

Mixing Probabilistic and non-Probabilistic Objectives in Markov Decision Processes

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Abstract

In this paper, we consider algorithms to decide the existence of strategies in MDPs for Boolean combinations of objectives. These objectives are omega-regular properties that need to be enforced either *surely*, *almost surely*, *existentially*, or with *non-zero probability*. In this setting, relevant strategies are *randomized infinite memory strategies*: both infinite memory and randomization may be needed to play optimally. We provide algorithms to solve the general case of Boolean combinations and we also investigate relevant subcases. We further report on complexity bounds for these problems.

CCS Concepts: • Software and its engineering → General programming languages; • Social and professional topics → History of programming languages.

Keywords: Markov Decision Processes, synthesis, omega-regular

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1 Introduction

Recently, there have been several works on how to mix the semantics of games and Markov decision processes [1, 4, 10, 11, 15]. This setting provides means to model the interaction between a system and its environment that is uncontrollable but obeys stochastic dynamics. The setting is then used to reason on strategies of the system that ensure for example some properties with *certainty* and others with *high probability*.

Here, we extend this line of work by studying a general setting where objectives for the system are Boolean combinations of atoms. These atoms are omega-regular properties, expressed as parity conditions, that need to be ensured either *surely* (A), *almost surely* (AS), *existentially* (E), or with *non-zero probability* (NZ). Sure (A) and existential (E) atoms are *non-probabilistic* while almost sure (AS) and non-zero (NZ) atoms are *probabilistic*. The coexistence of atoms of both types that need to be satisfied by a unique strategy makes this problem out of reach of classical techniques used to solve MDPs with CTL objectives or with PCTL objectives for example.

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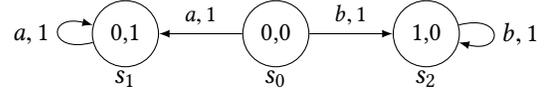


Figure 1. The MDP depicted here has 3 states and two parity functions p_1 and p_2 . The numbers assigned by the parity functions to the states are depicted by the integers inside the states. A parity condition is enforced if the maximum value of states that appear infinitely often is even. By taking a from s_0 , state s_1 is reached with probability one, and by taking b , s_2 is reached with probability one. Clearly on this example, a randomized strategy is needed to win for $NZ(p_1) \wedge NZ(p_2)$ from state s_0 .

Infinite memory and randomization. In some previous works on models that mix games and MDPs [1, 4, 10, 11], In [16], combination of parity conditions are studied. In that paper, randomization is necessary, but not infinite memory. In the setting that is considered in the current paper, relevant strategies for the systems are *randomized infinite memory strategies*: both infinite memory and randomization may be needed to play optimally. This implies that the techniques used here are more complex than for the previous works. Note that randomization is already necessary when considering conjunctions of two NZ atoms. The example we give above in Figure 1 is encompassed by the formalism of [16], and shows why we need to add randomization when compared to the work in [4]. In the MDP of Figure 1, there does not exist a deterministic choice from state s_0 between action a and action b that ensures $NZ(p_1) \wedge NZ(p_2)$, while a randomized strategy can enforce this objective by taking a with probability α ($0 < \alpha < 1$) and taking b with probability $1 - \alpha$.

Main contributions. Our main contributions are summarized in Table 1. We provide a Σ_2^P algorithm to decide the existence of a strategy to enforce a Boolean combination of atomic objectives. We also show that this problem is both NP and coNP hard. Then we provide additional results for relevant subclasses of Boolean combinations. For the conjunctive case, we prove the existence of a polynomial algorithm that uses an NP oracle while the problem is shown to be coNP hard. For conjunctions that contain only one sure atom (1A) and a number of other atoms, the complexity goes

	Hardness	Membership
$\wedge(\text{AS}, \text{NZ}, \text{E})$	P	P (Thm. 7.2)
$\wedge(1\text{A}, \text{AS}, \text{NZ}, \text{E})$	parity	$\text{NP} \cap \text{coNP}$ (Thm. 7.4)
$\wedge(\text{A}, \text{AS}, \text{NZ}, \text{E})$	coNP	$\text{P}^{\text{NP}} (= \Delta_2^{\text{P}})$ (Thm. 7.1)
$\mathbb{B}(\text{A}, \text{AS}, \text{NZ}, \text{E})$	NP and coNP (Thm. 7.6)	$\text{NP}^{\text{NP}} (= \Sigma_2^{\text{P}})$ (Thm. 7.5)

Table 1. Table of the main complexity results.

down to $\text{NP} \cap \text{coNP}$ and it is at least as hard as solving parity games. The complexity of this algorithm is dominated by the complexity of solving parity games. A polynomial time solution to parity games would lead to a polynomial time solution for our problem. Also the recent quasi-polynomial time solutions for parity games, see e.g. [12], can be used to obtain a quasi-polynomial time solution to our problem. Finally, for conjunctions that do not contain sure atoms, the problem can be solved in polynomial time.

Related works. Logical formalisms to express properties of transition systems and Markov decision processes were plentifully studied in the literature. But most of the results in the literature only consider either logics based on *non-probabilistic* atoms, e.g. CTL, or logics based on *probabilistic* atoms only, e.g. PCTL. The logic PCTL is used to express constraints on the probability of events that are temporal properties of paths. In [9], the strategy synthesis problem for MDPs with PCTL objectives is studied. The full logic, i.e. with arbitrary probabilistic thresholds, is undecidable but the qualitative fragment of the logic (thresholds 0 and 1, corresponding to NZ and AS in our setting) is decidable in EXPTIME. This high complexity is due to the succinctness of PCTL. As PCTL cannot express our non-probabilistic atoms, the two formalisms have incomparable expressive power. Settings that mix both non-probabilistic properties, such as A or E, with probabilistic ones such as AS or NZ, are more recent. We make now a more detailed review of the recent relevant works in that direction.

In [10, 11], MDPs with mean-payoff and shortest path objectives are considered. This work was, to the best of our knowledge, the first work to consider the synthesis of strategies that optimize an *expectation* (a probabilistic property) while satisfying a long-run *worst-case objective* (a non-probability objective). Similarly, the authors of [1] consider the synthesis of strategies that ensure a parity condition *surely* and at the same time an ϵ -optimal *expected* mean-payoff. Those works introduce refinements of the notion of *end-components* that we need to further refine here.

The authors of [18] study an extension of MSO, called $\text{MSO}+\nabla$ which uses a probabilistic second order quantifier.

The logic $\text{MSO}+\nabla$ is expressive enough to encode the problem we study here, but this logic has been proved to be undecidable [3, 7]. In [5, 6], a fragment of $\text{MSO}+\nabla$, called thin MSO has been introduced. The logic Thin MSO is expressive enough to encode the model-checking problem of the qualitative fragment $\text{CTL}^* + \text{PCTL}^*$ (union of CTL^* and PCTL^*) over Markov chains. Their algorithm has non-elementary complexity. The algorithm was recently improved in [17] where a model-checking algorithm with $3\text{NEXPTIME} \cap \text{co-}3\text{NEXPTIME}$ complexity is proposed. The works in [5, 17] do not consider the richer model of Markov decision processes as we do here.

In [13], the authors study qualitative tree automata, that is automata with a probabilistic acceptance condition. The non-emptiness problem of nondeterministic tree automaton with such acceptance condition has been proved decidable, but the problem has been proved undecidable for nondeterministic tree automaton with such acceptance condition [3]. There is a deep connection between tree automata and Markov Decision Processes, as the existence of a strategy on an MDP corresponds to deciding the non-emptiness of a qualitative tree automaton with unary alphabet.

In [6, 19], the authors study *subzero* automata: a class of tree automata with an acceptance condition that mixes the classical Rabin acceptance condition with probabilistic constraints. The problem of determining if a subzero automaton accepts some regular tree is decidable. This class of automaton can in turn be used to solve synthesis problem for *finite-memory* strategies (that are equivalent to regular trees) that enforce a first parity condition p_1 surely (A) and a second parity condition p_2 almost-surely (AS). Our work consider more general properties (both E and NZ in addition to A and AS, and their Boolean combinations) and more general strategies: randomized infinite memory strategies, and not only finite memory deterministic strategies (regular trees).

In this paper, we provide non-trivial extensions of results in [4] where only the case of one sure parity objective (1A) and one almost-sure parity objective (1AS) is considered. An $\text{NP} \cap \text{coNP}$ algorithm is provided there for this special case. In this paper, in addition to a Σ_2^{P} algorithm for the general case $\mathbb{B}(\text{A}, \text{AS}, \text{NZ}, \text{E})$, we also provide an algorithm that solves conjunctions of one sure parity objective (1A) and any number of almost-sure (AS), existential (E), and non-zero probability (NP) parity objectives with the same worst-case complexity as that of [4]. Algorithms in [4] heavily rely on notions of very good end-components (VGEC) and ultra good end-components (UGEC). Here, we need generalization of VGEC and UGEC, and additional technical results to build algorithms for our more general setting.

Finally, the authors of [15] consider the synthesis of *finite-memory strategies* for MDPs with a sure parity (S) and an almost-sure parity (AS) objectives. The restriction to finite memory strategies leads to simpler algorithms but the complexity is similar, i.e. $\text{NP} \cap \text{coNP}$. The authors of [15] also

consider the case of $2\frac{1}{2}$ -player games. In that setting the problem is coNP-complete.

Structure of the paper. In Section 2, we introduce necessary preliminaries about MDPs, and we formally define the class of properties that we consider, i.e. Boolean combinations of A, AS, E, and NZ atoms. In Sections 3, 4 and 5, we study notions of end-components that are the main technical ingredients of our algorithms. Section 6 introduces additional techniques needed to handle E and NZ atoms. In Section 7, we study the complexity of algorithms for the general case, and several relevant fragments.

Due to lack of space, full proofs are provided in

2 Preliminaries

For $k \in \mathbb{N}$, we denote by $[k]_0$ and $[k]$ the set of natural numbers $\{0, \dots, k\}$ and $\{1, \dots, k\}$ respectively. Given a finite set A , a (rational) *probability distribution* over A is a function $\text{Pr}: A \rightarrow [0, 1] \cap \mathbb{Q}$ such that $\sum_{a \in A} \text{Pr}(a) = 1$. We denote the set of probability distributions on A by $\mathcal{D}(A)$. The *support* of the probability distribution Pr on A is $\text{Supp}(\text{Pr}) = \{a \in A \mid \text{Pr}(a) > 0\}$.

Markov chain. We denote by \mathbb{N} the set $\{1, 2, \dots\}$, and by \mathbb{N}_0 the set $\mathbb{N} \cup \{0\}$. A *Markov chain* (MC, for short) is a tuple $\mathcal{M} = (S, E, \text{Pr})$, where S is a set of states, $E \subseteq S \times S$ is a set of edges (we assume in this paper that the set $E(s)$ of outgoing edges from s is nonempty and finite for all $s \in S$), and $\text{Pr}: S \rightarrow \mathcal{D}(E)$ assigns a probability distribution – on the set $E(s)$ of outgoing edges from s – to all states $s \in S$. In the following, $\text{Pr}(s, (s, s'))$ is denoted $\text{Pr}(s, s')$, for all $s \in S$. The Markov chain \mathcal{M} is *finite* if S is finite.

For $s \in S$, the set of infinite paths in \mathcal{M} starting from s is $\text{Paths}^{\mathcal{M}}(s) = \{\pi = s_0 s_1 \dots \in S^{\omega} \mid s_0 = s, \forall n \in \mathbb{N}_0, \text{Pr}(s_n, s_{n+1}) > 0\}$. The set of all infinite paths in \mathcal{M} is $\text{Paths}^{\mathcal{M}} = \bigcup_{s \in S} \text{Paths}^{\mathcal{M}}(s)$. For $\pi = s_0 s_1 \dots \in \text{Paths}^{\mathcal{M}}$, we denote by $\pi(i, l)$ the sequence of $l - i + 1$ states (or $l + 1$ edges) $s_i \dots s_{i+l}$, and for simplicity, we denote $\pi(i, 0)$ by $\pi(i)$. The infinite suffix of π starting in s_n is denoted by $\pi(n, \infty) \in \text{Paths}^{\mathcal{M}}$. The set of finite paths starting from a state $s \in S$ is defined as $\text{Fpaths}^{\mathcal{M}}(s) = \{\pi = s \dots s' \in S^+ \mid \exists \bar{\pi} \in \text{Paths}^{\mathcal{M}}, \pi \bar{\pi} \in \text{Paths}^{\mathcal{M}}(s)\}$ and $\text{Fpaths}^{\mathcal{M}} = \bigcup_{s \in S} \text{Fpaths}^{\mathcal{M}}(s)$. For $\pi = s \dots s'$, we denote by $\text{Last}(\pi)$, the last state s' in π . As in [20], we extend the probability distribution to the space of infinite paths by considering cylinders defined by finite prefixes and using Carathéodory's extension theorem. We denote this probability distribution over the set of infinite paths beginning from some initial state s by $\text{Pr}_{\mathcal{M}}^s$. When s is clear from the context, we omit it and only denote this distribution by $\text{Pr}_{\mathcal{M}}$.

