interdomain routing for example, this could be used to model a malicious router that advertises lies to selected neighbours. Advertising a non-existent or non-feasible path would allow for example an attacker to attract the packets of an opponent’s network. Second, we could investigate a better way to model asynchronicity (useful for the routing problem) than the concurrent dynamics we have studied here. Third, we chose to model fairness via a qualitative property which ensures that all the players will eventually have the opportunity to update their strategies if they want to. An alternative way could be the use of probabilities: indeed, there are games for which a dynamics \( \rightarrow \) does not fairly terminate (with the definition of fairness of the present paper), but where an equilibrium is reached almost surely when interpreting \( G(\rightarrow) \) has a finite Markov chain (with uniform distributions). Finally, we could apply the dynamics of graph-based games to other problems than interdomain routing, like load sensitive routing.


Appendix to Section 4

A.1 Missing proofs

Proof of Theorem 8. Since simulations are preorders, they are transitive relations (if \( G \) simulates \( G' \) and \( G' \) simulates \( G'' \), then \( G \) simulates \( G'' \)). Therefore, it is sufficient to only consider that \( G' \) has been obtained from \( G \) either by deleting a single edge, or by deleting a single state.

First, let us consider the case where \( G' \) is obtained by the deletion of an edge \( e_0 \). For the dynamics \( 1 \rightarrow \), we define a relation \( \sqsubseteq \), subset of \( \Sigma(G') \times \Sigma(G) \), by letting \( \sigma' \sqsubseteq \sigma \) if for all histories \( h \in \text{Hist}(G') \) (i.e. \( h \) does not go through \( e_0 \)), we have \( \sigma(h) = \sigma'(h) \). This relation has \( \Sigma(G') \) as a domain. We are going to show that this is a simulation relation. To do so, consider \( \sigma' \) and \( \tau' \) in \( \Sigma(G') \) such that \( \sigma' \rightharpoonup \tau' \), and a strategy \( \sigma \in \Sigma(G) \) such that \( \sigma' \sqsubseteq \sigma \). Let \( \tau \in \Sigma(G) \) be defined for a history \( h \) of \( G \) by \( \tau(h) = \tau'(h) \) if \( h \in \text{Hist}(G') \), and \( \tau(h) = \sigma(h) \) otherwise.

We must prove that \( \sigma \rightharpoonup \tau \). Let \( h' \in \text{Hist}(G') \) be the history s.t. (1) \( \sigma'(h') \neq \tau'(h') \); and (2) for all histories \( h \in \text{Hist}(G') \setminus \{h'\} : \sigma'(h) = \tau'(h) \). Such a history \( h' \) exists since \( \sigma' \rightharpoonup \tau' \).

Moreover, if \( i = \text{player}(h') \), \( \text{Outcome}(\sigma',h') \prec_i \text{Outcome}(\tau',h') \), again because \( \sigma' \rightharpoonup \tau' \).

Then, we have \( \tau(h') = \tau'(h') \neq \sigma'(h') = \sigma(h') \). Moreover, for all \( h \in \text{Hist}(G') \setminus \{h'\} \), if \( h \notin \text{Hist}(G') \), then \( \tau(h) = \sigma(h) \) by definition, and otherwise \( \tau(h) = \tau'(h) = \sigma'(h) = \sigma(h) \).

Therefore, \( \text{Outcome}(\sigma,h') \prec \text{Outcome}(\tau,h') \), so that \( \sigma \rightharpoonup \tau \). We thus have proved that \( G'(\rightharpoonup_1) \sqsubseteq G(\rightharpoonup_1) \).

