Dynamics and coalitions in sequential games

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We consider n-player non-zero sum games played on finite trees (i.e., sequential games), in which the players have the right to repeatedly update their respective strategies (for instance, to improve their payoff wrt to the current strategy profile). This generates a dynamics in the game which may eventually stabilise to a Nash Equilibrium (as with Kukushkin's lazy improvement), and it is thus interesting to study the conditions that guarantee such a dynamics to terminate.

We build on the works of Leroux and Pauly who have studied extensively the Lazy Improvement Dynamics. We extend these works by first defining a turn-based dynamics, proving that it terminates on subgame perfect equilibria, and showing that several variants do not terminate. Second, we define a variant of Kukushkin's lazy improvement where the players may now form coalitions to change strategies. We show how properties of the players's preferences on the outcomes affect the termination of this dynamics and thereby characterise classes of games where it always terminates (in particular two-player games).

1 Introduction

Since the seminal works of Morgenstern and von Neuman in the fifties [7], game theory has emerged as a prominent paradigm to model the behaviour of rational and selfish agents acting in a competitive setting. The first and main application of game theory is to be found in the domain of economics where the agents can model firms, investors, who are competing for profits, gain access to market, etc. Since then, game theory has evolved into a fully developed mathematical theory and has recently found many applications in computer science. In this setting, the agents usually model different components of a computer system (and of its environment), that have their own objective to fulfil. For example, game theory has been applied to analyse peer-to-peer file transfer protocols [8] where the agents want to maximise the amount of downloaded data in order to obtain, say, a given file; while minimising the amount of uploaded data to save bandwidth. Another application is that of controller synthesis where the two, fully antagonistic, agents are the system and its environment, and where the game objective models the control objective.

The most basic model to describe the interaction between the players is that of games in *strategic form* (aka matrix games), where all the players chose simultaneously an action (or strategy) from a finite set, and where they all obtain a payoff (usually modelled by a real value) which depends on their joint choice of actions (or strategy profile). Strategic games are one-shot games, in the sense that the players play only one action. An alternative model where players play repeatedly is that of *sequential games*, which we consider in this work. Such a game is played on a finite tree whose inner nodes are labelled by players, whose edges correspond to the possible actions of the players, and whose leaves are associated with *outcomes*. The game starts at the root, and, at each step of the game, the player who owns the current node picks an edge (i.e. an action) from this node, and the game moves to the destination node of the edge. The outcome the players obtain is that which is associated with the leaf that the game reaches at the end of the play, and each player has a *preference* over the possible outcome (each player's aim is thus to obtain the best outcome according to this preference relation).

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Arguably the most natural question about these games is to determine how rational and selfish players would act (assuming that they have full knowledge of the other players's possible actions and of the outcomes). Several answers to this question have been provided in the literature, such as the famous notion of Nash equilibrium [6], which is a situation (strategy profile) in which no player has an incentive to change his choice of action alone (because such a choice would not be profitable to him). Apart from Nash equilibria, other such *solution concepts* have been proposed like subgame perfect equilibria. This traditional view on game theory can be qualified as *static*, in the sense that all players chose their strategies (thereby forming a strategy profile that can qualify as one of the equilibria listed above), and the game is then played *once* according to these strategies. It is also natural to consider a more *dynamic* setting, in which the players can play the game repeatedly, updating their respective strategies after each play, in order to try and improve the outcome at the next play.

Contribution In this paper, we continue a series of works that aim at connecting these static and dynamic views. That is, we want to study (in the case of extensive form games) the long term behaviour of the dynamics in which players update repeatedly their strategies and characterise when such dynamics converge, and to what kind of strategy profiles (i.e., do these stable profiles correspond to some notions of equilibria). Obviously, this will depend on how the players update their strategies between two plays of the game. Our results consist in identifying minimal conditions on the updates (modelling potential rational behaviours of the players) for which we can guarantee convergence to some form of equilibria after a bounded number of updates. Our work is an extension of a previous paper by Le Roux and Pauly [3], where they study extensively the so-called *Lazy Improvement Dynamics*, in which, intuitively, only one player at a time can update his strategy, in order to improve his outcome, and *only* in nodes that have an influence on the final outcome (lazy update). Their main result is that this dynamics terminate on Nash equilibria when the preferences of the players are acyclic. We consider a broader family of dynamics and characterise their termination. More precisely:

- We start (Section 3) by considering all dynamics that respect the *subgame improvement* property, where players update their strategies only if this yields a better outcome in all subgames that are rooted in a node where a change has been performed. We argue that this can be regarded as a rational behaviour of the players: improving in the subgames can be regarded as an incentive, even when this does not improve the global outcome of the game. Indeed, such an improvement can turn out to be profitable if one of the other players later deviates from its current choice (this is the same intuition as the one behind the notion of Subgame Perfect Equilibrium). Note that such dynamics have not been considered at all in [3]. We show that, in all games where the preferences of the players are acyclic, these dynamics terminate and the terminal profiles are exactly the Subgame Perfect Equilibria of the game.
- Then, we consider (Section 4) all the dynamics that respect the *improvement* property, where all players that change their respective strategy improve the outcome (from the point of view of their respective preferences). Among these dynamics are several ones that have already been studied by Le Roux and Pauly [3] such as the Lazy Improvement Dynamics. We complete the picture (see Table 1), in particular we consider the dynamics that satisfies the *improvement* and the *laziness* property but does not restrict the update to be performed by a single player, contrary to the *Lazy Improvement Dynamics* of Le Roux and Pauly. Thus in our dynamics, players play lazily but are allowed to form coalitions to obtain an outcome which is better for all the players of the coalition. We give necessary and sufficient conditions on the preferences of the players, to ensure termination, in several classes of games (among which 2 player games).

Related works The most related works is the paper by Le Roux and Pauly [3] that we extend here, as already explained. This work is inspired by the notions of *potential* and *semi-potential* introduced respectively by Monderer and Shapley [5]; and Kukushkin [1]. Note also that the idea of repeatedly playing a game and allowing the players to update their strategies between two plays is also behind evolutionary game theory, but in this case, the rules governing the updates are inspired from Darwinian evolution [4, 10].