Markov decision process. A finite *Markov decision process* (MDP, for short) is a tuple $\Gamma = (S, E, \text{Act}, \text{Pr})$, where S is a finite set of states, Act is a finite set of actions, and $E \subseteq S \times \text{Act} \times S$ is a set of edges, and $\text{Pr}: S \times \text{Act} \rightarrow \mathcal{D}(E)$ is

a partial function that assigns a probability distribution – on the set $E(s, a)$ of outgoing edges from s – to all states $s \in S$ if action $a \in \text{Act}$ is taken from s . For all $s \in S$ there exists at least one $a \in \text{Act}$ such that $E(s, a)$ is defined. Given $s \in S$ and $a \in \text{Act}$, we define $\text{Post}(s, a) = \{s' \in S \mid \text{Pr}(s, a, s') > 0\}$. Then, for all state $s \in S$, we denote by $\text{Act}(s)$ the set of actions $\{a \in \text{Act} \mid \text{Post}(s, a) \neq \emptyset\}$. We assume that, for all $s \in S$, we have $\text{Act}(s) \neq \emptyset$. Given an MDP $\Gamma = (S, E, \text{Act}, \text{Pr})$, and a set of states $C \subseteq S$, we define the restriction of Γ to C , denoted $\Gamma \upharpoonright C$, as the MDP $(C, E', \text{Act}, \text{Pr}')$ where $E' = \{(s, a, s') \mid s, s' \in C, a \in \text{Act}, \text{Post}(s, a) \subseteq C, \text{ and } (s, a, s') \in E\}$, and Pr' is a partial function defined as $\text{Pr}'(s, a) = \text{Pr}(s, a)$ if $(s, a, s') \in E'$ for $a \in \text{Act}$, and $s, s' \in C$, and is undefined otherwise.

A *strategy* in Γ is a function $\sigma: S^+ \rightarrow \mathcal{D}(\text{Act})$ such that for all $s_0 \dots s_n \in S^+$, we have $\text{Supp}(\sigma(s_0 \dots s_n)) \subseteq \text{Act}(s_n)$. A strategy σ can be encoded by a transition system $\Upsilon = (Q, S, \text{act}, \delta, \iota)$ where Q is a (possibly infinite) set of states, called *modes*, $\text{act}: Q \times S \rightarrow \mathcal{D}(\text{Act})$ selects a distribution on actions such that, for all $q \in Q$ and $s \in S$, we have, $\text{act}(q, s) \in \mathcal{D}(\text{Act}(s))$. The function $\delta: Q \times S \rightarrow Q$ is a mode update function and $\iota: S \rightarrow Q$ selects an initial mode for each state $s \in S$. If the current state is $s \in S$, and the current mode is $q \in Q$, then the strategy chooses the distribution $\text{act}(q, s)$, and the next state s' is chosen according to the distribution $\text{act}(q, s)$. Formally, $(Q, S, \text{act}, \delta, \iota)$ defines the strategy σ such that $\sigma(\rho \cdot s) = \text{act}(\delta^*(\iota(\rho(0)), \rho), s)$ for all $\rho \in S^*$, and $s \in S$, where δ^* extends δ to sequence of states starting from ι as expected, i.e., $\delta^*(\iota(\rho(0)), \rho \cdot s) = \delta(\delta^*(\iota(\rho(0)), \rho), s)$, and $\delta^*(\iota(\rho(0)), \varepsilon) = \iota(\rho(0))$. We denote by Υ_{σ} a transition system with minimal number of modes that corresponds to a strategy σ . A strategy is said to be *memoryless* if there exists a transition system encoding the strategy with $|Q| = 1$, that is, the choice of action only depends on the current state. A memoryless strategy can be seen as a function $\sigma: S \rightarrow \mathcal{D}(\text{Act})$. Formally, a strategy σ is memoryless if for all finite sequences of states ρ_1 and ρ_2 in S^+ such that $\text{Last}(\rho_1) = \text{Last}(\rho_2)$, we have $\sigma(\rho_1) = \sigma(\rho_2)$. A strategy is called a *finite memory* strategy if there exists a transition system encoding the strategy in which Q is finite. A strategy is *deterministic* if $\sigma: S^+ \rightarrow \text{Act}$. For deterministic strategies, we have $\text{act}: Q \times S \rightarrow \text{Act}$ such that for all $q \in Q$ and $s \in S$, we have $\text{act}(q, s) \in \text{Act}(s)$. Note that the state space of $\Gamma^{[\sigma]}$ is $Q \times S$. For a sequence π of states in $\Gamma^{[\sigma]}$, we denote by $\text{proj}_S(\pi)$ the corresponding sequence of states in the MDP Γ . Once we fix a strategy σ encoded by the transition system $(Q, S, \text{act}, \delta, \iota)$ in an MDP $\Gamma = (S, E, \text{Act}, \text{Pr})$, we obtain an MC $\Gamma^{[\sigma]} = (S', E', \text{Pr}')$, where $S' = Q \times S$ is the set of states, $E' = \{(q \times s) \times (q' \times s') \mid q, q' \in Q, s, s' \in S, \delta(q, s) = q', \exists a \in \text{Act}, a \in \text{Supp}(\text{act}(q, s)) \text{ and } (s, a, s') \in E\}$ is the set of edges, and for $q, q' \in Q, s, s' \in S$ we have the probability distribution $\text{Pr}'(q, s)(q', s') = \sum_{a \in \text{Supp}(\text{act}(q, s))} \text{act}(q, s)(a) \cdot \text{Pr}(s, a, s')$ if $q' = \delta(q, s)$ and is not defined otherwise. In the sequel, by

abuse of notation, we write the projection onto the second component, that is s instead of (q, s) , for a state of this MC, unless specifically stated.

One and two-player games. For a given objective, an MDP $\Gamma = (S, E, Act, Pr)$ can also be considered to have the semantics of a zero-sum two-player turn-based game where the game is played for infinitely many rounds and the exact probabilities are not important (this is the case when we will consider A and E atoms). The first round starts from a designated initial state $s_{init} \in S$. In each round, Player 1 chooses an action $a \in Act(s)$ from a state s while Player 2 that is adversarial resolves the non-determinism by choosing a state s' such that $Pr(s, a, s') > 0$. We denote by $G_\Gamma = (S, E, Act)$ the two-player game that is obtained from an MDP $\Gamma = (S, E, Act, Pr)$. When the players resolve the non-determinism co-operatively, we have a one-player game. Equivalently, in a one-player game, Player 1 chooses both action a as well as the state s' .

Given a target set T , we define the attractor of T , denoted $Attr_1(T)$ as the set of states from which there exists a strategy for Player 1 to reach T with certainty. This corresponds to reachability in a classical “and-or” graph. For a two-player game, given T , an algorithm to obtain its attractor computes a sequence of sets of states $(Attr_1^n(T))_{n \geq 0}$ defined as follows: (i) $Attr_1^0(T) = T$; and (ii) for all $n \geq 0$: $Attr_1^{n+1}(T) = Attr_1^n(T) \cup \{s \in S \mid \exists a \in Act, Post(s, a) \subseteq Attr_1^n(T)\}$. Clearly $Attr_1^{n+1}(T) \supseteq Attr_1^n(T)$. If S is finite, then there exists an $m \in \mathbb{N}_0$ such that $Attr_1^m(T) = Attr_1^n(T)$ for all $n \geq m$. The algorithm for the case of one-player game only changes in the induction step where we have for all $n \geq 0$: $Attr_1^{n+1}(T) = Attr_1^n(T) \cup \{s \in S \mid \exists a \in Act, Post(s, a) \cap Attr_1^n(T) \neq \emptyset\}$. The algorithm for the one-player case corresponds to classical graph reachability.

We denote the size of an MC \mathcal{M} , MDP Γ and two-player game G by $|\mathcal{M}|$, $|\Gamma|$ and $|G|$ respectively. For each case, the size is the sum of the number of states, the number of edges, and the size of the representation of the transition matrix, that is, $|S| + |E| + |Pr|$.

Parity conditions and qualitative parity logic. Given an MDP Γ , a parity condition is a function $p: S \rightarrow \mathbb{N}_0$. Given a path $\pi \in S^\omega$, the set $\text{inf}(\pi) = \{s \in S \mid \forall i \geq 0, \exists j \geq i, \text{ such that } \pi(j) = s\}$ is the set of states visited infinitely often on this path. A path satisfies a parity condition p iff $\max\{p(s) \mid s \in \text{inf}(\pi)\}$ is even. Given a parity condition p , its dual is the condition $\bar{p}: s \mapsto 1 + p(s)$. We denote by parity the set of parity conditions. A path satisfies \bar{p} iff it does not satisfy p . We now define qualitative parity logic (QPL) which is defined by the following grammar.

$$\text{atom} = A(p) \mid E(p) \mid AS(p) \mid NZ(p) \quad (p \in \text{parity})$$

$$\varphi = \text{atom} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg \varphi$$

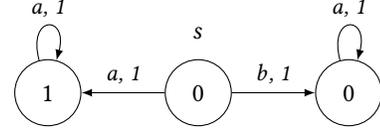


Figure 2. MDP in the proof of Remark 2.1

Given an MDP $\Gamma = (S, E, Act, Pr)$, a state $s \in S$, and a parity condition p , for the atomic formulas, we say that s under strategy σ

- *surely* satisfies p , denoted $s, \sigma \models_\Gamma A(p)$, iff $\forall \pi \in \text{Paths}^{\Gamma[\sigma]}(s)$, we have that π satisfies p .
- *almost-surely* satisfies p , denoted $s, \sigma \models_\Gamma AS(p)$, iff $\Pr_{\Gamma[\sigma]}(\{\pi \in \text{Paths}^{\Gamma[\sigma]}(s) : \pi \text{ satisfies } p\}) = 1$.
- satisfies p with *non-zero probability*, denoted $s, \sigma \models_\Gamma NZ(p)$, iff $\Pr_{\Gamma[\sigma]}(\{\pi \in \text{Paths}^{\Gamma[\sigma]}(s) : \pi \text{ satisfies } p\}) > 0$.
- *existentially* satisfies p , denoted $s, \sigma \models_\Gamma E(p)$, iff $\exists \pi \in \text{Paths}^{\Gamma[\sigma]}(s)$, such that π satisfies p .

Given two QPL formulas φ and ψ , and a strategy σ we define the semantics of Boolean connectives as follows:

- $s, \sigma \models_\Gamma \varphi \wedge \psi$ iff $s, \sigma \models_\Gamma \varphi$ and $s, \sigma \models_\Gamma \psi$
- $s, \sigma \models_\Gamma \varphi \vee \psi$ iff $s, \sigma \models_\Gamma \varphi$ or $s, \sigma \models_\Gamma \psi$
- $s, \sigma \models_\Gamma \neg A(p)$ iff $s, \sigma \models_\Gamma E(\bar{p})$
- $s, \sigma \models_\Gamma \neg E(p)$ iff $s, \sigma \models_\Gamma A(\bar{p})$
- $s, \sigma \models_\Gamma \neg AS(p)$ iff $s, \sigma \models_\Gamma NZ(\bar{p})$
- $s, \sigma \models_\Gamma \neg NZ(p)$ iff $s, \sigma \models_\Gamma AS(\bar{p})$
- $s, \sigma \models_\Gamma \neg(\varphi \wedge \psi)$ iff $s, \sigma \models_\Gamma \neg \varphi \vee \neg \psi$
- $s, \sigma \models_\Gamma \neg(\varphi \vee \psi)$ iff $s, \sigma \models_\Gamma \neg \varphi \wedge \neg \psi$

Given a formula φ , we will use $s \models_\Gamma \varphi$ to denote $\exists \sigma : s, \sigma \models_\Gamma \varphi$. Given a formula φ , let $\llbracket \varphi \rrbracket \stackrel{\text{def}}{=} \{s \in S \mid s \models_\Gamma \varphi\}$. We note that satisfying surely a parity condition is the same as winning the parity objective in the two-player game corresponding to the MDP Γ . Satisfying existentially is the same as finding a satisfying path in the one-player game associated to this MDP.

Given an MDP $\Gamma = (S, E, Act, Pr)$, a state $s \in S$, and a QPL formula φ , the QPL-synthesis problem is to find a strategy σ such that $s, \sigma \models_\Gamma \varphi$. The QPL-realizability problem is to decide whether $s \models_\Gamma \varphi$. In what follows, we focus on the QPL-realizability problem, but the algorithms we provide give all the elements necessary to build a winning strategy when such a strategy exists, and so they can be easily extended to solve QPL-synthesis.