The same kind of reasoning can be made for \( \rightharpoonup_{P1} \) and \( \rightharpoonup_{PC} \). The situation is easier in the case of positional strategies, since \( G \) and \( G' \) have the same set of states (we have assumed we have only deleted an edge). Let \( \sqsubseteq \), subset of \( \Sigma^P(G') \times \Sigma^P(G) \) be such that \( \sigma' \sqsubseteq \sigma \) if \( \forall v \in V, \sigma'(v) = \sigma(v) \) (note that, although \( \sigma \) and \( \sigma' \) take the same decisions in the same states, they are, formally, different objects. Indeed, they map histories, which are different in \( G \) and \( G' \), to states). This relation has \( \Sigma^P(G') \) as domain, and for all \( \sigma' \in \Sigma^P(G') \) there is one and only one \( \sigma \in \Sigma^P(G) \) such that \( \sigma' \sqsubseteq \sigma \). We are going to show that this is a simulation relation. To do so, consider \( \sigma' \) and \( \tau' \) in \( \Sigma^P(G') \) such that \( \sigma' \rightharpoonup_{P1} \tau' \) (resp. \( \sigma' \rightharpoonup_{PC} \tau' \)), and a strategy \( \sigma \in \Sigma^P(G) \) such that \( \sigma' \sqsubseteq \sigma \). Let \( \tau \in \Sigma^P(G) \) be the (unique) strategy profile such that \( \tau' \sqsubseteq \tau \). By definition of \( \rightharpoonup_{P1} \) (resp., \( \rightharpoonup_{PC} \)) and \( \sqsubseteq \), we have \( \sigma \rightharpoonup_{P1} \tau \) (resp., \( \sigma \rightharpoonup_{PC} \tau \)).

Consider then the case where \( G' \) is obtained by the deletion of a state \( v \). For the dynamics \( \rightarrow \), if \( h \in \text{Hist}(G) \) such that \( \text{last}(h) \neq v \), we can construct a corresponding play \( f(h) \) of \( G' \) by replacing a sequence \( wvw' \) of \( h \) by \( uv' \). Notice that \( f \) is a bijection from \( \text{Hist}(G) \setminus \{h \mid \text{last}(h) = v \} \) to \( \text{Hist}(G') \), by the conditions over the deletion of \( v \) in the definition of minors. Then, the relation on strategy profiles is given by \( \sqsubseteq \sqsubseteq \sigma \) if:

\[
\text{for all } h \in \text{Hist}(G) \setminus \{h \mid \text{last}(h) = v \} \colon \sigma'(f(h)) = \begin{cases} \sigma(h) & \text{if } \sigma(h) \neq v \\ \sigma'(f(h)) = v' & \text{otherwise.} \end{cases}
\]

Since \( f \) is bijective and since, for all \( \sigma \) and \( h \): \( \text{last}(h) = v \) implies \( \sigma(h) = v' \); then the relation \( \sqsubseteq \) is also a bijection. This enables us to prove as before that it is a simulation relation (indeed even a bisimulation relation), so that \( \sqsubseteq \) is a simulation relation of \( G'(\rightarrow) \) by \( G(\rightarrow) \), in that case too.

We can make the same reasoning for \( \rightharpoonup_{P1} \) and \( \rightharpoonup_{PC} \). Let \( \sqsubseteq \), subset of \( \Sigma^P(G') \times \Sigma^P(G) \) such that \( \sigma' \sqsubseteq \sigma \) if:

\[
\text{for all } u \in V \setminus \{v \} \colon \sigma'(u) = \begin{cases} \sigma(u) & \text{if } \sigma(u) \neq v \\ v' & \text{if } \sigma(u) = v. \end{cases}
\]
It is clear that this relation is a bisimulation between $G'(\xrightarrow{P_1})$ and $G'(\xrightarrow{PC})$ (resp. $G'(\xrightarrow{P_1})$ and $G'(\xrightarrow{SPC})$). □

Proof of Theorem 9. We follow the same proof as in Theorem 8. We have already seen that if $G'$ is obtained by the deletion of a state, then $G$ and $G'$ are bisimilar, and we can show easily that a fair cycle in $G(\rightarrow)$ implies the existence of a fair cycle in $G'(\rightarrow)$.