2 Preliminaries

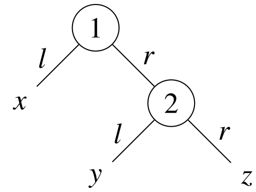
Sequential games We consider *sequential games*, which are n-player non-zero sum games played on finite trees. Figure 1 shows an example of such games, with two players denoted 1 and 2. Intuitively, each node of the tree is controlled by either of the players, and the game is played by moving a token along the branches of the tree, from the root node, until the leaves, which are labeled by a payoff (in this case, x, y or z). The tree edges are labeled by the actions that the player controlling the origin node can play. For example, in the root node, Player 1 can chose to play r, in which case the game reaches the second node, controlled by Player 2, who can chose to play l. The payoff for both players is then y. We also associate a preference relation with each player that indicates how he ranks the payoffs. In the example of Figure 1, player 1 prefers z to x and x to y (noted $y \prec_1 x \prec_1 z$), and Player 2 prefers y to z and z to x.

Let us now formalise the basic notions about these games. The definitions and notations of this section are inspired from [9].

Definition 1. *A* sequential *or* extensive form *game G is a tuple $\langle N, A, H, O, d, p, (\prec_i)_{i\in N}\rangle$ where:*

- *N is a non-empty finite set of* ***players****;*
- *A is a non-empty finite set of* ***actions****;*
- *H is a finite set of finite sequences of A which is prefix-closed. That is, the empty sequence ε is a member of H; and $h = a^1 \dots a^k \in H$ implies that $h' = a^1 \dots a^\ell \in H$ for all $\ell < k$. Each member of H is called a node. A node $h = a^1 \dots a^k \in H$ is* terminal *if $\forall a \in A$, $a^1 \dots a^k a \notin H$. The set of terminal nodes is denoted by Z.*
- *O is the non-empty set of* ***outcomes****,*
- *$d : H \setminus Z \to N$ associates a player with each nonterminal node;*
- *$p : Z \to O$ associates an outcome with each terminal node;*
- *For all $i \in N$: $\prec_i$ is a binary relation over O, modelling the preferences of Player i. We write $x \prec_i y$ and $x \not\prec_i y$ when $(x,y) \in \prec_i$ and $(x,y) \notin \prec_i$ respectively.*

From now on, we fix a sequential game $G = \langle N, A, H, O, d, p, (\prec_i)_{i\in N}\rangle$. Then, we let $H_i = \{h \in H \setminus Z \mid d(h) = i\}$ be the set of **nodes** belonging to player i. A **strategy** $s_i : H_i \to A$ of player i is a function associating an action with all nodes belonging to player i, s.t. for all $h \in H_i$: $hs_i(h) \in H$, i.e. $s_i(h)$ is a legal action from h. Then, a tuple $s = (s_i)_{i\in N}$ associating one strategy with each player is called a **strategy profile** and we let $Strat_G$ be the set of all strategy profiles in G. For all strategy profiles s, we denote by $\langle s\rangle$ the **outcome** of s, which is the outcome of the terminal node obtained when all players play according to s. Formally, $\langle s\rangle = p(h)$ where $h = a^1 \dots a^k \in Z$ is s.t. for all $0 \le \ell < k$: $d(a^1 \dots a^\ell) = i$ implies $a^{\ell+1} = s_i(a^1 \dots a^\ell)$. Let $s = (s_i)_{i\in N}$ be a strategy profile, and let s^*_j be a Player j strategy. Then, we denote by (s_{-j}, s^*_j) the strategy profile $(s_1, \dots, s_{j-1}, s^*_j, s_{j+1}, \dots, s_{|N|})$ where all players play s_i, except



$y \prec_1 x \prec_1 z$
$x \prec_2 z \prec_2 y$

Figure 1: A sequential game with two players.

Player j who plays s^*_j. Let s be a strategy profile, and let $h \in H \setminus Z$ be a nonterminal node. Then, we let: (1) $H|_h = \{h' \mid hh' \in H\}$ be the **subtree** of H from h; (2) $s|_h$ be the **substrategy profile** of s from h which is s.t. $\forall hh' \in H \setminus Z$: $s(hh') = s|_h(h')$; and (3) $G|_h = \langle N, A, H|_h, O, d|_h, p|_h, (\prec_i)_{i \in N}\rangle$ be the **subgame** of G from h. Since a strategy profile $s = (s_i)_{i\in N}$ fixes a strategy for all players, we abuse notations and write, for all nodes $h \in H$, $s(h)$ to denote the action $s_{d(h)}(h)$, i.e. the action that the owner of h plays in h according to s. Then, we say that a node $h = a^1 \dots a^k$ **lies along** the play induced by s if $\forall 1 \leq \ell < k$, $s(a^1 \dots a^\ell) = a^{\ell+1}$. As an example, let us consider again the game in Figure 1. In this game, both players can chose either l or r in the nodes they control. So, possible strategies for Player 1 and 2 are s_1 s.t. $s_1(\varepsilon) = r$ and s_2 s.t. $s_2(r) = l$ respectively. Then, $\langle (s_1, s_2)\rangle = y$. As we consider only simple games in our examples, we denote a profile of strategy by the actions taken by the players. For example, we denote the profile $s = (s_1, s_2)$ by rl. With this notation, this game has four strategy profiles : rr, rl, lr and ll.

Equilibria Now that we have fixed the notions of games and strategies, we turn our attention to two classical notions of *equilibria*. First, a strategy profile s^* is a **Nash equilibrium** (NE for short) if for all players $i \in N$, for all strategies s_i of player i: $\langle (s^*_{-i}, s^*_i)\rangle \not\prec_i \langle (s^*_{-i}, s_i)\rangle$. It means that, in an NE s^*, no player has interest to deviate alone from his current choice of strategy (because no such possible deviation is profitable to him). On the other hand, a strategy profile s^* is a **Subgame Perfect Equilibrium** (SPE for short) if, for all players $i \in N$, for all strategies s_i of player i, for all nonterminal nodes $h \in H_i$ of player i: $\langle (s^*_{-i}|_h, s^*_i|_h)\rangle \not\prec_i \langle (s^*_{-i}|_h, s_i|_h)\rangle$. In other words, s^* is an NE in every subgame of G.

For example, in the game in Figure 1, the only NE is ll. Moreover, it is also an SPE because l is an NE in the only subgame of G. If we consider The same game with following preferences for player 2 : $x \prec_2 y \prec_2 z$, then ll is still a NE, but not a SPE. The only SPE of this game is rr (which is also a NE).

Dynamics: general definitions Let us now introduce the central notion of the work, i.e. dynamics in extensive form games. As explained in the introduction, we want to study the behaviour of the players when they are allowed to repeatedly update their current strategy in a strategy profile, and characterise the cases where such repeated updates (i.e. dynamics) converge to one of the equilibria we have highlighted above. More specifically, we want to characterise when a dynamics terminates for a game G, i.e., when players cannot infinitely often update their strategy.