Remark 2.1. We define the negation of the formulas using classical De Morgan’s laws. We note that the logic QPL is closed under negation. It is also important to note that in this semantics, $s \models_\Gamma \neg \varphi$ is not equivalent to $s \not\models_\Gamma \varphi$. Indeed, $s \models_\Gamma \neg \varphi$ implies that there exists a strategy σ such that $s, \sigma \models_\Gamma \neg \varphi$ whereas $s \not\models_\Gamma \varphi$ implies that for all strategies σ , we have $s, \sigma \not\models_\Gamma \varphi$.

2.1 Proof of Remark 2.1

Proof. Note we have that $s \not\models_{\Gamma} \varphi$ means that for all strategies σ , we have that $s, \sigma \not\models_{\Gamma} \varphi$, that is $s, \sigma \models_{\Gamma} \neg\varphi$. Hence $s \models_{\Gamma} \neg\varphi$.

We now show that the converse does not hold using the example in Figure 2. On each edge, the name of the action and the corresponding probability is written and inside parentheses in each state, we write the value that the parity function assigned to the state. Consider the formula $\varphi = A(p)$. Now $s \models_{\Gamma} \varphi$ since there is a strategy (choosing action b from s) such that $s, \sigma \models_{\Gamma} \varphi$. Also $s \models_{\Gamma} \neg\varphi$, where $\neg\varphi$ is the formula $E(\bar{p})$ since there is a strategy (choosing action a from s) such that $s, \sigma \models_{\Gamma} \neg\varphi$.

Now for $s \not\models_{\Gamma} \varphi$ to be true, we need that for all strategies σ , we have that $s, \sigma \models_{\Gamma} \neg\varphi$, that is, for all strategies σ , we need that $s, \sigma \models_{\Gamma} E(\bar{p})$ which is clearly not true since as we show above that there indeed exists a strategy σ such that $s, \sigma \models_{\Gamma} \varphi$, and hence the result. \square

Similarly, even though $s \models_{\Gamma} \varphi \vee \psi$ is equivalent to $s \models_{\Gamma} \varphi$ or $s \models_{\Gamma} \psi$, we note that $s \models_{\Gamma} \varphi \wedge \psi$ is not the same as $s \models_{\Gamma} \varphi$ and $s \models_{\Gamma} \psi$. Also using De Morgan's laws, the negation can be applied only to the parity objectives to get their duals, for example, $\neg(A(p_1) \wedge A(p_2))$ is the same as $E(\bar{p}_1) \vee E(\bar{p}_2)$. We can indeed define a negation free normal form that can be obtained by taking the DNF and pushing negations down to the atoms. In the rest of the paper, we thus restrict our attention to the subclass of the logic that is free of negation and disjunction.

Additional objectives. We define the following additional objectives, introduced for technical reasons, even though they are not part of QPL¹. A parity condition is called a *Büchi condition* if it is defined as $p: S \rightarrow \{1, 2\}$. A path $\pi \in S^{\omega}$ satisfies a *conjunction of parity conditions* $\bigwedge_{x \in X} p_x$ if for all $x \in X$ we have $\max\{p_x(s) \mid s \in \text{inf}(\pi)\}$ is even. It is not hard to see that conjunctions of parity conditions can be expressed as *Streett conditions*. A path π satisfies a *reachability condition* towards a set $R \subseteq S$, denoted $\diamond R$, if there exists $i \in \mathbb{N}_0$ such that $\pi(i) \in R$.

Given an MDP, we can define, in the same way as previously the sure, almost-sure, non-zero, and existential objectives for these conditions, as well as conjunctions and disjunctions of these objectives.

End-components. An *end-component* (EC, for short) $M = (C, A)$ such that $C \subseteq S$, and $A: C \rightarrow 2^{Act}$ is a *sub-MDP* of Γ (for all $s \in C$, we have $A(s) \subseteq Act(s)$, and for all $a \in A(s)$, we have $\text{Post}(s, a) \subseteq C$) that is strongly connected. We denote by $EC(\Gamma)$ the set of end-components of MDP Γ . By abuse of notation, in the sequel, we often refer to a set $C \subseteq S$ to be an end-component when there exists a function $A: C \rightarrow 2^{Act}$ such that (C, A) is an end-component. A *maximal EC* (MEC, for short) is an EC that is not included in any other EC. For

¹Given an MDP Γ , we could have expressed these conditions using QPL, but this would involve constructing a larger and more complex MDP from the given MDP Γ .

every strategy in an MDP, the set of states seen infinitely often during a path form an end-component with probability 1. Formally:

Proposition 2.1 ([2]). *Given an MDP Γ , for all strategies σ , for all states $s \in S$, we have $\Pr_{\Gamma|\sigma}(\{\pi \in \text{Paths}^{\Gamma|\sigma}(s) \mid \text{inf}(\pi) \in EC(\Gamma)\}) = 1$.*

3 Type I end-components

In this section, we define Type I ECs that are a generalization of super-good end components as defined in [1].

Lemma 3.5 is the main result of the section, where we state that we can compute the set of maximal Type I ECs. This will be used later, to compute the set of maximal ECs of other kinds, namely Type II and Type III that are used in Sections 4 and 5 to solve satisfiability of formulas of the form $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$ and $\bigwedge(\mathcal{A}, \mathcal{AS}) \wedge NZ(p_{nz})$ respectively. Lemmas 3.2, 3.3 and 3.4 are technical lemmas that are required in the proof of Lemma 3.5. The proof of Lemma 3.2 uses the notion of Streett-Büchi games.

Given two sets of parity conditions $\{p_a \mid a \in \mathcal{A}\}$ and $\{p_{as} \mid as \in \mathcal{AS}\}$, an end-component C of Γ is Type I $(\mathcal{A}, \mathcal{AS})$ if the following property holds:

- $(I_1) \forall s \in C, s \models_{\Gamma|C} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(\diamond C_{\text{even}}^{\max}(p_{as}))$,

where

$$C_{\text{even}}^{\max}(p_{as}) = \{s \in C \mid (p_{as}(s) \text{ is even}) \wedge (\forall s' \in C,$$

$$p_{as}(s') \text{ is odd} \implies p_{as}(s') < p_{as}(s))\}$$

contains the states with even priorities that are larger than any odd priority in C (this set can be empty for arbitrary ECs but needs to be non-empty for Type I $(\mathcal{A}, \mathcal{AS})$ ECs);

We write Type I $(\mathcal{A}, \mathcal{AS})$ EC as Type I EC when the parity sets are clear from the context. We introduce the following notations: $\mathcal{EC}_I(\Gamma, \mathcal{A}, \mathcal{AS})$ is the set of all Type I $(\mathcal{A}, \mathcal{AS})$ ECs, and $\mathcal{T}_{I, \mathcal{A}, \mathcal{AS}} = \bigcup_{U \in \mathcal{EC}_I(\Gamma, \mathcal{A}, \mathcal{AS})} U$ is the set of states belonging to some Type I EC. Given an EC C , we say a state $s \in C$ is of Type I for C if C is Type I. In this paper, we only consider Type I $(\mathcal{A}, \mathcal{AS})$ ECs where \mathcal{AS} is either \mathcal{A} or $\{a\}$.

Intuitively, within a Type I EC, there is a strategy to visit all $C_{\text{even}}^{\max}(p_{as})$ for all $as \in \mathcal{AS}$ with probability 1 while guaranteeing $A(p_a)$ for all $a \in \mathcal{A}$. We note that this property must hold while staying inside the end-component C . This notion strengthens the notion of *super-good end-component* (SGEC in [1]), that are defined for some parity condition p_a , and are Type I $(\{a\}, \{a\})$ ECs. In the case of SGEC, it has been shown in [1] that the existence of a strategy to enforce condition I_1 in Γ can be reduced to checking the existence of a winning strategy in a game, constructed in polynomial time from Γ , with a conjunction of one parity objective and one Büchi objective. The existence of a winning strategy in such a game is in $NP \cap \text{coNP}$. The structure of this game is

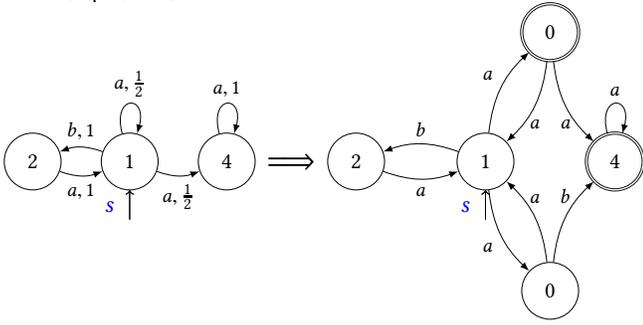


Figure 3. An example of a Type I EC at left, and a game associated to it at right.

different from the one of the original MDP, as its size is polynomially increased to transform the qualitative reachability condition into a sure Büchi. In the sequel, we generalize this result to multiple parity conditions. We illustrate the reduction by the following example.

Example 3.1. Consider the example in Figure 3 where an MDP (on the left side of the figure) that is a Type I $\{\{a\}, \{a\}\}$ EC for a parity condition p_a is transformed into a game (on the right side of the figure) that satisfies $A(p_a) \wedge A(\Box\Diamond R)$. In order to convert the $AS(\Diamond C_{\text{even}}^{\max}(p_a))$ condition of (I_1) into the $A(\Box\Diamond R)$ condition, we add two states to the game: The top-most state and the bottom-most state. In the MDP on the left of Figure 1, a strategy that alternates between playing action a and playing action b at state s indeed satisfies the condition (I_1) .

Now consider the game on the right side of Figure 3. The top-most state and the right-most state shown in double circles form the set R . A strategy to satisfy $A(p_a) \wedge A(\Box\Diamond R)$ is as follows: When in state s , alternate between playing action a and playing action b . When in the bottom-most state, play action b . The $A(p_a)$ atom is clearly satisfied. The sure Büchi $A(\Box\Diamond R)$ holds for the following reason. When action a is chosen in state s , if player 2 chooses to go to the bottom-most state, then from this state player 1 plays action b , and reaches the right-most state that is absorbing and in R . If player 2 chooses to go to the top-most state always, as this state is in R , the Büchi condition is again satisfied.

We now state the first step of the reduction. The result below relies on a reduction to a two-player game $G_{R, \{p_a \mid a \in \mathcal{A}\}}^\Gamma$ with a conjunction of one Büchi and multiple parity conditions, that we call a Streett-Büchi game. The approach to the proof is similar to Lemma 3 of [1], that studies the case where \mathcal{A} is a singleton.

Lemma 3.2. *Given an MDP $\Gamma = (S, E, Act, Pr)$, a state $s_0 \in S$, a set of parity conditions $\{p_a \mid a \in \mathcal{A}\}$, and a target set $R \subseteq S$, it can be decided if $s_0 \models_{\Gamma} \bigwedge_{a \in \mathcal{A}} (A(\neg(\Diamond R)) \rightarrow p_a) \wedge AS(\Diamond R)$. If the answer is YES, then there exists a finite-memory witness strategy. This decision problem is coNP complete.*

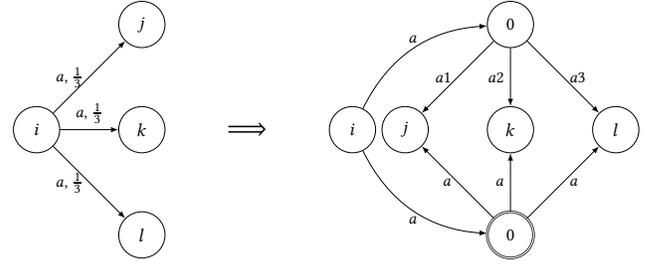


Figure 4. Modifying edges associated to action a on a state s with priority i by adding new edges and new states. The bottom-most state in the figure at right corresponds to $(s, a, 0)$ and is Büchi winning. The top-most state in the figure corresponds to $(s, a, 1)$.

Proof. To establish this lemma, given an MDP Γ , a set of parity conditions $\{p_a \mid a \in \mathcal{A}\}$, and a target set $R \subseteq S$, we construct a game $G_{R, \{p_a \mid a \in \mathcal{A}\}}^\Gamma$ with a conjunction of a Büchi and multiple parity conditions. We call this game a Streett-Büchi game, and its formal definition is as follows:

- The state space of $G_{R, \{p_a \mid a \in \mathcal{A}\}}^\Gamma$ is a copy of the state space of Γ where the states in $S \setminus R$ have been copied $2 \cdot |S| \cdot |R| \cdot |Act|$ times as shown in Figure 4. S' is defined as $S' = S \cup ((S \setminus R) \times Act \times \{0, 1\})$.
- We consider an arbitrary set of actions Act' such that $Act \subseteq Act'$ and for all $s \in S \setminus R$ and $a \in Act$, we have $|Act'| \geq |\{s' \mid (s, a, s') \in E\}|$. This second condition is to make sure that in the states labelled $(s, a, 1)$, we always have enough actions available to choose the successor state.
- The set E' of edges is defined according to Figure 4 with the additional property that states in R are made absorbing. That is, E' is the union of the following sets:
 - $\{(s, a, (s, a, 0)) \mid s \in S \setminus R, a \in Act(s)\}$,
 - $\{(s, a, (s, a, 1)) \mid s \in S \setminus R, a \in Act(s)\}$,
 - $\{((s, a, 0), a, s') \mid (s, a, s') \in E \cap ((S \setminus R) \times a \times S)\}$,
 - $\{((s, a, 1), a_{s, a, s'}, s') \mid (s, a, s') \in E \cap ((S \setminus R) \times Act \times S)$, and for all $(s, a, s'), (s, a, s'') \in S$ with $s' \neq s''$, we have $a_{s, a, s'} \neq a_{s, a, s''}$,
 - $\{(s, a, s) \mid s \in R \text{ for one } a \in Act\}$.
- The parity conditions p'_a are defined as $p'_a(s) = p_a(s)$ for all $s \in S \setminus R$, $p'_a((s, a, i)) = 0$ for all $s \in S \setminus R$, $a \in Act$, $i \in \{0, 1\}$, and $p'_a(s) = 0$ for all $s \in R$. We have to change the priority of states of R , because as they are made absorbing they may be losing for the parity condition otherwise, as on Figure 5.
- The set of Büchi states is $B = R \cup \{(s, a, 0) \mid s \in S \setminus R, a \in Act\}$.