If $G'$ is obtained by the deletion of a dominated edge $e$, then we construct the same simulation relation $\sqsubseteq$ as before (in the cases $\xrightarrow{P_1}$ and $\xrightarrow{PC}$), i.e. $\sigma' \sqsubseteq \sigma$ if $\forall v \in V: \sigma'(v) = \sigma(v)$. This is still a simulation relation, even with the best-reply dynamics, since we remove a dominated edge. We can also show similarly that $\sqsubseteq^{-1}$ is a partial simulation, by using once again that a change of profile towards $e$ is not possible since it is dominated. If there is a fair cycle in $G'(\rightarrow)$ then the simulation gives a cycle in $G(\rightarrow)$: this is also a fair cycle since the edge $e$ is dominated and thus cannot be the only cause of non-fairness. Reciprocally, the partial simulation $\sqsubseteq^{-1}$ allows one to reconstruct from a fair cycle in $G(\rightarrow)$ a cycle in $G'(\rightarrow)$: if this cycle is not fair, this means that the edge $e$ is chosen infinitely often in the cycle of $G'$, which violates the best-reply condition of the dynamics $\rightarrow$ in $G$. □

A.2 Examples related to Theorem 8 and Theorem 9

Figure 4 illustrates Theorem 8, and shows why it does not work with $\xrightarrow{SPC}$ and $\xrightarrow{SPC'}$. Assume that this game $G$ has the following preferences: $(v_1 v_2)^\omega \prec_1 v_1 v_\perp \prec_1 v_1 v_2 v_\perp \prec_1 v_1 v_3 v_\perp$. Clearly, $G^{DIS}$ is a minor of $G$, and $G'(\xrightarrow{SPC})$ simulates $G^{DIS}(\xrightarrow{SPC})$. However, it is not the case for $G'(\xrightarrow{SPC'})$: in particular, there is no cycle in $G'(\xrightarrow{SPC'})$ although $G^{DIS}(\xrightarrow{SPC'})$ has a cycle. The example can be adapted to $\xrightarrow{SPC'}$.

Figure 5 shows why Theorem 8 does not work with fair termination (instead of termination). The game $G$ on the left of the figure is a 3-player game, where Player $i$ owns state $v_i$ (for $i = 1, 2, 3$), and where the players' preferences are as follows:

- $v_1 v_\perp \prec_1 v_1 v_2 v_\perp$;
- $v_2 v_\perp \prec_2 v_2 v_3 v_\perp$; and

The game $G$ on the left of the figure is a 3-player game, where Player $i$ owns state $v_i$ (for $i = 1, 2, 3$), and where the players' preferences are as follows:

- $v_1 v_\perp \prec_1 v_1 v_2 v_\perp$;
- $v_2 v_\perp \prec_2 v_2 v_3 v_\perp$; and
Figure 6: A game $\mathcal{G}$ where $\mathcal{G}^{\overleftrightarrow{PC}}$ does not terminate. When removing the edge $e$ (which is dominated by $f$), we obtain a new game $\mathcal{G}'$ which is a dominant minor of this game, in which $\mathcal{G}'^{\overleftrightarrow{PC}}$ terminates.

Player 3 prefers $v_3v_\perp$ to all other paths.

Clearly, $\mathcal{G}^{DIS}$ (Figure 1) is a minor of $\mathcal{G}$. We already know (Figure 1 again) that $\mathcal{G}^{DIS}(\overleftrightarrow{PC})$ admits a fair cycle, so, if Theorem 9 were to hold on $\overleftrightarrow{PC}$, $\mathcal{G}(\overleftrightarrow{PC})$ should admit a fair cycle too. Let us explain why it is not the case. The graph $\mathcal{G}(\overleftrightarrow{PC})$ is sketched on the right-hand side of Figure 5. All the nodes in the left gray boxes are those where Player 3 plays $c_3$, and those in the right box are where he plays $s_3$. Several edges are not drawn for clarity: they are all the edges where Player 3 changes from $c_3$ to $s_3$. Given his preferences, he will always choose to do so if he decides to update his strategy, so there is an implicit edge from $c_1c_2c_3$ to $c_1c_2s_3$, but also one from $c_1c_2c_3$ to $c_1s_3s_3$ for example. Then, as soon as Player 3 has decided to play $s_3$, the other two players will both prefer to play $s_1$ and $s_2$ respectively. As a consequence, the only cycle is $(c_1c_2c_3,s_1s_2c_3)^\omega$, where Player 3 never plays although he could. This cycle is not fair. Hence, $\mathcal{G}(\overleftrightarrow{PC})$ admits no fair cycle.