Formally, for a sequential game G, a **dynamics** $\underset{G}{\rightharpoonup}$ is a binary relation over $Strat_G \times Strat_G$ (where, as usual, we write $s \underset{G}{\rightharpoonup} s'$ whenever $s, s' \in Strat_G$, $(s, s') \in \underset{G}{\rightharpoonup}$). Intuitively, $s \underset{G}{\rightharpoonup} s'$ means that the dynamics $\underset{G}{\rightharpoonup}$ under consideration allows the strategy profile s to be updated into s', by the change of strategy of 0, 1 or several players. When the context is clear, we drop the name of the game G and write $\rightharpoonup$ instead of $\underset{G}{\rightharpoonup}$. Given this definition, it is clear that a dynamics $\rightharpoonup$ corresponds to a directed graph $(Strat_G, \rightharpoonup)$, where $Strat_G$ is the set of graph nodes, and $\rightharpoonup$ is its set of edges. For example, the graphs in Figure 2 represent some associated graph for the game in Figure 1. Then, we say that a dynamics $\rightharpoonup$ over a sequential game G **terminates** if there is no infinite sequence of strategy profiles $(s^k)_{k\in\mathbb{N}}$ such that $\forall k \in \mathbb{N}$, $s^k \rightharpoonup s^{k+1}$. Equivalently, $\rightharpoonup$ terminates iff its associated graph is acyclic. Intuitively, a dynamics terminates if players can not update their strategy infinitely often, which means that the game will eventually reach stability. We say that a strategy profile s is **terminal** iff there is not s' s.t. $s \rightharpoonup s'$ (i.e., s is a deadlock in the associated graph). Finally, given a pair of strategies s and s', we let $H(s,s') = \{h \in H | s(h) \neq s'(h)\}$ (resp. $H_i(s,s') = \{h \in H_i | s(h) \neq s'(h)\}$) the set of nodes (resp. the set of nodes belonging to player i) where the player who owns the node plays differently according to s and s'.

Let us now identify peculiar families of dynamics (of whom we want to characterise the terminal profiles), by characterising how players have update their strategies from one profile to another.

Definition 2 (Properties of strategy updates)**.** *Let s, s' be two strategy profiles of a game G. Then:*

1. *(s,s') verifies the **Improvement Property** (written $(s,s') \models I$) if $\forall i \in N$: $s_i \neq s'_i$ implies $\langle s\rangle \prec_i \langle s'\rangle$. That is, every player that changes his strategy improves his payoff.*
2. *(s,s') verifies the **Subgame Improvement Property**, written $(s,s') \models SI$, if $\forall i \in N$, $\forall h \in H_i(s,s')$: $\langle s|_h\rangle \prec_i \langle s'|_h\rangle$. That is, every player that changes his strategy improves his (induced) payoff in all the subgames rooted at one of the changes.*
3. *(s,s') verifies the **Laziness Property**, written $(s,s') \models L$, if $\forall h \in H(s,s')$, h lies along the play induced by s'. Intuitively, we consider such updates as* lazy *because we require that the players do not change their strategy in nodes which does not influence the payoff.*
4. *(s,s') verifies the **One Player Property**, written $(s,s') \models 1P$, if $\exists i \in N$ such that $\forall j \neq i$, $s_j = s'_j$. That is, at most one player updates its strategy (but he can perform changes in as many nodes as he wants).*
5. *(s,s') verifies the **Atomicity Property**, written $(s,s') \models A$, if $\exists h^* \in H$ such that $\forall h \neq h^*$, $s(h) = s'(h)$. That is, the change affects at most one node of the tree.*

Note that, except for the first two properties, those requirements do not depend on the outcome. We argue that the first three properties (Improvement, Subgame Improvement and Laziness) correspond to some kind of rational behaviours of the players, who seek to improve the outcome of the game, while performing a minimal amount of changes. On the other hand, the One Player is relevant because updates satisfying this property disallow players from forming coalitions to improve their outcomes. Finally, the Atomicity Property is interesting *per se* because it corresponds to some kind of *minimal update*, where only one choice of only one player can be changed at a time. Because of that, dynamics respecting this property will be useful in the rest of the papers to establish general results on Dynamics.

Based on these properties, we can now define the dynamics that we will consider in this paper. For all $x \in \{I, SI, L, 1P, A\}$, we define the x-Dynamics as the set of all pairs (s,s') s.t. $(s,s') \models x$. Intuitively, the x-Dynamics is the most permissive dynamics where the players update their strategies respecting x. For a set $X \subseteq \{I, SI, L, 1P, A\}$, we define the X-dynamics as the intersection of all the x-dynamics for $x \in X$. Throughout the paper, we denote by $\overset{X}{\rightharpoonup}$ the X-Dynamics.

Observe that, following Definition 2, any update satisfying the Atomicity Property also satisfy the One Player Property. However, no such implication exist in general between the other properties:

Lemma 3. *For s, s', two strategies of a game G: $(s,s') \models A$ implies $(s,s') \models 1P$.*

Moreover, for all $x, y \in \{I, SI, L, 1P, A\}$ s.t. $(x,y) \neq (A, 1P)$, there exists a pair of strategies s and s' of some game G s.t.: $(s,s') \models x$ does not imply $(s,s') \models y$.

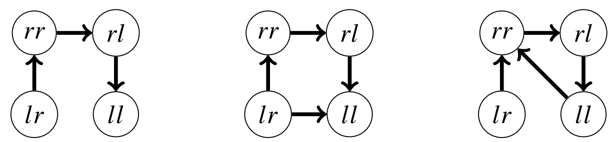


Figure 2: The graphs corresponding to the $\{I,L,1P\}$-Dynamics (left), $\{SI,A\}$-Dynamics (middle) and $\{I,L\}$-Dynamics (right), respectively, for the game in Figure 1

Proof. The first point follows from Definition 2. Indeed, since only one move can be made between s and s' (if $(s,s') \models A$), then only one player can have updated his strategy.