Figure 5. We change the parity value of the target state, since they are made absorbing, otherwise there would be no winning strategy in the parity-Büchi game while there is a parity winning strategy ensuring to reach R in the initial MDP.

It is established in [1] that for one parity condition p we have $s_0 \models A(\neg(\Box\Diamond R) \rightarrow p) \wedge AS(\Box\Diamond R)$ if and only if there exists a winning strategy in the Büchi parity game $G_{R, \{p_a \mid a \in \mathcal{A}\}}^\Gamma$ from state s_0 . The proof of the result can be generalized to our setting: $s_0 \models_\Gamma \bigwedge_{a \in \mathcal{A}} A(\neg(\Diamond R) \rightarrow p_a) \wedge AS(\Diamond R)$ iff $s_0 \models_{G_{R, \{p_a \mid a \in \mathcal{A}\}}^\Gamma} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge A(\Box\Diamond B)$ where B has been defined above. Finding a strategy for the second objective above is solving a Streett game, which is classically co-NP complete. We do not give the full proof of this result, as there would be no change from [1], but give a quick outline. Note that corresponding to every path in MDP Γ , there exists a path in the game $G_{R, \{p_a \mid a \in \mathcal{A}\}}^\Gamma$, and vice versa.

Given a finite-memory strategy σ for the Streett-Büchi game, we obtain a strategy σ' for the MDP by playing exactly as in the game. We have that every path in the game under strategy σ either reaches R or satisfies for all $a \in \mathcal{A}$ the condition p_a . Since for every path in the MDP under strategy σ' , there exists a corresponding path in the game under strategy σ , every path in the MDP under strategy σ' also either reaches R or satisfies for all $a \in \mathcal{A}$ the condition p_a . Since σ' is a finite-memory strategy, in every state of the MDP there is a non-zero probability of reaching R in some fixed k steps. This probability is also always greater than some lower bound, hence the probability of reaching R is 1.

Reciprocally, given a winning strategy σ in an MDP Γ , there exists a winning strategy σ' in $G_{R, \{p_a \mid a \in \mathcal{A}\}}^\Gamma$. The strategy σ' consists in behaving like σ in Γ , and in states $(s, a, 1)$, where σ is not defined, strategy σ' takes the action which corresponds to a probabilistic transition in Γ that would lead to the shortest possible path to R . In the MDP under strategy σ every path that does not reach R satisfies for all $a \in \mathcal{A}$ the condition p_a . Since corresponding to every path in the game under strategy σ' there exists a path in the MDP under strategy σ , we have that in the game under strategy σ' every path that does not reach R satisfies for all $a \in \mathcal{A}$ the condition p_a . Now, for condition B , there are two possibilities. The first one is that some $(s, a, 0)$ is visited infinitely often, and the condition B is satisfied. The second one is that eventually the states $(s, a, 0)$ are not visited any more. In this case in every $(s, a, 1)$, the action leading to the shortest path to R is taken. Then the set R is eventually reached and B is also satisfied. \square

The following lemma relates the $\bigwedge_{a \in \mathcal{A}} A(\neg(\Diamond R) \rightarrow p_a) \wedge AS(\Diamond R)$ objective of Lemma 3.2 and the $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\Diamond R)$ objective of Type I $(\mathcal{A}, \{a\})$ ECs under the condition that for all $s \in S$ it holds that $s \models_\Gamma \bigwedge_{a \in \mathcal{A}} A(p_a)$.

Lemma 3.3. *Given an MDP $\Gamma = (S, E, Act, Pr)$, a state $s_0 \in S$, a set of parity conditions $\{p_a \mid a \in \mathcal{A}\}$, and a target set $R \subseteq S$, if for all $s \in S$ it holds that $s \models_\Gamma \bigwedge_{a \in \mathcal{A}} A(p_a)$ then we have that $s_0 \models_\Gamma \bigwedge_{a \in \mathcal{A}} A(\neg(\Diamond R) \rightarrow p_a) \wedge AS(\Diamond R)$ if and only if $s_0 \models_\Gamma \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\Diamond R)$.*

Proof. The right to left implication is simple, as $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\Diamond R)$ implies $\bigwedge_{a \in \mathcal{A}} A(\neg(\Diamond R) \rightarrow p_a) \wedge AS(\Diamond R)$. For the left to right implication, consider a strategy σ_0 such that $s_0, \sigma_0 \models_\Gamma \bigwedge_{a \in \mathcal{A}} A(\neg(\Diamond R) \rightarrow p_a) \wedge AS(\Diamond R)$. Recall that from every $s \in S$ we have σ_s such that $s, \sigma_s \models_\Gamma \bigwedge_{a \in \mathcal{A}} A(p_a)$. Consider the following strategy σ : it plays like σ_0 , but when it reaches some $r \in R$, it switches to play like σ_r forever. It clearly satisfies $AS(\Diamond R)$. It also satisfies $\bigwedge_{a \in \mathcal{A}} A(p_a)$: indeed, on a path π , either R is never reached, and thanks to the use of σ_0 we have that $A(\neg(\Diamond R) \rightarrow p_a)$ implying p_a holds on this path for all $a \in \mathcal{A}$. Otherwise, if π eventually reaches R , σ switches to some strategy σ_r that satisfies $\bigwedge_{a \in \mathcal{A}} A(p_a)$ by assumption. \square

As I_1 implies $s \models_\Gamma \bigwedge_{a \in \mathcal{A}} A(p_a)$, pruning states that do not satisfy $\bigwedge_{a \in \mathcal{A}} A(p_a)$ before using Lemma 3.2 and Lemma 3.3 is always possible. Lemma 3.2 and Lemma 3.3 can only be used to compute Type I $(\mathcal{A}, \{a\})$ ECs. We have the following lemma to relate Type I $(\mathcal{A}, \{a\})$ ECs and Type I $(\mathcal{A}, \mathcal{A})$ ECs.

Lemma 3.4. *In an EC C , for all $s \in C$ we have that $s \models_{\Gamma \downarrow C} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{a \in \mathcal{A}} AS(\Diamond C_{\text{even}}^{\max}(p_a))$, iff for all $a_i \in \mathcal{A}$, and for all $s \in C$, we have that $s \models_{\Gamma \downarrow C} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\Diamond C_{\text{even}}^{\max}(p_{a_i}))$.*

Proof. We prove here the right to left implication as the other direction is obvious. Assume that for $i \in [|\mathcal{A}|]$, we have a strategy $\sigma_{a_i, s}$ for $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\Diamond C_{\text{even}}^{\max}(p_{a_i}))$. We define the following strategy σ_s :

1. Let $s_0 = s$
2. For i going from 1 to $|\mathcal{A}|$, play $\sigma_{a_i, s_{i-1}}$ until reaching some state s_i of $C_{\text{even}}^{\max}(p_{a_i})$

As in all possible paths we end up having some i such that $\sigma_{a_i, s_{i-1}}$ is followed forever, and all these strategy satisfy $\bigwedge_{a \in \mathcal{A}} A(p_a)$, we have that σ_s also satisfies $\bigwedge_{a \in \mathcal{A}} A(p_a)$. As every $\sigma_{a_i, s_{i-1}}$ has probability 1 of reaching $C_{\text{even}}^{\max}(p_{a_i})$, the strategy σ_s satisfies $\bigwedge_{a \in \mathcal{A}} AS(\Diamond C_{\text{even}}^{\max}(p_a))$. As a conclusion $s, \sigma_s \models_{\Gamma \downarrow C} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{a \in \mathcal{A}} AS(\Diamond C_{\text{even}}^{\max}(p_a))$ hence the result holds. \square

The following lemma states that we can compute the maximal Type I $(\mathcal{A}, \mathcal{A})$ ECs. The remaining part of this section is devoted to the proof of the following lemma. The proof of this lemma involves a detailed algorithmic procedure for computing the set of maximal Type I $(\mathcal{A}, \mathcal{A})$ ECs. In this procedure we iteratively compute the maximal Type I $(\mathcal{A}, \{a_i\})$ ECs for all $a_i \in \mathcal{A}$. The combination of Lemma 3.2 and Lemma 3.3 is used for the computation of the set maximal Type I $(\mathcal{A}, \{a_i\})$ ECs. Every time we do this computation, we prune all the states that do not belong to at least one of these ECs and solve Streett games again. We note that computing the maximal Type I $(\mathcal{A}, \{a_i\})$ ECs follows an approach similar to the procedure in [1] that computes the set of maximal SGECs. The difference is that we add an additional step in our algorithm, and use Lemma 3.4 to be able to combine the different $\{p_{a_i}\}$. We note that a naive generalization of the algorithm in [1] to compute the set of maximal Type I $(\mathcal{A}, \mathcal{A})$ ECs results in an EXPTIME complexity, while we end up with a P^{NP} complexity as we show later in Section 7.

Lemma 3.5. *Given an MDP $\Gamma = (S, E, Act, Pr)$, it is possible to compute the set of maximal Type I $(\mathcal{A}, \mathcal{A})$ ECs. This can be done by solving iteratively a number of Streett games that are polynomial in $|\mathcal{A}|$ and $|S|$.*

Given a set C of end-components, we denote by S_C the set of states belonging to some end-component in C . Formally: $S_C = \{s \in S, \exists C \in C \mid s \in C\}$. At the end, in Algorithm 4, we compute the set of maximal Type I $(\mathcal{A}, \mathcal{A})$ ECs C^2 . To find these ECs C , we look at the possible even parity value that may appear in states of $C_{even}^{max}(p)$, and to do so efficiently we look at all even values that p may take. For each of these values, we solve a parity-Büchi game. These games can be solved by using a linear number of calls to an $NP \cap coNP$ oracle, and hence the problem is in $P^{NP \cap coNP} = NP \cap coNP$ [8]. A naive generalization of [1] would result in an EXPTIME complexity. Indeed, if we consider objectives of the form $s \models_{\Gamma|C} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{a \in \mathcal{A}} AS(\diamond C_{even}^{max}(p_a))$, we would consider every tuple of length $|\mathcal{A}|$ of possible even parity values that may appear in states of $C_{even}^{max}(p_a)$. The number of such tuples is exponential in $|\mathcal{A}|$, which leads to the EXPTIME complexity. Our approach is to use Lemma 3.4 to check for every $a_i \in \mathcal{A}$ if $s' \models_{\Gamma|C} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\diamond C_{even}^{max}(p_{a_i}))$. This means we can test separately for every $a_i \in \mathcal{A}$ the possible even parity values of states in $C_{even}^{max}(p_{a_i})$, which only considers a polynomial number of possibilities. We can check this condition by converting it to a game with multiple parity, and a single Büchi condition. Since a Büchi condition can be regarded as a parity condition, this game can be regarded as a multiple-parity game, and thus can be translated to a Streett game. We solve a number of Streett games linear in $|S|$ and

²Note that in [1] the case where \mathcal{A} is a singleton $\{p\}$ has been solved

$|\mathcal{A}|$ to check if C is a Type I $(\mathcal{A}, \{a_i\})$ EC. Towards this, we modify the algorithms defined in [1], where the case when \mathcal{A} is singleton has been solved. We begin with Algorithm 1 to check if an EC C of MDP Γ is a Type I $(\mathcal{A}, \{a_i\})$.