Finally, we exhibit a third example of a game $\mathcal{G}$ in which $\mathcal{G}(\overleftrightarrow{PC})$ does not terminate, and from which we can extract a dominant minor $\mathcal{G}'$ s.t. $\mathcal{G}'(\overleftrightarrow{PC})$ terminates. This shows why Theorem 9 does not hold anymore when the ‘best reply’ hypothesis is dropped. This game is showed in Figure 6, with the following preferences for the players:

- $v_1v_2v_3v_\perp \prec_1 v_1v_\perp \prec_1 v_1v_4v_\perp \prec_1 v_1v_2v_\perp$;
- $v_2v_3v_4v_\perp \prec_2 v_2v_\perp \prec_2 v_2v_3v_1v_\perp \prec_2 v_2v_3v_\perp$;
- $v_3v_4v_\perp \prec_3 v_3v_1v_2v_\perp \prec_3 v_3v_\perp \prec_3 v_3v_1v_\perp$.

Then, one can check that the following sequence of (positional) strategy profiles forms a cycle for $\overleftrightarrow{PC}$ (but not for $\overleftrightarrow{bPC}$). Note that we do not indicate $s_4$ which is always equal to $v_\perp$, and we write in bold the change from the previous profile:

1. $\sigma_1 = v_2$, $\sigma_2 = v_\perp$, and $\sigma_3 = v_\perp$;
2. $\sigma_1 = v_2$, $\sigma_2 = v_3$, and $\sigma_3 = v_\perp$;
3. $\sigma_1 = v_\perp$, $\sigma_2 = v_3$, and $\sigma_3 = v_\perp$;
4. $\sigma_1 = v_\perp$, $\sigma_2 = v_3$, and $\sigma_3 = v_1$;
5. $\sigma_1 = v_4$, $\sigma_2 = v_3$, and $\sigma_3 = v_1$;
6. $\sigma_1 = v_4$, $\sigma_2 = v_\perp$, and $\sigma_3 = v_1$;
7. $\sigma_1 = v_4$, $\sigma_2 = v_\perp$, and $\sigma_3 = v_\perp$;
8. $\sigma_1 = v_2$, $\sigma_2 = v_\perp$, and $\sigma_3 = v_\perp$;

However, when removing from $\mathcal{G}$ the edge $e$ (which is dominated by $f$), one obtains a new game $\mathcal{G}'$ that is a dominant of $\mathcal{G}$ and in which the $\overleftrightarrow{PC}$ terminates.
Proof of Theorem 10. We prove the contrapositive of the Theorem, i.e. we assume that $G$ has more than one equilibrium, and we deduce that $G(\mathcal{BPC})$ has a fair cycle.

To prove that $G(\mathcal{BPC})$ has a fair cycle, we proceed in two steps. First, we introduce a new dynamics $\rightharpoonup$ and prove, by means of Lemma 11, that $G(\rightharpoonup)$ has an $L$-fair cycle. Then, we define a partial simulation $\sqsubseteq$ of $G(\mathcal{BPC})$ by $G(\rightharpoonup)$ and use Proposition 5 to conclude that $G(\mathcal{BPC})$ has a cycle. Finally, we show that this cycle is fair.