Let us give a counter example for the lack of implication between I and SI (the other cases follow immediately from Definition 2). Let us consider the game in Figure 1. Then, we claim that $(ll,rr) \models I$ but $(ll,rr) \not\models SI$. Indeed, for both players, $\langle ll \rangle = x \prec \langle rr \rangle = z$, but $\langle ll|_r \rangle = y \not\prec_2 \langle rr|_r \rangle = z$. Then $(s,s') \models I \not\Rightarrow (s,s') \models SI$. On the other hand, $(lr,ll) \models SI$ but $\not\models I$, because $\langle lr \rangle = x = \langle ll \rangle$. Then $(s,s') \models SI \not\Rightarrow (s,s') \models I$. □

As a consequence, we obtain direct inclusions between some dynamics. For instance $\xrightarrow{A} \subseteq \xrightarrow{1P}$ in all games. In [3], Le Roux and Pauly consider the so-called Lazy Improvement Dynamics which corresponds to our $\{I,L,1P\}$-Dynamics. The underlying idea is to disallow players from making changes in nodes that are irrelevant (because they will not appear along the play generated by the profile), while ensuring that the payoff improves. In [3], Le Roux and Pauly prove that this dynamics terminates for all games that do not have cyclic preferences. Moreover, the terminal profiles are exactly the Nash Equilibria.

Examples of graphs associated with dynamics of particular interest (for the game in Figure 1) are displayed in Figure 2: the $\{I,L,1P\}$-Dynamics (or Lazy Improvement Dynamics) on the left; the $\{SI,A\}$-Dynamics in the middle (which will be particularly relevant to the discussion in Section 3); and the $\{I,L\}$-Dynamics on the right (which will be particularly relevant in Section 4).

The rest of the paper will be structured as follows: in Section 3, we will consider dynamics which are subsets of the SI-Dynamics (like the $\{SI,A\}$-Dynamics). In Section 4, we will consider dynamics which are subsets of the I-Dynamics, in order to complete the results obtained by Leroux and Pauly in [3].

3 Subgame Improvement Dynamics

In this section we will focus on dynamics that respect the *Subgame Improvement Property* (see Definition 2), i.e., dynamics which are subsets of the SI-Dynamics (note that these dynamics have not been considered at all in [3]). More precisely, we will consider all the X-Dynamics s.t.$\{SI\} \subseteq X \subseteq \{SI,A,1P\}$. Let us notice that we do not consider here the L property, because we argue that there is little interest in the $\{SI,L\}$-Dynamics. Indeed, let us consider the game in Figure 1, with the following preferences instead: $y \prec_1 x \prec_1 z$ and $y \prec_2 z \prec_2 x$. Then, we can update ll into rr with the $\{SI,L\}$-Dynamics. Observe that in this update, both players update their strategy and thus agree to form a coalition to perform it. However, this update is not profitable for player 2 who obtains a worse outcome: z instead of x. This example shows that the $\{SI,L\}$-Dynamics yields strategy updates that are not rational.

The central result of this section is that all those dynamics terminate when the preferences of the players are acyclic[1] and converge to subgame perfect equilibria, as stated in the follow theorem:

Theorem 4. *Let $N, O, (\prec_i)_{i\in N}$ be respectively set of players, set outcomes and preferences, and let X be s.t. $SI \in X$. Then, the two following statements are equivalent:*

1. *In* all *games built over $N, O, (\prec_i)_{i\in N}$ the X-Dynamics terminates;*
2. *The preferences $(\prec_i)_{i\in N}$ are acyclic.*

Moreover, for all $\{SI\} \subseteq X \subseteq \{SI, 1P, A\}$, the set of terminal profiles of the X-Dynamics is the set of SPEs of the game.

An important consequence of this theorem is *sufficient condition* to ensure termination (on SPEs) of one of those dynamics in a game G, i.e. that the preferences of G are acyclic. Observe that this condition is very weak because it constrains only the preferences, and not the structure (tree) of the game. We argue that it is also very reasonable, as a cyclic preference relation seems to make little sense in practice. This condition is, however, not necessary: Theorem 4 tells us that when the preferences are cyclic, the dynamics does not terminate in *all* games. Actually, one can find examples of games with cyclic preferences where the dynamics still terminate, and examples where they do not. If we consider the game in Figure 1, such that preferences of player 1 are $x \prec_1 y \prec_1 z \prec_1 y$ (and player 2 as before), then the dynamics will terminate. However, if we decide now that player 2 has also cyclic preferences : $x \prec_2 y \prec_2 z \prec_2 y$, in this case player 2 will infinitely often change his strategy, because he improves it by changing from y to z *and* from z to y.

This section will be mainly devoted to proving Theorem 4. Our proof strategy works as follows. First of all, we establish termination of the $\{SI\}$-Dynamics when the preferences are acyclic (Proposition 5). This guarantees that all the X-Dynamics terminate in the same case when $SI \in X$, since all these dynamics are more restrictive. Second, we show that all SPEs appear as terminal profiles of the $\{SI\}$-Dynamics (without guaranteeing, at that point that all terminal profiles are SPEs). Finally, we show that all the terminal profiles of the $\{SI, A\}$-Dynamics are SPEs. We then argue, using specific properties of this dynamics, and relying on Definition 2, that this implies that the set of terminal profiles of all the dynamics we consider coincide with the set of SPEs.

The $\{SI\}$-Dynamics As announced, we start by two properties of the $\{SI\}$-Dynamics. The first proposition states that, for a fixed set of preferences, the $\{SI\}$-Dynamics terminates in all games built on these preferences iff the preferences are acyclic. The second shows that, in these cases, the SPEs are contained in the terminal profiles of the $\{SI\}$-Dynamics.

Proposition 5. *Let $N, O, (\prec_i)_{i\in N}$ be respectively set of players, set outcomes and preferences. Then, the two following statements are equivalent: (1) In all games G built over $N, O, (\prec_i)_{i\in N}$, the $\{SI\}$-Dynamics terminates; (2) The preferences $(\prec_i)_{i\in N}$ are acyclic*

Sketch of proof. Given a cyclic preference, we build a one player game where all the outcomes can be reached from the root (in one step). Since the preference is cyclic, we clearly have that $\{SI\}$-Dynamics does not terminate.

On the other hand, to prove that when preferences are acyclic, the dynamics terminates in all games, we make an induction over the number of nodes of the game. We consider a game G with $n+1$ nodes, and in which we have an infinite sequence of strategy profiles such that $s^1 \xrightarrow{SI} s^2 \xrightarrow{SI} s^3 \xrightarrow{SI} \dots$ Then, we

[1] Actually, as we show at the end of the section, in the presence of players who have cyclic preferences and play lazily, the players who have acyclic preferences are still guaranteed to perform a finite number of updates only.

consider a node $h^* \in H_i$, such that every successor of h^* is in Z. By the Subgame Improvement Property, if player i changes in this place from x^1 to x^2, he will never come back to x^1 (as preferences are acyclic and he has to improves his payoff at this place). Moreover, as the number of successors of h^* is finite, we can consider that, at after a finite number of steps in the $s^1 \xrightarrow{SI} s^2 \xrightarrow{SI} s^3 \xrightarrow{SI} \ldots$ sequence, player i will not change anymore in node h^*, and definitely choose some successor x of h^*. Then we can replace h^* by this successor x. We are then in a game with n nodes, in which the dynamics terminates by induction hypothesis. The key ingredient is to prove that the dynamics coincide in the two games. □

Proposition 6. *Let G be a sequential game. Then, all SPEs of G are terminal profiles of* $\xrightarrow{SI}$.