Algorithm 1

Input : An EC C of MDP Γ , parity conditions $\{p_a \mid a \in \mathcal{A}\}$, $a_i \in \mathcal{A}$

Output : Yes if and only if for all $s \in C$ we have

$$s \models_{\Gamma} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\diamond C_{even}^{max}(p_{a_i}))$$

- 1: Compute G_C the Streett game associated to $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\diamond C_{even}^{max}(p_{a_i}))$.
 - 2: **if** all states win in G_C **then**
 - 3: return “yes”.
 - 4: **else**
 - 5: return “no” and the set of winning states.
-

Now, given an MDP Γ , an objective $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge$

$AS(\diamond C_{even}^{max}(p_{a_i}))$ and an odd number k , Algorithm 2 computes the set \mathcal{MEC}_1^k of maximal Type I $(\mathcal{A}, \{a_i\})$ ECs of Γ whose maximum odd value for parity p_{a_i} is k . Similarly to [1], we focus on maximum odd ranks because we remove the attractor of states that have an odd rank greater than k . If we tried to obtain the maximum even rank, we would look for a decomposition in EC, such that every EC in this decomposition may have a different maximum odd rank. We proceed by removing the attractor of states having an odd rank greater than this maximum. This is not possible without already having the decomposition, and so we would end up guessing the decomposition and the maximum odd value in every EC; this would be less efficient.

Algorithm 2

Input : An MDP Γ , parity conditions $\{p_a \mid a \in \mathcal{A}\}$, $a_i \in \mathcal{A}$, $k \in odd$.

Output : The set \mathcal{MEC}_1^k of maximal Type I $(\mathcal{A}, \{a_i\})$ ECs with maximum odd parity value k

- 1: Compute C the maximal EC decomposition of Γ .
 - 2: **for all** $C \in C$ **do**
 - 3: **if** $odd_{>k}(C) \neq \emptyset$ **then**
 $odd_{>k}(C) := \{s \in C \mid p_{a_i}(s) > k \wedge p_{a_i}(s) \in odd\}$
 - 4: $\Gamma := Attr_1(odd_{>k}(C))$, and begin again from Step 1.
 - 5: **if** C is not Type I **then** // Call to Algorithm 1
 - 6: $\Gamma := Attr_1(S \setminus \mathcal{MEC}_1^k)$, and begin again from Step 1 // \mathcal{MEC}_1^k is the set of winning states in Algorithm 1.
 - 7: **return** C .
-

We get the set of maximal Type I $(\mathcal{A}, \{a_i\})$ ECs by choosing the maximal elements from the set $\bigcup_{k \in odd} \mathcal{MEC}_1^k$. This is

done in Algorithm 3. Note that it is possible that for some odd k , for $C \in \mathcal{MEC}_1^k$, there may exist $C' \in \mathcal{MEC}_1^\ell$, where ℓ is odd, and $\ell < k$ such that $C' \subsetneq C$. Hence we take the maximal elements of the union of \mathcal{MEC}_1^k over all k to obtain the set of maximal Type I $(\mathcal{A}, \{a_i\})$ ECs.

Algorithm 3

Input : An MDP Γ , parity conditions $\{p_a \mid a \in \mathcal{A}\}$, $a_i \in \mathcal{A}$.

Output : The set of maximal Type I $(\mathcal{A}, \{a_i\})$ ECs

- 1: Return the set of maximal elements (for the inclusion) of the set $\bigcup_{k \text{ odd}} \mathcal{MEC}_1^k$.
-

For an MDP $\Gamma = (S, E, Act, Pr)$, we now write Algorithm 4 that computes the maximal Type I $(\mathcal{A}, \mathcal{A})$ ECs.

Algorithm 4

Input : An MDP Γ , parity conditions $\{p_a \mid a \in \mathcal{A}\}$.

Output : The set \mathcal{C} of maximal Type I $(\mathcal{A}, \mathcal{A})$ ECs

- 1: Initialize $\mathcal{M} = \Gamma$
 - 2: For i going from 1 to $|\mathcal{A}|$, compute the decomposition into maximal Type I $(\mathcal{A}, \{a_i\})$ EC of the MDP associated to S_C . We call this decomposition C_i which is the output of Algorithm 3. If $S_{C_i} \subsetneq S_{C_1}$, take $\mathcal{M} = (S_{C_i}, E, Act, Pr)$ and begin again from $i = 1$ // Note that if we reach C_i , we have that $C_1 = C_2 = \dots = C_{i-1}$
 - 3: Return $\mathcal{C} = C_1$.
-

We use the following lemmas to show the correctness of Algorithm 4.

Lemma 3.6. *If C_1 and C_2 are two Type I ECs, and $C_1 \cup C_2$ is an EC, then $C_1 \cup C_2$ is a Type I EC.*

Proof. Observe that we have strategy σ_i in C_i such that for all $s \in C_i$, we have that $s, \sigma_i \models_{C_3-i} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\diamond(C_1 \cup C_2)_{\text{even}}^{\max}(p))$. If $C_1^{\max}(p) = C_2^{\max}(p)$ then strategy σ playing as σ_i in C_i is a witness that $C_1 \cup C_2$ is an Type I EC. Now w.l.o.g., we assume $(C_1 \cup C_2)_{\text{even}}^{\max}(p) = C_1^{\max}(p) > C_2^{\max}(p)$. We define strategy σ as follows: if the initial state belongs to C_1 it plays as σ_1 . Otherwise, if the initial state is in C_2 , it is defined as follows:

1. Play for $|C_2|$ steps uniformly at random while staying in $C_1 \cup C_2$.
2. If a state $q \in C_1$ is reached, then play as σ_1 forever.
3. Else play as σ_2 until reaching $C_2^{\max}(p)$ and start again from 1.

Either this strategy reaches C_1 at some point, and then it satisfies both $\bigwedge_{a \in \mathcal{A}} A(p_a)$ and $AS(\diamond(C_1 \cup C_2)_{\text{even}}^{\max}(p))$. This happens with probability one. In the remaining cases, we stay in C_2 forever. This means that it will either play σ_2 forever at some point, satisfying $A(p)$, or alternate between bounded length

random walks and parts where it follows σ_2 until reaching $C_2^{\max}(p)$ infinitely often, also satisfying $A(p)$. \square

We now have the tools to prove Lemma 3.5.

Proof. We denote by C^* the correct decomposition we want to compute, and by \mathcal{C} the decomposition obtained by applying Algorithm 4. We show equality between C^* and \mathcal{C} by proving $\mathcal{C} \subseteq C^*$ and $C^* \subseteq \mathcal{C}$.

Consider some $C \in C^*$. Since all states in C satisfy $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\diamond C_{\text{even}}^{\max}(p_{a_i}))$ for all $a_i \in \mathcal{A}$, none of these states gets removed by Algorithm 4. Also Algorithm 4 computes the maximal ECs such that all states inside each such maximal EC satisfies the property. Thus $C \in \mathcal{C}$. Hence $C^* \subseteq \mathcal{C}$.

Now we prove that for all $C \in \mathcal{C}$, there exists $C^* \in C^*$ such that $C = C^*$. Let us denote by $s_1 \dots s_n$ the states of C . For all $i \in [n]$, since s_i belongs the Type I EC C_i , it also belongs to some maximal Type I EC C_i^* . It is easy to see that for all $i, j \in [n]$, we have that $C_i^* \cup C_j^* \cup C$ is an EC, and in general that $C \cup \bigcup_{i \in [n]} C_i^*$ is an EC. By Lemma 3.6 we have that $C \cup \bigcup_{i \in [0, n]} C_i^*$ is a Type I, and by maximality of the C_i^* and of C we have that $C = C_1^* = \dots = C_n^* = C \cup \bigcup_{i \in [n]} C_i^*$ and thus there exists $C^* \in C^*$ such that $C = C^*$.

For the number of Streett games solved, we look at the steps of Algorithm 4. Algorithm 1 solves a single Streett game. Thus we compute the number of calls to Algorithm 1. From Algorithm 4, we call Algorithm 3 for $O(|\mathcal{A}| \cdot |S|)$ times. For a fixed MDP \mathcal{M} , we call Algorithm 3 for $|\mathcal{A}|$ times, and we change \mathcal{M} for $O(|S|)$ times, every time a state is removed. From Algorithm 3, for a given a_i , we call Algorithm 2 for $\min(k, S)$ times where k is the maximum odd value less than or equal to $\max_{s \in S} p_{a_i}(s)$. Note that the number of odd values of parity p_{a_i} is no more than the number of states in the MDP. From Algorithm 2, every time a state is removed, we call Algorithm 1. Thus the number of Streett games solved is $O(|\mathcal{A}| \cdot |S|^3)$. \square

4 Type II end-components

In this section, we define Type II ECs that are a generalization of UGEC defined in [4]. In the setting of the current paper, they are a generalization of Type I ECs, that have an additional condition. The main result of the section, Lemma 4.2, shows an equivalence between solving the realizability problem for formulas of the form $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$ and solving the realizability problem for formulas involving sure parity conditions and almost-sure reachability of the Type II end-components. To do so, the two directions of the proof are done separately. For the right to left direction, we are given a strategy σ_T . First Lemma 4.8 shows that inside a Type II EC, there always exists a strategy σ for $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$. We rely on Lemmas 4.3, and Lemma 4.4 to prove that such a σ can be constructed, and then we use

Lemma 4.6 to show that σ satisfies $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$.

In Lemma 4.10, we use the strategy σ_T to reach a Type II EC and thereafter play σ , completing the proof of this direction.

Lemma 4.15 states the result in the other direction. Towards this, we first show that for a strategy satisfying the formula $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$, all the states that are visited under this strategy satisfy this particular formula. We then introduce the notion of density in Definition 4.12 to relate in Lemma 4.13 the states satisfying the formula $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$ to the Street-Büchi game of Section 3. Since Street-Büchi games are related to Type I ECs, and Type II ECs are extensions of Type I ECs, it then remains to prove that there exists at least one Type II EC in the MDP. This is done in Lemma 4.14.

Given two sets of parity conditions $\{p_a \mid a \in \mathcal{A}\}$ and $\{p_{as} \mid as \in \mathcal{AS}\}$, an end-component C of Γ is Type II $(\mathcal{A}, \mathcal{AS})$ if the following two properties hold:

- $(II_1) \forall s \in C, s \models_{\Gamma|C} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(\diamond_{\text{even}}^{C^{\max}}(p_a))$
- $(II_2) \forall s \in C, s \models_{\Gamma|C} \bigwedge_{a \in \mathcal{A}} AS(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$

We note that condition (II_1) is exactly the one defining a Type I $(\mathcal{A}, \mathcal{A})$ EC. We write Type II $(\mathcal{A}, \mathcal{AS})$ EC as Type II EC when the parity sets are clear from the context. We introduce the following notations: $\mathcal{EC}_{II}(\Gamma, \mathcal{A}, \mathcal{AS})$ is the set of all Type II $(\mathcal{A}, \mathcal{AS})$ ECs, and $\mathcal{T}_{II, \Gamma, \mathcal{A}, \mathcal{AS}} = \bigcup_{U \in \mathcal{EC}_{II}(\Gamma, \mathcal{A}, \mathcal{AS})} U$ is the set of states belonging to some Type II EC.

Intuitively, a Type II $(\mathcal{A}, \mathcal{AS})$ EC, is a Type I $(\mathcal{A}, \mathcal{A})$ EC where there also exists an additional strategy staying within the EC and almost-surely satisfying all parity conditions p_a and p_{as} . This notion generalizes the notion of a *ultra-good end-component* (UGEC in [4]) which is a Type II EC where both \mathcal{A} and \mathcal{AS} are singletons. In the sequel, we use the following notation: $\bigwedge(\mathcal{A}, \mathcal{AS}) = \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$.

Finding solutions for (II_2) is done in [16], and it is shown in [4] how to use simple techniques from [2]. Winning strategies for (II_2) may require either randomization or deterministic finite memory. We showed how to compute ECs satisfying (II_1) (Type I ECs) in Lemma 3.5. In the sequel we relate Type II ECs to the formula $\bigwedge(\mathcal{A}, \mathcal{AS})$. In particular, from every state belonging to a Type II ECs there exists a strategy satisfying the formula $\bigwedge(\mathcal{A}, \mathcal{AS})$. We illustrate this by the following example.

Example 4.1. Consider the example in Figure 6. We show a strategy satisfying $A(p_1) \wedge AS(p_2)$ in an MDP that is also a Type II EC. Indeed every state satisfies condition (II_1) when action a is chosen from state s and every state satisfies condition (II_2) when action b is chosen from state s .

Now the strategy from s to satisfy $A(p_1) \wedge AS(p_2)$ is the following. In *round 1*, action b is chosen some i_0 times. If the state with parity $(2, 2)$ is not visited when action b is chosen i_0 times, then action a is chosen until the state with

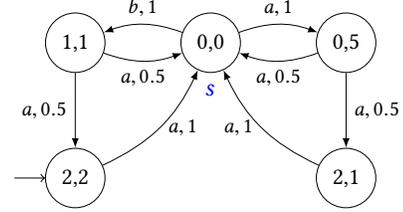


Figure 6. An example of a Type II EC

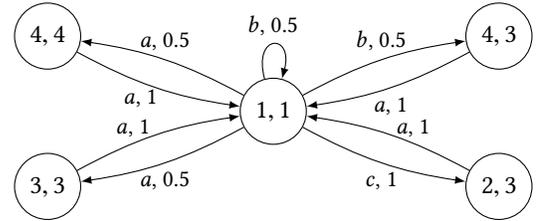


Figure 7. Finite memory may be needed for condition (II_1) .

parity $(2, 1)$ is visited. Once state $(2, 1)$ is visited, we proceed to round 2 in which action b is chosen $i_1 = i_0 + 1$ times. From every round, we proceed to the next round if either the state with parity $(2, 2)$ is visited in the current round, or otherwise if the state with parity $(2, 1)$ is visited and so on, resulting in action b being chosen $i_j = i_0 + j$ times at round j . Now we compute the probability of not choosing action a from s during n rounds. The probability of not choosing a from s after the first round is $1 - 2^{-i_0}$. The probability of not choosing a from s after the first and the second round is $(1 - 2^{-i_0}) \cdot (1 - 2^{-(i_0+1)})$, and thus the probability of never choosing a when n rounds already happened is $p(n) = \prod_{j=n}^{\infty} (1 - 2^{-(i_0+j)})$. In [4], it has been shown that $\lim_{n \rightarrow \infty} p(n) = 1$, implying that with probability 1 action a will eventually stop being played.