The new dynamics we consider takes explicitly into account the fact that, when a player updates its strategy, it might take some time for the other players to be notified. Hence, each state in this new dynamics stores what each Player $j$ believes about the current strategy of all Player $i$. Formally, let $G(\rightharpoonup) = (V, \rightharpoonup, L)$ be the labelled graph defined as follows.
the set of states $V$ contains all states $v$ of the form $v = (\sigma_{i,j})_{1 \leq i,j \leq n}$, i.e. a state associated a strategy $\sigma_{i,j}$ to each pair of players $i$ and $j$. Intuitively, $\sigma_{i,j}$ represents what Player $j$ believes is the current strategy of Player $i$. In other words, for all $1 \leq j \leq n$, $\sigma_j = (\sigma_{i,j})_{1 \leq i \leq n}$ represents the belief of the player $j$ about the strategies of the other players. In particular, $\sigma_{j,j}$ is the current strategy of player $j$. We let

$$V_0 = \{v \in V | \forall j, j' \in N : (\sigma_{i,j})_{i \in N} = (\sigma_{i,j'})_{i \in N}\}$$

be the set of states where every player knows the strategies of the other players;

$L = \{0,1,\ldots,n\}$; and

$v \xrightarrow{\ell} v'$ iff:

- Either $\ell = 0$, then $v' \in V_0$ and for all $1 \leq j \leq n$, $\sigma_{i,j} = \sigma'_{i,j}$. The $0$ change is an update of knowledge. All players learn the strategies of the other players;
- Or $\ell \neq 0$, then for all $j \neq \ell$: $(\sigma_{i,j})_{1 \leq i \leq n} = (\sigma'_{i,j})_{1 \leq i \leq n}$ and the update of Player $\ell$ is performed according to $\xrightarrow{P_1}$ if possible. Formally, $(\sigma_{i,j})_{1 \leq i \leq n} = (\sigma'_{i,j})_{1 \leq i \leq n}$ with $\sigma_{\ell,\ell} \neq \sigma'_{\ell,\ell}$ if this is permitted by $\xrightarrow{P_1}$, or $(\sigma_{i,j})_{1 \leq i \leq n} = (\sigma'_{i,j})_{1 \leq i \leq n}$ otherwise. Such a move in the dynamics thus corresponds to Player $\ell$ updating his strategy according to $\xrightarrow{P_1}$ if possible, or no update at all if not. This change is not yet learned by the other players.

The rest of the proof will be as follows:

1. We check that $G(\xrightarrow{\sim})$ verifies the conditions in Lemma 11, and then has a $L$-fair cycle;
2. We define a partial simulation $\subseteq$ of $G(\xrightarrow{\sim})$ by $G(\xrightarrow{bPC})$ of domain $V_0$;
3. We deduce that $G(\xrightarrow{\sim})$ has a cycle containing elements of $V_0$;
4. Using Proposition 5, we deduce that $G(\xrightarrow{bPC})$ has a cycle;
5. Then, we conclude that $G(\xrightarrow{bPC})$ has a fair cycle.

1. $G(\xrightarrow{\sim})$ is deterministic because the dynamics is Best Reply and the preferences are strict for a different neighbour.
2. $G(\xrightarrow{\sim})$ is complete by definition.
3. Let $v \in V$ and let $i,j \in L$. We will show that $\exists I, J$ and $v^*$ such that $v \xrightarrow{ij} v^*$ and $v \xrightarrow{ji} v^*$.
4. If $i = j$, let $I = J = \emptyset$.
5. If $i = 0$ and $j \neq 0$. Let $I = ji$ and $J = ji$. Intuitively, $ij$ means that player $j$ changes his strategy, there is a knowledge update, then $j$ changes again, then there is a knowledge update. It is clear that if we remove the first change of $j$, the result will be exactly the same, i.e. $ij \xrightarrow{ij} ij$.
6. If $j = 0$ and $i \neq 0$. Let $I = ji$ and $J = j$. This is symmetrical to the previous point, then we have that $ij \xrightarrow{ji} ji$.
7. If $i \neq 0$ and $j \neq 0$. Then let $I = j$ and $J = i$. Since these updates are performed without any knowledge update, it is clear that $ij = ji$.
8. For $\tau \in \Sigma^P$ and $\bar{\tau} \in \Sigma^P$ two different equilibria of $G(\xrightarrow{bPC})$. Then $t = (\tau_{i,j})_{1 \leq i,j \leq n}, \bar{t} = (\bar{\tau}_{i,j})_{1 \leq i,j \leq n} \in V_0$ such that $\forall j, \tau_{i,j} = \tau_i$ and $\bar{\tau}_{i,j} = \bar{\tau}_i$ are the terminal nodes of $G(\xrightarrow{\sim})$. Let $v = (\sigma_{i,j})_{1 \leq i,j \leq n} \in V$ such that $\forall 1 \leq i \leq n, \sigma_{i,i} = \tau_i$ and $\forall j \neq i, \sigma_{i,j} = \bar{\tau}_i$. In this state, all players play the first equilibrium $\tau$ but believe that the other players play the second equilibrium $\bar{\tau}$. Clearly, $v \xrightarrow{\sim} t$ and $v \xrightarrow{\sim} \bar{t}$. Then the set of states that can reach at least two terminal nodes is non empty.