Sketch of proof. The property follows directly from the Subgame Improvement Property, and the definition of SPEs. Indeed, the SI Property requires to improves the outcome in the subgame where players change; and SPEs require the players to choose the best response in all subgames. Then, when a profile is an SPE, it cannot be updated without violating SI. □

The $\{SI, A\}$-Dynamics We now turn our attention to the $\{SI, A\}$-Dynamics, and show that all its terminal profiles are necessarily SPEs. As announced, this result will be sufficient to establish that the other dynamics we consider here terminate on SPEs too.

Proposition 7. *Let G be a sequential game. Then, all terminal profiles of* $\xrightarrow{SI,A}$ *are SPEs of G.*

Sketch of proof. The proof is by contradiction: consider a strategy profile s which is not an SPE. Then, there is a node where a player does not make the best choice for his payoff. He can thus update his choice in this node, and this unique update will improve the outcome, it is thus allowed in the $\{SI, A\}$-Dynamics. Thus, all profiles which are not SPEs are not terminal for this dynamics. □

Although this will not serve to prove Theorem 4, we highlight now an interesting property of the $\{SI, A\}$-Dynamics: in some sense, it weakly simulates the $\{SI\}$-Dynamics, in the sense that, in all games, every update of the $\{SI\}$-Dynamics can be split into a sequence of updates of $\{SI, A\}$-Dynamics.

Lemma 8. *For all sequential games G, for all* (s, s') *s.t.* $s \xrightarrow{SI} s'$*: there are* $s^1, \ldots, s^k$ *s.t.* $s \xrightarrow{SI,A} s^1 \xrightarrow{SI,A} \ldots \xrightarrow{SI,A} s^k \xrightarrow{SI,A} s'$.

Sketch of proof. The proof is by induction over $|H(s, s')|$, the number of changes between s and s'. First of all, we prove that there is $h^* \in H(s, s')$ such that nothing changes in $s'|_{h^*}$, and there is no node higher than h^* where we leave the path towards h^* between s and s'. Formally, if $h^* = a^1 \cdots a^k$, $s|_{s'(h^*)} = s'|_{s'(h^*)}$ and $\nexists h' \in H(s, s')$, $h' = a^1 \ldots a^\ell$ $(1 < \ell < k)$, such that $s(h) = a^{\ell+1}$ and $\forall \ell < \ell' < k, s(a^1 \ldots a^{\ell'}) = a^{\ell'+1}$. The point of the existence of such a h^* is that, for s^1 such that $s^1(h^*) = s'(h^*)$ and for $h \neq h^*, s^1(h) = s'(h)$, we have $s \xrightarrow{SI,A} s^1$, and $(s^1, s') \models SI$. Then, as $|H(s^1, s')| = |H(s, s')| - 1$, we conclude by ind. hyp. □

Proof of Theorem 4 Equipped with these three propositions, we can now prove our theorem:

Proof of Theorem 4. Let us consider a set of players N, a set of outcomes O, preferences $(\prec_i)_{i \in N}$ and X s.t. $SI \in X$. The X-Dynamics terminates for every game G built over $N, O, (\prec_i)_{i \in N}$ if and only if the preferences are acyclic by Proposition 5, because $\xrightarrow{X} \subseteq \xrightarrow{SI}$ by definition.

Next, let us consider X s.t. $\{SI\} \subseteq X \subseteq \{SI, A, 1P\}$. By definition, and using the fact that Property A implies Property $1P$ (see Lemma 3), we have: $\xrightarrow{SI,A} \subseteq \xrightarrow{X} \subseteq \xrightarrow{SI}$. Let s be a terminal profile of $\xrightarrow{X}$. Then, it is also terminal in $\xrightarrow{SI,A}$ since $\xrightarrow{SI,A} \subseteq \xrightarrow{X}$. By proposition 7, s is thus an SPE of G. On the other hand, let s' be an SPE of G. Then by Proposition 6, s' is a terminal node of $\xrightarrow{SI}$. Since $\xrightarrow{X} \subseteq \xrightarrow{SI}$, s' is also a terminal node of $\xrightarrow{X}$. Thus, we have shown that all SPEs of G are terminal nodes of G and vice-versa. □

Termination in the presence of irrational players We close this section by answering the following question: 'what happens when some players have cyclic preferences and some have not?' We call irrational the players who have cyclic preferences and show that, although their presence is sufficient to prevent termination of the whole dynamics, players with acyclic preferences can still be guaranteed a bounded number of updates in their choices, provided that the irrational players play lazily. Thus, in this case, any infinite sequence of updates will eventually be made up of updates from the irrational players only. This provides some robustness to our termination result.

Let $N, O, (\prec_i)_{i \in N}$ containing irrational players. We denote by N_r the set of rational players (who have acyclic preferences), and N_c the set of irrational players (with cyclic preferences). Let us consider the dynamics $\rightsquigarrow$ such that $s \rightsquigarrow s'$ iff: (i) either $H(s,s') \cap N_c = \emptyset$ and $s \xrightarrow{SI} s'$; (ii) or $H(s,s') \cap N_c \neq \emptyset$ and $s \xrightarrow{\{I,L,1P\}} s'$. It means that the rational players play according to $\{SI\}$-Dynamics, while the irrational players have to play according to $\{I,L,1P\}$-Dynamics. In this case, we say that the dynamics terminates for rational players if there does exists an infinite sequence of strategy profiles $(s^k)_{k \in \mathbb{N}}$ such that: (1) $\forall k \in \mathbb{N}, s^k \rightsquigarrow s^{k+1}$ and (2) $\forall j, \exists k > j$ s.t.$H(s^k, s^{k+1}) \cap N_r \neq \emptyset$.

Moreover, let us notice that, if we allow irrational players to play with the $\{SI\}$-Dynamics, the result does not hold. Indeed, if we consider the game in Figure 1, and consider player 2 as the irrational one, the graph associated to the $\{SI\}$-Dynamics is represented in Figure 3 (left), where dotted lines represent moves of the irrational player. Clearly, this graph contains a cycle in which rational player changes infinitely many times.