The strategies for conditions (II_1) and (II_2) in the example above are deterministic memoryless strategies. However, in general, the strategy for (II_1) may require finite memory and the strategy for (II_2) may require memory or randomization. We illustrate this below. In Figure 7, note that starting from the state with parity $(1, 1)$, we need memory to satisfy condition (II_1) . A strategy from the state with parity $(1, 1)$ that alternates between actions b and c satisfies condition (II_1) .

In Figure 8, starting from the state with parity $(0, 0)$, a randomized memoryless strategy that chooses action a with probability $p > 0$ and action b with probability $1-p$ will visit both of the other two states infinitely often almost-surely, and hence a randomized memoryless strategy suffices here.

We now state the main result of this section.

Lemma 4.2. *Given an MDP Γ , and two sets of parity conditions $\{p_a \mid a \in \mathcal{A}\}$ and $\{p_{as} \mid as \in \mathcal{AS}\}$, for all states s_0 , we have $s_0 \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS})$ iff $s_0 \models_{\Gamma} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\diamond_{\text{even}} \mathcal{T}_{II, \Gamma, \mathcal{A}, \mathcal{AS}})$.*

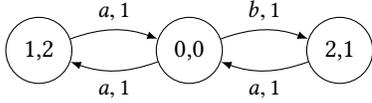


Figure 8. Randomisation or finite-memory may be needed for condition (\mathbf{II}_2) .

A strategy that enforces such conditions may require infinite memory.

We outline a sketch of proof of Lemma 4.2. We show each direction separately. We fix some $s_0 \in S$ and use it throughout this section. The necessity of using infinite memory is proved in Theorem 18 of [4], in the subcase where \mathcal{A} and \mathcal{AS} are singletons.

We start with the right-to-left implication. Since $s_0 \models_{\Gamma} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\diamond \bigwedge_{\Gamma} \mathcal{T}_{\Pi, \Gamma, \mathcal{A}, \mathcal{AS}})$, there exists a witness strategy σ_T . By definition, paths in $\text{Paths}^{\Gamma[\sigma_T]}(s_0)$ eventually reaches some Type II EC $C \in \mathcal{EC}_{\text{II}}(\Gamma, \mathcal{A}, \mathcal{AS})$ with probability one. Since C is a Type II EC, there exist two strategies, σ_1 and σ_2 respectively ensuring condition (\mathbf{II}_1) and (\mathbf{II}_2) . In what follows we define a strategy σ_C from σ_1 and σ_2 such that for all $s \in C$ we have $s, \sigma_C \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS})$. Finally, we compute a strategy σ such that from s_0 , we play σ_T until reaching a Type II EC C , where we play σ_C . This strategy σ satisfies $\bigwedge (\mathcal{A}, \mathcal{AS})$ from s .

We construct the infinite memory strategy σ_C from σ_1 and σ_2 . To do so, we first show some technical lemmas ensuring σ_C can be effectively computed.

Lemma 4.3. *Given an EC C , it can be decided in polynomial time if condition (\mathbf{II}_2) holds in C , and a randomized memoryless strategy suffices.*

Proof. Given an EC C , we claim that (i) the existence of a sub-EC D such that for all $i \in \mathcal{A} \cup \mathcal{AS}$ we have $D_{\text{even}}^{\max}(p_i) \neq \emptyset$ is not only necessary but also sufficient for C to satisfy condition (\mathbf{II}_2) , and (ii) the existence of such a set can be decided in polynomial time.

For (i), it is obvious that the existence of such an EC is necessary. To prove it is also sufficient, we assume we have a sub-EC D such that for all $i \in \mathcal{A} \cup \mathcal{AS}$ we have $D_{\text{even}}^{\max}(p_i) \neq \emptyset$. For all sub-EC of C , in particular for D , we can build a uniform randomized memoryless strategy σ such that

$$\Pr_{\Gamma|_C}^{\sigma} \left[\{ \pi \in \text{Paths}^{\Gamma|_C}(\sigma)(s) \mid \text{inf}(\pi) = D \} \right] = 1.$$

Since for all $i \in \mathcal{A} \cup \mathcal{AS}$ we have $D_{\text{even}}^{\max}(p_i) \neq \emptyset$, we thus have that all p_a and p_{as} are almost-surely satisfied by σ , hence that σ is a witness for $s \models_{\Gamma|_C} \bigwedge_{a \in \mathcal{A}} AS(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$, and that condition (\mathbf{II}_2) holds in C .

It remains to check the existence of such a set $D \subseteq C$ in polynomial time. First, we check if all $C_{\text{even}}^{\max}(p_a) \neq \emptyset$ and $C_{\text{even}}^{\max}(p_{as}) \neq \emptyset$. If this holds, then $D = C$ and the answer is YES (it takes linear time obviously). If it does not hold, then

we compute the sets

$$C_{\text{odd}}^{\max}(p_i) = \{ s \in C \mid (p_i(s) \text{ is odd}) \wedge$$

$$(\forall s' \in C, p_i(s') \text{ is even} \implies p_i(s') < p_i(s)) \}$$

and we iterate this procedure in the sub-EC $C' \subset C$ defined as

$$C' = \text{Attr}_1 \left(\bigcup_{i \in \mathcal{A} \cup \mathcal{AS}} C_{\text{odd}}^{\max}(p_i) \right).$$

It is easy to see that a suitable D exists if and only if this procedure stops before $C' = \emptyset$. In addition, this procedure takes at most $|C|$ iterations (as we remove at least one state at each step) and each iteration takes linear time. This implies our result and concludes our proof. \square

Lemma 4.4. *Let C be an EC of Γ . If condition (\mathbf{II}_2) holds then there exists a randomized memoryless witness strategy σ_2 and a sub-EC $D \subseteq C$ such that for all $i \in \mathcal{A} \cup \mathcal{AS}$ we have $D_{\text{even}}^{\max}(p_i) \neq \emptyset$, and for all $s \in C$, we have that $\Pr_{\Gamma|_C}^{\sigma_2} \left[\{ \pi \in \text{Paths}^{\Gamma|_C}(\sigma_2)(s) \mid \text{inf}(\pi) = D \} \right] = 1$.*

Proof of Lemma 4.4. Let C be an EC that satisfies condition (\mathbf{II}_2) , and a corresponding witness strategy σ (whose existence is granted by Lemma 4.3). By Proposition 2.1, we know that for all states $s \in C$,

$$\Pr_{\Gamma|_C}^{\sigma} \left[\{ \pi \in \text{Paths}^{\Gamma|_C}(\sigma)(s) \mid \text{inf}(\pi) \in EC(\Gamma|_C) \} \right] = 1,$$

where $EC(\Gamma|_C)$ is the set of ECs of $\Gamma|_C$. We claim that for all $D \in \Gamma|_C$ for which

$$\Pr_{\Gamma|_C}^{\sigma} \left[\{ \pi \in \text{Paths}^{\Gamma|_C}(\sigma)(s) \mid \text{inf}(\pi) = D \} \right] > 0,$$

and for all $i \in \mathcal{A} \cup \mathcal{AS}$, we necessarily have that $D_{\text{even}}^{\max}(p_i) \neq \emptyset$. Indeed, assume this is false for p_i . Then, with probability strictly greater than zero, σ induces a path π such that $\max_{s' \in \text{inf}(\pi)} p_i(s')$ is odd (as the maximal priority in $\text{inf}(\pi) = D$ is odd). This contradicts the fact that σ is a witness strategy for $s \models_{\Gamma|_C} \bigwedge_{a \in \mathcal{A}} AS(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$.

We know that such a D exists. As every state of an EC can almost-surely be visited using a uniform randomized memoryless strategy [2], we can conceive a witness strategy σ_2 that reaches and stays only in D with probability 1. \square

Proposition 4.5 (Optimal reachability [2]). *Given an MDP $\Gamma = (S, E, \text{Act}, \text{Pr})$, and a target set $T \subseteq S$, we can compute for each state $s \in S$ the maximal probability v_s^* to reach T , in polynomial time. There is an optimal deterministic memoryless strategy σ^* that enforces v_s^* from all $s \in S$. Now, fix $s \in S$ and $c \in \mathbb{Q}$ such that $c < v_s^*$. Then there exists $k \in \mathbb{N}$ such that by playing σ^* from s for k steps, we reach T with probability larger than c .*

Lemma 4.6. *Let C be an EC of Γ . Every strategy σ_2 satisfies the following property: for all $s \in C$, for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that:*

$$\Pr_{\Gamma_C^{|\sigma_2|}}[\{\rho \in \text{Fpaths}_{\Gamma_C^{|\sigma_2|}} \mid \forall a \in \mathcal{A}, \exists k_a \leq n, \\ \rho(k_a) \in C_{\text{even}}^{\max}(p_a)\}] \geq 1 - \varepsilon.$$

Proof. This result is a consequence of Proposition 4.5. See that for all $a \in \mathcal{A}$ we have that σ_2 reaches $D_{\text{even}}^{\max}(p_a)$ almost-surely from all state $s \in C$. Thus we can apply Proposition 4.5 to all $a \in \mathcal{A}$ and find some $k_a \in \mathbb{N}$ such that by playing σ_2 from any $s \in S$ for k_a steps, we reach $D_{\text{even}}^{\max}(p_a)$ with probability larger than $(1 - \varepsilon)^{\frac{1}{|\mathcal{A}|}}$ (there exists such a k_a for every $s \in S$, and as S is finite, taking the maximum of them works for every $s \in S$). We take $n = \sum_{a \in \mathcal{A}} k_a$. We consider a disjoint episodes labelled by i for every $a \in \mathcal{A}$, of a duration of k_a steps. For all $a \in \mathcal{A}$, during episode i we have a probability of visiting $D_{\text{even}}^{\max}(p_a)$ greater than $(1 - \varepsilon)^{\frac{1}{|\mathcal{A}|}}$. Since these episodes are independent, the probability that for all $a \in \mathcal{A}$, we visited $D_{\text{even}}^{\max}(p_a)$ during episode i is greater than $((1 - \varepsilon)^{\frac{1}{|\mathcal{A}|}})^{|\mathcal{A}|} = (1 - \varepsilon)$, and hence the desired property. \square

Definition 4.7. Let $C \in \mathcal{EC}_{\text{II}}(\Gamma)$. Let $(n_i)_{i \in \mathbb{N}}$ be a sequence of naturals n_i such that for all $i \in \mathbb{N}$ we have:

$$\Pr_{\Gamma_C^{|\sigma_2|}}[\{\rho \cdot \rho' \in \text{Paths}_{\Gamma_C^{|\sigma_2|}} \mid \rho \in \text{Fpaths}_{\Gamma_C^{|\sigma_2|}},$$

$$\forall a \in \mathcal{A}, \exists k_a < n_i, \rho(k_a) \in C_{\text{even}}^{\max}(p_a)\}] \geq 1 - 2^{-i}$$

whose existence is granted by Lemma 4.6. The strategy σ_C is defined as follows.

1. Play σ_2 for n_i steps. Then $i = i + 1$ and go to 2.
2. If for all $a \in \mathcal{A} : C_{\text{even}}^{\max}(p_a)$ was visited in phase 1, then go to 1.
Else, play σ_1 until all $C_{\text{even}}^{\max}(p_a)$ are reached, and then go to 1.

Strategy σ_C can be effectively constructed, as the construction of σ_1 comes from Section 3 and the construction of σ_2 comes from Lemma 4.4. The following lemma states that for all $s \in C$, the strategy σ_C indeed satisfies $\bigwedge(\mathcal{A}, \mathcal{AS})$:

Lemma 4.8. Let $C \in \mathcal{EC}_{\text{II}}(\Gamma, \mathcal{A}, \mathcal{AS})$. For all $s \in C$, it holds that $s, \sigma_C \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS})$.