Then, by Lemma 11, $G(\xrightarrow{\sim})$ has an $L$-fair cycle.
2. We define a partial simulation $\sqsubseteq$ of $G(\Rrightarrow^+)$ by $G(\frac{b\text{PC}}{\rightarrow})$ of domain $V_0$ as follows:

\[(\sigma_{i_j})_{1 \leq i_j \leq n} \sqsubseteq (\tau_{i_j})_{1 \leq i_j \leq n} \text{ if } \forall j, (\sigma_{i_j})_{1 \leq i_j \leq n} = (\tau_{i_j})_{1 \leq i_j \leq n} \]  

The proof that $\sqsubseteq$ is a partial simulation of $G(\Rrightarrow^+)$ by $G(\frac{b\text{PC}}{\rightarrow})$ comes directly from the definition of $\frac{b\text{PC}}{\rightarrow}$ and $\Rrightarrow$.

3. Clearly, if $G(\Rrightarrow)$ has an L-fair cycle, it means that this cycle contains elements of $V_0$ by definition of $\Rrightarrow$. Then $G(\Rrightarrow^+)$ contains a cycle with only states of $V_0$. Intuitively, this means that the cycle visits states where all the Players are perfectly informed about the strategies of all other players, which are the states that ‘corresponds’ to those in $G(\frac{b\text{PC}}{\rightarrow})$.

4. By Proposition 5, it means that $G(\frac{b\text{PC}}{\rightarrow})$ has a cycle.

5. As the first cycle built in $G(\Rrightarrow)$ was L-fair, it means that, during this cycle, every player has changed his strategy, or at least has had the opportunity to change it. It means that, in $G(\frac{b\text{PC}}{\rightarrow})$, the cycle is fair.

\begin{proof}

Proof of Theorem 14. In [7], Griffin et al. prove a stronger result than Theorem 12, showing that if $G(\frac{\text{PC}}{\rightarrow})$ has a fair cycle, then $G$ has a DW such that:

\begin{enumerate}

\item $\forall u_i \in U$, $u_i \not\in \pi_j \in P$, if $j \neq i$;
\item $\forall v \not\in U$, $v \not\in \pi_i \cap \pi_j$, for $\pi_i \in P$ and $h_j \in H$;
\item $\forall v \in (\pi_i \cap \pi_j), \pi(v) = \pi(v)$.
\end{enumerate}

We call such a dispute wheel a DW+. We show that if a game has a DW+, this implies the existence of a SDW by showing the remaining properties:

\begin{enumerate}

\item $\forall u_i \in U$, $u_i \not\in h_j \in H$, if $j \neq i$ and $j \neq i - 1$.
\item $\forall v \not\in U$, $v \not\in h_i \cap h_j$, for $h_i, h_j \in H$ with $i \neq j$.
\end{enumerate}

Moreover, Theorem 13 tells us that having an SDW implies that $G(\frac{\text{PC}}{\rightarrow})$ has a cycle. We conclude the proof by showing that if $G(\frac{\text{PC}}{\rightarrow})$ has a cycle, it has a fair cycle, because if a player which was not allowed to change during the non fair cycle decides to change, it will not influence the choices of the other players, then the cycle remains.