Proposition 9. *Let $N, O, (\prec_i)_{i \in N}$ such that $N = N_r \cup N_c$ as before. Then, the dynamics $\rightsquigarrow$ terminates for rational players in all games built over $N, O, (\prec_i)_{i \in N}$.*

Sketch of proof. In [3, Section 5], Le Roux and Pauly provide an alternative proof of the termination for the Lazy Improvement Dynamics (here $\{I,L,1P\}$-Dynamics). They associate a function per player which decrease when the associated player update his strategy, and does not otherwise (i.e. when other players update). The termination of the $\{I,L,1P\}$-Dynamics is then a consequence of the decrease of the functions, together with the finiteness of strategies (per player). When irrational players are added, as they do not affect the functions of rational players, we clearly have that the $\{I,L,1P\}$-Dynamics terminates for rational players.

In order to obtain the desired result, we need adapt the proof of [3, Section 5], by introducing a global function (i.e. for all rational players). This new global function has the property to decrease when rational players update their strategies. Moreover, one also shows that the latter function is not affected by the updates of the irrational players, as their strategies follows the $\{I,L,1P\}$-Dynamics. This implies that rational players do not update their strategy an infinite number of time. □

4 Improvement dynamics and coalitions

While Section 3 was devoted to characterising the X-Dynamics with $SI \in X$, we turn now attention to those where $I \in X$. Recall that Le Roux and Pauly have studied in [3] the Lazy Improvement Dynamics,

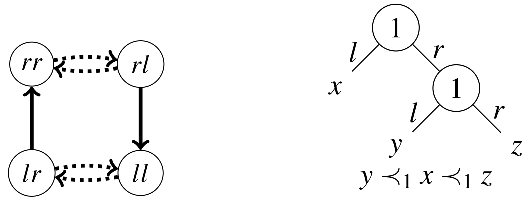


Figure 3: Associated graph for first game with irrational player (left) and Terminal profiles of $\{I,A\}$-Dynamics are not always NE (right)

Table 1: Characterisation of the X-Dynamics with $I \in X$. The 'Games' column characterises a class of games for which the result holds. The 'Termination' column gives necessary (n.) and/or sufficient (s.) conditions to ensure that the dynamics terminates on *all* games in the class. '?' means 'indifferent'.

A	L	$1P$	Games	Termination	Final Profiles	Reference
✓	?	?	acyclic prefs	$\top$	$\supseteq$ NEs	Corollary 11
×	×	×		×	not appl.	[3]
×	×	✓		×	not appl.	[3]
×	✓	×	swo prefs	prefs can be layered (s.)	= CNEs	Theorem 15
			swo prefs	prefs out of pattern (n.)		Theorem 15
			slo prefs	prefs out of pattern (n. & s.)		Corollary 16
			swo prefs, 2 player	out of pattern (n. & s.)		Corollary 16
×	✓	✓	acyclic prefs	$\top$	= NEs	[3]

which corresponds to our $\{I,L,1P\}$-Dynamics and shown that it terminates when the preferences of the players are acyclic, and terminates to Nash Equilibria. Their study of the $\{I,L,1P\}$-Dynamics was motivated by the fact that less restrictive dynamics (that are still contained in $\overset{I}{\rightharpoonup}$) do not always terminate, namely the $\{I\}$-Dynamics and the $\{I,1P\}$-Dynamics. These results appear as the grey lines in Table 1.

Our contribution in the present section is to fill in Table 1 by the following results. First, for all the X-Dynamics for $\{I,A\} \subseteq X$, we show that acyclic preferences guarantee termination, and that the final profiles contain the Nash Equilibria. Second, we consider the $\{I,L\}$-Dynamics and characterise families of games and conditions on the preferences where termination can be guaranteed (with the terminal profiles being exactly the so-called *Coalitional Nash Equilibria* that we introduce). The conditions that we present consists in identifying *patterns* in the preferences.

Observe that the $\{I,L\}$-Dynamics can be regarded as a *coalition dynamics*, where several players can change their respective strategies at the same time in order to obtain a better outcome. For example, for the game in Figure 1, players can make a coalition to change from ll to rr, because they prefer z to x.

4.1 The $\{I,A\}$-Dynamics

To complete Table 1, we consider now the first line which represents all the X-Dynamics with $\{I,A\} \subseteq X$. All these results can be grouped because adding more properties does not change the dynamics as stated by the next proposition:

Proposition 10. *All the X-Dynamics with $\{I,A\} \subseteq X \subseteq \{SI,I,A,L,1P\}$ are equal.*

Proof. To understand these equalities, we must focus on $\{I,A\}$-Dynamics. This dynamics allows only one update between two profiles, and the outcome must be better for the player that has changed his strategy. If we want that the outcome of the game changes, the atomic move must have occurred along the play induced by the strategy. Thus, $\{I,A\}$-Dynamicsverifies the Lazy Property (L). Moreover, by Lemma 3, it also verifies the One Player Property ($1P$). Finally, as the outcome is improved, and only one change has been done, in particular the payoff is improved in the subgame rooted at the change. Thus, the $\{I,A\}$-Dynamics also verifies the Subgame Improvement Property (SI). □

By Theorem 4, as $\{I,A\}$-Dynamics verifies the Subgame Improvement Property, it terminates for every game over some $N,O,(\prec_i)_{i\in N}$ if and only if the preferences are acyclic:

Corollary 11. *Let $N,O,(\prec_i)_{i\in N}$ be respectively set of players, set outcomes and preferences. Then, the two following statements are equivalent: (1) in all games G built over $N,O,(\prec_i)_{i\in N}$, the $\{I,A\}$-Dynamics terminates; (2) the preferences $(\prec_i)_{i\in N}$ are acyclic.*

Now that we have established termination, let us turn our attention to the terminal profiles. It turns out that they contain all Nash Equilibria of the game:

Proposition 12. *Let G be a sequential game. Then all NEs of G are terminal profile of $\xrightarrow{\{I,A\}}$.*

Proof. If s is an NE, then we know that no player can not improve the outcome alone from s. Then, s must be a terminal profile of $\xrightarrow{\{I,A\}}$. □

Let us notice that some non-NE profiles can also be a terminal profile of $\{I,A\}$-Dynamics. For example, in the game in Figure 3 (right), ll is a terminal profile, while it is not a NE.

4.2 The $\{I,L\}$-Dynamics

Let us now turn our attention to the $\{I,L\}$-Dynamics. As can be seen from Table 1, the conditions to ensure termination are more involved and require a finer characterisation of the preferences. We start by discussing these conditions.

Orders Let us begin with the definition of strict linear and weak order, before introducing patterns required for the termination of this dynamics.