Proof. Let us first look at the $\bigwedge_{a \in \mathcal{A}} A(p_a)$ condition. Each path π has to follow one of these three cases:

- Strategy σ_1 is only played a finite number of times, and for a finite duration: This means that eventually for some i_0 , in each round $i > i_0$, in episodes of n_i steps, $C_{\text{even}}^{\max}(p_a)$ was visited for all $a \in \mathcal{A}$. This also means that eventually only σ_2 is played and π stays in C , hence all p_a are satisfied on π .
- Strategy σ_1 is eventually played for an infinite duration without coming back to 1: By definition of σ_1 , path π satisfies all p_a .

- Strategy σ_1 and σ_2 are both played infinitely often: The only way to stop strategy σ_1 is to have visited all $C_{\text{even}}^{\max}(p_a)$. As σ_2 and σ_1 were both played infinitely often, σ_1 was stopped infinitely often, and so $C_{\text{even}}^{\max}(p_a)$ was visited infinitely often for all $a \in \mathcal{A}$. As π has to stay in C , it implies that π satisfies all p_a .

For the $\bigwedge_{as \in \mathcal{AS}} A(p_{as})$ conditions, we can prove that with probability 1, eventually only σ_2 is played. As σ_2 has itself probability 1 of ensuring all p_{as} , we get that σ_C satisfies $\bigwedge_{as \in \mathcal{AS}} A(p_{as})$. \square

Now we construct a strategy σ from σ_T and σ_C .

Definition 4.9. Based on strategies σ_T and σ_C for all $C \in \mathcal{EC}_{\text{II}}(\Gamma)$, we build the global strategy σ as follows.

1. Play σ_T until a Type II EC C is reached, then go to 2.
2. Play σ_C forever.

The following lemma concludes this direction of the proof of Lemma 4.2:

Lemma 4.10. It holds that $s_0, \sigma \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS})$.

Proof. First, we consider for all $a \in \mathcal{A}$ the objective $A(p_a)$. Along each run π consistent with σ , either (i) a Type II EC C is eventually reached and σ switches to σ_C , or (ii) σ behaves as σ_T forever. Since all strategies σ_C and strategy σ_T ensure $A(p_a)$ and the parity condition is prefix-independent, we have that $s_0, \sigma \models_{\Gamma} A(p_a)$.

Second, since with probability one, σ_T reaches some Type II EC C , in which σ_C ensures $AS(p_{as})$ for all $as \in \mathcal{AS}$. Again invoking prefix-independence, we conclude that $s_0, \sigma \models_{\Gamma} AS(p_{as})$, which ends our proof. \square

Now we sketch the proof of the left-to-right implication of Lemma 4.2. We make use of the following lemma.

Lemma 4.11. Given an MDP $\Gamma = (S, E, Act, Pr)$, and two sets of parity conditions $\{p_a \mid a \in \mathcal{A}\}$ and $\{p_{as} \mid as \in \mathcal{AS}\}$, for all states $s, s' \in S$, and for all strategies σ the following holds: if $s, \sigma \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS})$ and $s' \notin \llbracket \bigwedge(\mathcal{A}, \mathcal{AS}) \rrbracket$, then $s' \notin \text{Paths}_{\Gamma^{[\sigma]}}(s)$.

Proof. The proof follows from the fact that for a strategy σ in Γ that satisfies $A(p)$ (resp. $AS(p)$), for all finite paths π from s in $\Gamma^{[\sigma]}$, if π leads to a state \tilde{s} , then it holds that for the set of paths originating from \tilde{s} , we have that $A(p)$ (resp. $AS(p)$) is satisfied. The details of the proof is as follows:

Assume by contradiction that $s' \in \text{Paths}_{\Gamma^{[\sigma]}}(s)$. Since it holds that $s' \not\models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS})$, there can be two possibilities: either (i) for some $a \in \mathcal{A}$, we have $s' \not\models A(p_a)$ or (ii) for some $as \in \mathcal{AS}$, we have $s' \not\models AS(p_{as})$.

For (i), since the condition p_a is prefix-independent, we have that any path going through s' does not satisfy p_a , thus $s \not\models A(p_a)$, and hence the contradiction. For (ii), if $s' \not\models AS(p_{as})$, and since by assumption $s' \in \text{Paths}_{\Gamma^{[\sigma]}}(s)$, we have

that s' can be reached from s with non-zero probability in the MC $\Gamma^{[\sigma]}$, and thus $s, \sigma \models_{\Gamma} \text{AS}(p_{as})$ does not hold true, and hence the contradiction. \square

We now introduce the following definition.

Definition 4.12 (Density). Let $\Gamma = (S, E, \text{Act}, \text{Pr})$ be an MDP, $s \in S$ an initial state, σ a strategy, and $R \subseteq S$. We say that R is *dense* in σ from s if and only if for all $\rho \in \text{Fpaths}^{\Gamma^{[\sigma]}}(s)$, there exists ρ' such that $\rho \cdot \rho' \in \text{Fpaths}^{\Gamma^{[\sigma]}}(s)$ and $\text{Last}(\rho') \in R$. That is, after all prefixes in the tree $\text{Paths}^{\Gamma^{[\sigma]}}(s)$, there is a continuation that visits R .

Now we state the following lemma that uses the above definition.

Lemma 4.13. *Given an MDP $\Gamma = (S, E, \text{Act}, \text{Pr})$, a state $s \in S$, a set of parity conditions $\{p_a \mid a \in \mathcal{A}\}$, a set $R \subseteq S$, if there exists a strategy σ such that $s, \sigma \models_{\Gamma} \bigwedge_{a \in \mathcal{A}} \text{A}(p_a)$, and R is dense in σ from s , then $s \models_{G_{R, \{p_a \mid a \in \mathcal{A}\}}^{\Gamma}} \bigwedge_{a \in \mathcal{A}} \text{A}(p_a) \wedge \text{A}(\square \Diamond B)$, with $G_{R, \{p_a \mid a \in \mathcal{A}\}}^{\Gamma}$ the Streett-Büchi game defined in Section 3 where the Büchi condition is B .*

Proof. We construct a strategy σ' to play in $G_{R, \{p_a \mid a \in \mathcal{A}\}}^{\Gamma}$ from strategy σ , and we show that σ' is winning for the Büchi parity condition.

Let $\alpha: S' \rightarrow S \cup \{\varepsilon\}$ be a mapping from the states in $G_{R, \{p_a \mid a \in \mathcal{A}\}}^{\Gamma}$ to the states in Γ such that $\alpha(s) = s$ for all $s \in S$, and $\alpha(s, a, i) = \varepsilon$ for all $(s, a, i) \in (S \setminus R) \times \text{Act} \times \{0, 1\}$. Then we extend α to map prefixes in $G_{R, \{p_a \mid a \in \mathcal{A}\}}^{\Gamma}$ to prefixes in Γ .

Now, we label each prefix $\rho \in \text{Fpaths}^{\Gamma^{[\sigma]}}(s)$ by a finite continuation $\text{lab}(\rho) = \rho'$ such that (i) $\rho \cdot \rho' \in \text{Fpaths}^{\Gamma^{[\sigma]}}(s)$ and $\text{Last}(\rho') \in R$; and (ii) if $\rho' = \rho'_1 \cdot s \cdot \rho'_2$ is the label of ρ then $\text{lab}(\rho \cdot \rho'_1 \cdot s) = \rho'_2$. It should be clear that due to the density of R (as given in Definition 4.12), such a labelling is always possible.

Now, we define the strategy σ' from σ and this labelling:

- (i) For a prefix ρ in the game such that $\text{Last}(\rho) \in R$, the only possible choice is the only action available as R is absorbing in the game.
- (ii) For a prefix ρ such that $\text{Last}(\rho) \in S \cup (S \setminus R) \times \{\text{Act}\} \times \{0, 1\}$ and such that $\alpha(\rho)$ is not consistent with σ , we choose any $a \in \text{Act}$.
- (iii) For a prefix ρ such that $\text{Last}(\rho) \in S$ and such that $\alpha(\rho)$ is consistent with σ , we define $\sigma'(\rho) = \sigma(\alpha(\rho))$.
- (iv) For a prefix ρ such that $\text{Last}(\rho) \in (S \setminus R) \times \{\text{Act}\} \times \{1\}$ and such that $\alpha(\rho)$ is consistent with σ , then $\alpha(\rho)$ in the tree $\text{Paths}^{\Gamma^{[\sigma]}}(s)$ is labelled with a finite path ρ' that leads to a state in R , i.e., $\text{lab}(\alpha(\rho)) = \rho'$, then we define $\sigma'(\rho) = \text{First}(\rho')$.
- (iv) For a prefix ρ such that $\text{Last}(\rho) \in (S \setminus R) \times \{\text{Act}\} \times \{0\}$ and such that $\alpha(\rho)$ is consistent with σ , then we play the only action available.

We now show that σ' wins the Büchi B and parity p'_a conditions for all $a \in \mathcal{A}$ defined in the game $G_{R, \{p_a \mid a \in \mathcal{A}\}}^{\Gamma}$ from state s . First, for $a \in \mathcal{A}$ as σ is enforcing p_a in Γ , we have that p'_a replicates the priorities given by p_a , and absorbing states in R have even priorities for p'_a , it is clear that σ' enforces p'_a in the game $G_{R, \{p_a \mid a \in \mathcal{A}\}}^{\Gamma}$. Now, for the Büchi condition, we consider the following case study on the paths π consistent with σ' in the game.

- If π ends up in R , then it is clearly Büchi accepting.
- If π is such that after a finite prefix ρ , we always visit states $s' \in S \times \text{Act} \times \{1\}$, then according to the definition of σ' , the play will follow a finite path ρ' to a state in R , and so π reaches R and as a consequence π is Büchi accepting.
- Finally, if π is such that it is eventually never the case that we visit any $s' \in S \times \text{Act} \times \{1\}$, then we visit infinitely often the copies $s' \in S \times \text{Act} \times \{0\}$, and because all $(s, a, 0) \in B$, we have that π is also Büchi accepting.

So in all cases, play π satisfies the Büchi condition, and we are done.

Let $\alpha: S' \rightarrow S \cup \{\varepsilon\}$ be a mapping from the states in $G_{R, \{p_a \mid a \in \mathcal{A}\}}^{\Gamma}$ to the states in Γ such that $\alpha(s) = s$ for all $s \in S$, and $\alpha(s, a, i) = \varepsilon$ for all $(s, a, i) \in (S \setminus R) \times \text{Act} \times \{0, 1\}$. Then we extend α to map prefixes in $G_{R, \{p_a \mid a \in \mathcal{A}\}}^{\Gamma}$ to prefixes in Γ .

Now, we label each prefix $\rho \in \text{Fpaths}^{\Gamma^{[\sigma]}}(s)$ by a finite continuation $\text{lab}(\rho) = \rho'$ such that (i) $\rho \cdot \rho' \in \text{Fpaths}^{\Gamma^{[\sigma]}}(s)$ and $\text{Last}(\rho') \in R$; and (ii) if $\rho' = \rho'_1 \cdot s \cdot \rho'_2$ is the label of ρ then $\text{lab}(\rho \cdot \rho'_1 \cdot s) = \rho'_2$. It should be clear that due to the density of R (as given in Definition 4.12), such a labelling is always possible.

Now, we define the strategy σ' from σ and this labelling:

- (i) For a prefix ρ in the game such that $\text{Last}(\rho) \in R$, the only possible choice is the only action available as R is absorbing in the game.
- (ii) For a prefix ρ such that $\text{Last}(\rho) \in S \cup (S \setminus R) \times \{\text{Act}\} \times \{0, 1\}$ and such that $\alpha(\rho)$ is not consistent with σ , we choose any $a \in \text{Act}$.
- (iii) For a prefix ρ such that $\text{Last}(\rho) \in S$ and such that $\alpha(\rho)$ is consistent with σ , we define $\sigma'(\rho) = \sigma(\alpha(\rho))$.
- (iv) For a prefix ρ such that $\text{Last}(\rho) \in (S \setminus R) \times \{\text{Act}\} \times \{1\}$ and such that $\alpha(\rho)$ is consistent with σ , then $\alpha(\rho)$ in the tree $\text{Paths}^{\Gamma^{[\sigma]}}(s)$ is labelled with a finite path ρ' that leads to a state in R , i.e., $\text{lab}(\alpha(\rho)) = \rho'$, then we define $\sigma'(\rho) = \text{First}(\rho')$.
- (iv) For a prefix ρ such that $\text{Last}(\rho) \in (S \setminus R) \times \{\text{Act}\} \times \{0\}$ and such that $\alpha(\rho)$ is consistent with σ , then we play the only action available.

We now show that σ' wins the Büchi B and parity p'_a conditions for all $a \in \mathcal{A}$ defined in the game $G_{R, \{p_a \mid a \in \mathcal{A}\}}^{\Gamma}$ from state s . First, for $a \in \mathcal{A}$ as σ is enforcing p_a in Γ , we have that p'_a replicates the priorities given by p_a , and absorbing states

in R have even priorities for p'_a , it is clear that σ' enforces p'_a in the game $G_{R, \{p_a \mid a \in \mathcal{A}\}}^\Gamma$. Now, for the Büchi condition, we consider the following case study on the paths π consistent with σ' in the game.