Let us prove this:

- If $G$ has a DW+, then we can build a DW such that $\forall u_i \in U$, $u_i \not\in h_j \in H$, if $j \neq i$ and $j \neq i - 1$. On the contrary, let us suppose that $\exists u_i \in U$ such that $\exists h_j \in H$ with $j \neq i - 1$; $j \neq i$ such that $u_i \in h_j$. It means that $\pi_j = u_j p_j \ldots 0 \prec_j u_j r_j \ldots u_{j+1} p_{j+1} \ldots 0 = h_j \pi_j + 1$. By conditions over the preferences of the game, it means that, for $h_j = u_j r_j \ldots p_i$, $\pi_j = u_j p_j \ldots 0 \prec_j u_j r_j \ldots u_{j+1} p_{j+1} \ldots 0 = h_j \pi_j + 1$. Let $H' = (\frac{b\text{PC}}{\rightarrow})$, where $\frac{b\text{PC}}{\rightarrow}$ is a partial simulation of $G(\frac{\text{PC}}{\rightarrow})$.

- If $G$ has a DW+ satisfying condition 4, then we can build a DW such that $\forall v \not\in U$, $v \not\in h_i \cap h_j$, for $h_i$, $h_j \in H$ with $i \neq j$. Suppose that $\exists v \not\in U$ such that $v \in h_i \cup h_j$. Let $h_i = u_i r_i \ldots v u_i \ldots u_{i+1}$ and $h_j = u_j r_j \ldots v u_j \ldots u_{j+1}$. In particular, as $\pi_j \prec_j u_j r_j \ldots v u_j \ldots u_{j+1} p_{j+1} \ldots 0 = h_j \pi_j + 1$, by conditions over the preferences, it means that, for $h_j = u_j r_j \ldots v u_j \ldots u_{j+1}$, then $\pi_j \prec_j u_j r_j \ldots v u_j \ldots u_{j+1} p_{j+1} \ldots 0 = h_j \pi_j + 1 = h_j' \pi_j$. Let $H' = (\frac{b\text{PC}}{\rightarrow})$, where $\frac{b\text{PC}}{\rightarrow}$ is a partial simulation of $G(\frac{\text{PC}}{\rightarrow})$.

- First of all, let $\sigma_1 \rightarrow \ldots \rightarrow \sigma_n$, a (not fair) cycle in $G$. Let us suppose that a player $i$ cannot change his strategy during this cycle. Let $\sigma_{i,j}$ be the best reply of player $i$ to $\sigma_i$. By hypothesis of Neighbour game, $\sigma_{i,j}$ is a fair cycle, with $\sigma_{i,j} = \sigma_{i,1}$ and $\forall k \neq i, \sigma_{j, k} = \sigma_{j, k}$. \end{proof}
Proof of Proposition 15. By Theorem 14, if $G^{DIS}$ is a minor of $G$, then $PC$ does not fairly terminates for $G$. Moreover, still by Theorem 14, if $PC$ does not fairly terminate, then it means that $G$ has an SDW. It remains to prove that if $G$ has an SDW, $G$ has $G^{DIS}$ as a minor.

Let $D = (U, P, H)$ be the SDW of $G$ with $U = (u_1, \ldots, u_k)$, $P = (\pi_1, \ldots, \pi_k)$ and $H = (h_1, \ldots, h_k)$. If $k > 2$, let $D' = ((u_1, u_2), (\pi_1, \pi_2), (h_1, h_2'))$ with $h_2' = h_2 h_3 \ldots h_k$. We do have that $\pi_1 \prec_1 h_1 \pi_2$ and $\pi_2 \prec_2 h_2' \pi_1 \sim_2 h_2 \pi_3$ because $G$ is an NITG. In particular, $G^{DIS}$ is a minor of $G$. \hfill \square