A **strict linear order** over a set O is a total, irreflexive and transitive binary relation over O. This is a natural way to see order. For example, usual orders over $\mathbb{N}$ or $\mathbb{R}$ are strict linear orders.

A **strict weak order** over a set O is an irreflexive and transitive binary relation over O that provides the transitivity of incomparability. Formally, for $x \neq y \in O$, if $\neg(x<y)$ and $\neg(y<x)$, we say that x and y are incomparable, and we write $x \sim y$ (this can happen because a strict weak order is not a total relation). Then, in a strict weak order, for all x, y, z, we have that $x \sim y$ and $y \sim z$ implies $x \sim z$.

We write $x \lesssim y$ if $x<y$ or $x \sim y$ (i.e.. if $\neg(x<y)$. Sometimes we will denote strict weak orders by $\lesssim$ to emphasise the possibility of incomparability, but this does not means that the relation is reflexive. We argue that strict weak orders are quite natural to consider in our context. Indeed, the incomparability of two outcomes for a player reflects the indifference of the player regarding these outcomes. We can easily imagine two outcomes x and y such that the first player prefer x from y but player 2 has no preference. This kind of vision justify the transitivity of incomparability. Indeed, if a player can neither choose between x and y, nor between y and z, it would not seem natural that for example, he prefers x to z. Finally, we say that $<'$ is a **strict linear extension** of a strict weak order $\lesssim$ if it is a strict linear order and $\forall x,y \in O$: $x<y$ implies $x<' y$.

Coalitional Equilibrium The terminal profiles of the Coalitional Improvement dynamics are called the **Coalitional Equilibria**. A strategy profile s^* is a Coalitional Equilibrium if, for all coalitions of players $I \subseteq N$, for all strategies s_i of player $i \in I$: $\langle (s^*_{-I}, s^*_I) \rangle \not<_i \langle (s^*_{-I}, s_I) \rangle$. In other words, no coalition can be in which all players improve the outcome. This is a stronger notion than that of Nash Equilibrium, and we believe it is a natural way to extend NEs to coalition of player. However, there is no link between Coalitional Equilibrium and SPEs, since Coalitional Equilibrium are not concerned with subgames.

Layerability While the notion of orders we have defined above make sense in our context, they are unfortunately not sufficient to ensure termination of $\{I, L\}$-Dynamics. Indeed, considering the game in Figure 1 and his associated graph with $\{I, L\}$-Dynamics in Figure 2 (right), we can see that, even when players have strict linear preferences, the dynamics does not terminate. We thus need to introduce more restrictions on outcomes and preferences to ensure termination. In [2], Le Roux considers a pattern over outcomes and preferences, and proves that the absence of this pattern induces some structure on the outcomes that we call *layerability* (in the case of strict linear order). Layerability, in turn, can be used to prove termination of dynamics as we are about to see. Our first task is thus to generalise the pattern and the definition of layerability of outcomes in this case in the case of *strict weak orders*.

Let O be a finite set of outcomes, N be a finite set of players and $(\lesssim_i)_{i\in N}$ be strict weak orders over O. For $(\lesssim_i)_{i\in N}$ a strict weak order, we say that $(\lesssim_i)_{i\in N}$ is: (i) **out of main pattern** for O if it satisfies (1); (ii) **out of secondary pattern** for O if it satisfies (2); (2); and (iii) **out of pattern** for O if it is out of main and secondary pattern, with:

$$\forall x, y, z \in O, \forall i, j \in N : \neg(x <_i y <_i z \text{ and } y <_j z <_j x) \tag{1}$$

$$\forall w, x, y, z \in O, \forall i, j \in N : \neg(w <_i x <_i y <_i z \text{ and } x \sim_j z <_j w \sim_j y) \tag{2}$$

Let us notice that, when $(<_i)_{i\in N}$ is a strict linear order, then $(<_i)_{i\in N}$ is out of pattern for O if and only if $(<_i)_{i\in N}$ is out of main pattern for O.

Moreover, we say that $(\lesssim_i)_{i\in N}$ **can be layered** for O if there is a partition $\{O_\lambda\}_{\lambda\in I}$ of O (whose elements are called *layers*) and a strict total order $<$ on I (i.e., the layers are totally ordered) s.t.:

1. $\lambda < \mu$ implies that $\forall i \in N, \forall x \in O_\lambda, y \in O_\mu, x \lesssim_i y$; and
2. $\forall \lambda \in I, \forall i, j \in N, \forall w, x, y, z \in O_\lambda$: $\neg(x <_i y \wedge x <_j y \wedge w <_i z \wedge z <_j w)$.

The intuition between the notion of layer is as follows: Point 1 tells us that the ordering of the layers is compatible with the preference relation of *all* players. That is, if we pick x in some layer O_λ and y in some layer O_μ, with $\lambda < \mu$ (i.e. O_μ is 'better' than O_λ), then *all* players will prefer y to x. However, with this point alone in the definition, one could put all the outcomes in the same layer (i.e., the partition would be trivially $\{O\}$). Point 2 ensures that the disagreement of the players on some outcomes is also reflected in the layers. That is, in all layers O_λ, we cannot find two players i and j that agree on a pair of elements x and y from O_λ (because they both prefer y to x) but disagree on pair of elements w and y from O_λ (because player i prefers z to w but player j prefers w to z). Given this intuition, it is not surprising that the notion of layer has a strong tie with the presence of the pattern in the preferences that prevents termination. Indeed, the following result, from [2] shows that these notions are *equivalent* in the case of *strict linear orders*:

Proposition 13 ([2]). *Let O be a finite set of outcomes, N a finite set of players and $(<_i)_{i\in N}$ strict linear orders. Then $(<_i)_{i\in N}$ is out of pattern for O if and only if $(<_i)_{i\in N}$ can be layered for O.*

As explained above, we seek to extend this result to strict weak orders, as this will be ancillary to establishing termination. Unfortunately, the result does not hold immediately in this case, as shown by the following example. Consider the followings preferences : $N = \{1,2,3\}$, $O = \{x,y,z\}$ and $x <_1 y <_1 z, y <_2 z \sim_2 x$ and $y \sim_3 z <_3 x$. These preferences are out of pattern (they satisfy both (1) and (2)), but they cannot be layered. However, if we restrict ourselves to two players, the absence of our main and secondary pattern is sufficient to ensure that the preferences can be layered, in the case of strict weak orders, as shown in the next Proposition. For the sake of clarity, let us denote $(\lesssim_i)_{i\in\{1,2\}}$ (resp. $(<_i)_{i\in\{1,2\}}$). by $\lesssim_{1,2}$ (resp. $<_{1,2}$). Then:

Proposition 14. *Let O be a finite set of outcomes, $N = \{1,2\}$ and $\lesssim_1, \lesssim_2$ strict weak orders. The following statements are equivalent: (1) $\lesssim_{1,2}$ is out of pattern for O; (2) there exists $<'_{1,2}$, a strict linear extension of $\lesssim_{1,2}$ that can be layered for O; (3) $\lesssim_{1,2}$ can be layered for O.*

Termination of the $\{I,L\}$-Dynamics Equipped with these preliminary results, we can now characterise termination of the $\{I,L\}$-Dynamics, as shown in Table 1. As we have seen with the example in Figure 1, the termination of the $\{I,L\}$-Dynamics is not as simple as the termination of $\{I,L,1P\}$-Dynamics or $\{SI,A\}$-Dynamics. When we consider that the preferences of the game are strict weak orders, we will see that 'can be layered' is a necessary condition, and 'out of pattern' is a sufficient condition. However, none of these characterisation are sufficient and necessary condition.

Theorem 15. *Let O be a finite set of outcomes, N be a finite set of players and $(\lesssim_i)_{i\in N}$ be strict weak orders. We have the following implications: $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$, $2 \not\Rightarrow 1$ and $4 \not\Rightarrow 3$ where: (1) $(\lesssim_i)_{i\in N}$ can be layered for O; (2) the $\{I,L\}$-Dynamics terminates in all game built over O, N and $(\lesssim_i)_{i\in N}$; (3) All games built over O, N and $(\lesssim_i)_{i\in N}$ admit a Coalitional Equilibrium; (4) $(\lesssim_i)_{i\in N}$ is out of pattern for O.*

Sketch of proof. $\boxed{1 \Rightarrow 2}$ For this point, the idea consists in reducing to a two player games where both players play lazily and do not form coalitions. This is the point of the proof where we exploit heavily the properties of layers: as $(\lesssim_i)_{i\in N}$ can be layered for O, we know that no player or coalition of players will make a change in order to reach an outcome which is in a lower layer than the current outcome. Indeed, by definition of layers, every player prefers every outcome of an upper layer to any outcome of a lower layer. Then, we can consider that, at some point along the dynamics, we will reach a layer and never leave it, because the game is finite. Let us denote it by O_λ.

Moreover, in that layer, we can make two teams of players. Those who agree with player 1, and the others. Indeed, inside a layer, we have $\neg(x <_i y \wedge x <_j y \wedge w <_i z \wedge z <_j w)$ for all $i, j \in N$ and for all $w,x,y,z \in O_\lambda$. We can then regard the first team as a single player and build a game G' with two players that never make coalitions, and play lazily. By [3], we know that this dynamics terminates for the game G' and conclude that the $\{I,L\}$-Dynamics terminates for the game G.

$\boxed{2 \Rightarrow 3}$ We know that the terminal profiles are the Coalitional Equilibria, by definition. Thus, if the dynamics terminates, the exists a Coalitional Equilibrium in the game.

$\boxed{3 \Rightarrow 4}$ If O is not out of the main pattern, let us consider the game in Figure 1. His associated graph, given by Figure 2 (right), has no terminal node. It means that there is no Coalitional Equilibrium. In the case where O is not out the secondary pattern, we consider the game and his associated graph in Figure 4 (left). The graph has no terminal node, so the game has no Coalitional Equilibrium.

$\boxed{2 \not\Rightarrow 1}$ A counter-example to this implication goes as follows. Consider $O = \{x,y,z\}$, $N = \{1,2,3\}$ and $\lesssim$ s.t. $y <_1 x \sim_1 z$, $z <_2 x <_2 y$ and $x <_3 z <_3 y$. Clearly, these preferences cannot be layered, but we claim that, in all games built over O, N and $\lesssim_{1,2,3}$, the $\{I,L\}$-Dynamics terminates. Indeed, observe that, as player 1 can not make any coalition, he will update his strategy only a finite number of times.

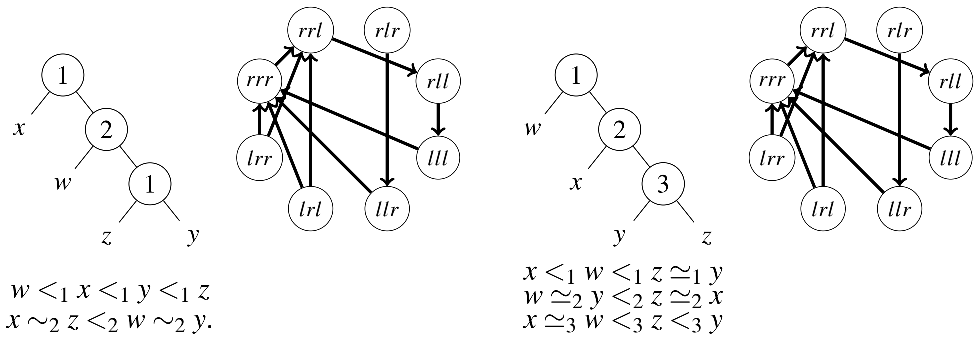


Figure 4: Two counter-examples with their associated graph

After that, either player 2 and player 3 will form a coalition in order to reach y, then will stop updating their strategy changing; or they will not. In this case, we can consider that they update according to $\{I,L,1P\}$-Dynamics. Then, by [3, Theorem 10], this dynamics terminate.

$\boxed{4 \not\Rightarrow 3}$ The game in Figure 4 (right) is an example where the preferences are out of pattern and where the graph associated to the $\{I,L\}$-Dynamics has no terminal node, hence the game admits no Coalitional Equilibrium. $\square$

Let us now finish by considering our two particular cases, in which we know that the 'layerability' of the order is *equivalent* to the absence of pattern, namely when preferences are strict linear order (Proposition 13), in two player games (Proposition 13). Then:

Corollary 16. *Let O be a finite set of outcomes, N be a finite set of players and $(<_i)_{i\in N}$ be preferences. When* either $N = \{1,2\}$ *and $\lesssim_1, \lesssim_2$ are strict weak orders;* or $(<_i)_{i\in N}$ *are strict linear orders, the following are equivalent: (1) The $\{I,L\}$-Dynamics terminates in all games built over O, N and $(<_i)_{i\in N}$; (2) All game built over O, N and $(<_i)_{i\in N}$ admit a Coalitional Equilibrium; (3) $(<_i)_{i\in N}$ is out of pattern for O.*

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