- If π ends up in R , then it is clearly Büchi accepting.
- If π is such that after a finite prefix ρ , we always visit states $s' \in S \times Act \times \{1\}$, then according to the definition of σ' , the play will follow a finite path ρ' to a state in R , and so π reaches R and as a consequence π is Büchi accepting.
- Finally, if π is such that it is eventually never the case that we visit any $s' \in S \times Act \times \{1\}$, then we visit infinitely often the copies $s' \in S \times Act \times \{0\}$, and because all $(s, a, 0) \in B$, we have that π is also Büchi accepting.

So in all cases, play π satisfies the Büchi condition, and we are done. \square

Lemma 4.11 implies that for all initial state s' , in $\Gamma_{|C}^{[\sigma]}$, after all finite path ρ beginning from some state (q', s') , and ending in (q'', s'') it holds that $(q'', s''), \sigma \models_\Gamma \bigwedge (\mathcal{A}, \mathcal{AS})$. Thus after every finite path, there exists a continuation that visits $\mathcal{T}_{\Pi, \Gamma, \mathcal{A}, \mathcal{AS}}$, hence $\mathcal{T}_{\Pi, \Gamma, \mathcal{A}, \mathcal{AS}}$ is dense from s' in σ , and so by Lemma 4.13, Lemma 3.2 and Lemma 3.3, we have that $s_0 \models_\Gamma \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\diamond \mathcal{T}_{\Pi, \Gamma, \mathcal{A}, \mathcal{AS}})$. Recall that s_0 is the initial state in Lemma 4.2. We detail below why $\mathcal{T}_{\Pi, \Gamma, \mathcal{A}, \mathcal{AS}}$ is non-empty.

Lemma 4.14. *If $s_0 \models_\Gamma \bigwedge (\mathcal{A}, \mathcal{AS})$ then $\mathcal{EC}_{\Pi}(\Gamma) \neq \emptyset$.*

Proof. We recall that given an MDP Γ , an initial state s , a strategy σ and a set of paths $\Pi \subseteq \text{Paths}^{\Gamma^{[\sigma]}}(s)$, we define

$$\text{States}(\Pi) = \{s \in S \mid \exists \pi \in \Pi, \exists n \in \mathbb{N}_0, \pi(n) = s\}.$$

Now consider the following set \mathcal{S} of subsets of S :

$$\mathcal{S} = \{R \subseteq S \mid \exists s \in S, \exists \sigma \text{ a strategy,}$$

$$(s, \sigma \models_\Gamma \bigwedge (\mathcal{A}, \mathcal{AS})) \wedge (R = \text{States}(\text{Paths}^{\Gamma^{[\sigma]}}(s)))\}.$$

We have that for each $R \in \min_{\subseteq}(\mathcal{S})$, the following properties hold:

1. R is an EC in Γ ,
2. $\forall s \in R, s \models_{\Gamma|_R} \bigwedge_{a \in \mathcal{A}} (A(p_a) \wedge AS(\diamond R_{\text{even}}^{\max}(p_a)))$,
3. $\forall s \in R, s \models_{\Gamma|_R} \bigwedge_{a \in \mathcal{A}} AS(p_a) \wedge AS(p_{as})$.

Before proving these three items, we claim that for all $R \in \min_{\subseteq}(\mathcal{S})$, and $s \in R$, there exists a strategy σ_R such that $s, \sigma_R \models_{\Gamma|_R} \bigwedge (\mathcal{A}, \mathcal{AS})$, i.e., σ_R satisfies the property without leaving R . This is a direct consequence of Lemma 4.11 and the minimality of R in \mathcal{S} for the \subseteq order. We use strategy σ_R in the rest of the proof.

Item 1). We first prove that R is strongly connected. By contradiction, assume it is not the case, i.e., that there exist $s, s' \in R$ such that there is no path in R from s to s' . Then, let R' be the set of states reachable with strategy σ_R from a prefix ρ ending in s . By Lemma 4.11, we have that $R' \in \mathcal{S}$.

But as there is no path from s to s' in R , we have that $s' \notin R'$ and $R' \subsetneq R$. This contradicts the minimality of R , hence we conclude that R is strongly connected. Then, for all state s of R , and $a \in Act(s)$, we have that $\text{Post}(s, a) \neq \emptyset$, as R is strongly connected. Hence, it remains to show that for all states $s \in R$, there exists an action a such that we have $\text{Post}(s, a) \subseteq R$. By contradiction, fix some $s \in R$, and assume that for all action a there exists $s_a \notin R$ such that $(s, s_a) \in E$. As R belongs to \mathcal{S} , recall that $R = \text{States}(\text{Paths}^{\Gamma^{[\sigma]}}(s''))$ for some strategy σ and state s'' . Since $s \in R$, there exists a prefix $\rho \in \text{Fpaths}^{\Gamma^{[\sigma]}}(s'')$ such that $\text{Last}(\rho) = s$. But then, for some a , prefix $\rho' = \rho \cdot s_a$ also belongs to $\text{Fpaths}^{\Gamma^{[\sigma]}}(s'')$, and $s_a \in R$. Thus, we conclude that R is indeed an EC in Γ .

Item 2). We fix any $s \in R$, and we prove that $s \models_{\Gamma|_R} \bigwedge_{a \in \mathcal{A}} (A(p_a) \wedge AS(\diamond R_{\text{even}}^{\max}(p_a)))$. As seen above, from the minimality of R and Lemma 4.11, we know that $s, \sigma_R \models_{\Gamma|_R} \bigwedge (\mathcal{A}, \mathcal{AS})$. Now, again from the minimality of R in \mathcal{S} , we know that in the subtree induced by $\text{Paths}^{\Gamma^{[\sigma_R]}}(s)$, every non-empty subset $R' \subseteq R$ is dense, otherwise, there would exist some $r \in R$ such that a subtree of $\text{Paths}^{\Gamma^{[\sigma_R]}}(s)$ only visits $R \setminus r$, and this subtree would itself define a winning strategy for $\bigwedge \mathcal{A} \mathcal{AS}$, breaking the minimality assumption. As a consequence, for all prefixes ρ , there exists a path beginning with ρ that eventually reaches a state of R' . Using the reduction to a parity-Büchi game which underlies Lemma 3.2, we deduce from the density argument presented in Lemma 4.13 that we can build σ from σ_R such that $s, \sigma \models_{\Gamma|_R} \bigwedge_{a \in \mathcal{A}} (AS(p_a) \wedge AS(\diamond R_{\text{even}}^{\max}(p_a)))$. It remains to argue that $R_{\text{even}}^{\max}(p_a)$ is non-empty to prove this item. This is necessarily true, otherwise $R_{\text{odd}}^{\max}(p_a)$ would be non-empty, and σ_R would not ensure $A(p_a)$ in R (as $R_{\text{odd}}^{\max}(p_a)$ would be a dense subset).

Item 3). This is trivial as, for $s \in R$, we have that σ_R enforces $s, \sigma_R \models_{\Gamma|_R} \bigwedge (\mathcal{A}, \mathcal{AS})$, a stronger property. \square

Finally, we state the following lemma. It refines Lemma 4.14 by showing that some Type II EC of $\mathcal{EC}_{\Pi}(\Gamma)$ can be reached almost-surely while satisfying $\bigwedge_{a \in \mathcal{A}} (A(p_a))$. We define

$\mathcal{T}_{\Pi, \Gamma, \mathcal{A}, \mathcal{AS}}^{\min} = \cup_{R \in \min_{\subseteq}(\mathcal{S})} R$, that is the set of all states that belong to a minimal set R of \mathcal{S} .

Lemma 4.15. *If $s_0 \models_\Gamma \bigwedge (\mathcal{A}, \mathcal{AS})$ then $s_0 \models_\Gamma \bigwedge_{a \in \mathcal{A}} (A(p_a) \wedge AS(\diamond \mathcal{T}_{\Pi, \Gamma, \mathcal{A}, \mathcal{AS}}^{\min}))$.*

Proof. Let σ be a witness for $s_0 \models_\Gamma \bigwedge (\mathcal{A}, \mathcal{AS})$. By Lemma 4.14, we know that $\mathcal{EC}_{\Pi}(\Gamma)$ is non-empty, and so is $\mathcal{T}_{\Pi, \Gamma, \mathcal{A}, \mathcal{AS}}^{\min}$. Furthermore, we claim that $\mathcal{T}_{\Pi, \Gamma, \mathcal{A}, \mathcal{AS}}^{\min}$ is dense in the tree induced by $\text{Paths}^{\Gamma^{[\sigma]}}(s_0)$. Indeed, by Lemma 4.11, after every prefix $\rho \in \text{Fpaths}^{\Gamma^{[\sigma]}}(s_0)$, the following property holds: $\text{Last}(\rho), \sigma[\rho] \models_\Gamma \bigwedge (\mathcal{A}, \mathcal{AS})$ (hence $\mathcal{T}_{\Pi, \Gamma, \mathcal{A}, \mathcal{AS}}^{\min}$ is reached, repeating the previous arguments). Since this holds for all prefixes of $\text{Fpaths}^{\Gamma^{[\sigma]}}(s_0)$, we have that $\mathcal{T}_{\Pi, \Gamma, \mathcal{A}, \mathcal{AS}}^{\min}$ is indeed

dense in the tree of σ . Hence, again using the density argument presented in Lemma 4.13, we can build σ' from σ such that $s_0, \sigma' \models_{\Gamma} \bigwedge_{a \in \mathcal{A}} (A(p_a) \wedge AS(\diamond \mathcal{T}_{II, \Gamma, \mathcal{A}, \mathcal{AS}}^{\min}))$. \square

We end this section with the following observation. Lemma 4.11 implies that winning strategies only visit states belonging to $\llbracket \bigwedge(\mathcal{A}, \mathcal{AS}) \rrbracket$. As a consequence, pruning the states that do not satisfy $\bigwedge(\mathcal{A}, \mathcal{AS})$ does not affect correctness. We use this pruned MDP in the rest of the paper. We state this formally:

Assumption 4.1. *For every state s of Γ , we have:*
 $s \in \llbracket \bigwedge(\mathcal{A}, \mathcal{AS}) \rrbracket$.

We detail how to do this pruning in Section 7, we use Lemma 4.2 to find the states that do not satisfy the objective.

5 Type III end-components

In this section, we define Type III ECs. These end-components are used to characterize the winning strategies for formulas of the form $\bigwedge(\mathcal{A}, \mathcal{AS}) \wedge NZ(p_{nz})$. To do so, we show in Lemma 5.2 that all Type III ECs satisfy such a formula. In Lemma 5.4, we use the previous lemma and the technical Lemma 5.3 to relate the formula $\bigwedge(\mathcal{A}, \mathcal{AS}) \wedge NZ(p_{nz})$ to the almost-sure reachability of the set of Type III ECs under the constraint $\bigwedge(\mathcal{A}, \mathcal{AS})$. We explain in Section 6 how to compute reachability of a set of states under the constraint $\bigwedge(\mathcal{A}, \mathcal{AS})$, and in Section 7 we explain how to compute the set of Type III ECs.

Given two sets of parity conditions $\{p_a \mid a \in \mathcal{A}\}$, $\{p_{as} \mid as \in \mathcal{AS}\}$ and another parity condition p_{nz} , an end-component C of Γ is Type III $(\mathcal{A}, \mathcal{AS}, p_{nz})$ if the following two properties hold:

- **(III₁)** $\forall s \in C, s \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS})$;
- **(III₂)** $\forall s \in C, s \models_{\Gamma|C} \bigwedge_{a \in \mathcal{A}} AS(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as}) \wedge AS(p_{nz})$

We write Type III $(\mathcal{A}, \mathcal{AS}, p_{nz})$ EC as Type III ECs when the parity sets are clear from the context. We note that condition **(III₁)** may require an infinite memory strategy (see Lemma 4.2), and can always be satisfied in the pruned MDP, due to Assumption 4.1. Condition **(III₁)** can be checked using [16]. Note that condition **(III₂)** is only about the parity conditions indexed by \mathcal{A} and \mathcal{AS} , and must hold while staying inside the end-component C , but the witness strategy for **(III₁)** may leave C . This notion strengthens in a non-trivial way the notion of *very-good end-component* (VGEC in [4]) which are Type III ECs where \mathcal{A} is a singleton, $\mathcal{AS} = \emptyset$, and the p_{nz} stays as it is. From condition **(III₁)** and Lemma 4.2 we know that if there exists a Type III EC in the MDP Γ then there also exists a Type II EC. We introduce the following notations: $\mathcal{EC}_{III}(\Gamma, \mathcal{A}, \mathcal{AS}, p_{nz})$ is the set of all Type III $(\mathcal{A}, \mathcal{AS}, p_{nz})$ ECs of Γ , and $\mathcal{T}_{III, \mathcal{A}, \mathcal{AS}, p_{nz}} = \bigcup_{C \in \mathcal{EC}_{III}(\Gamma, \mathcal{A}, \mathcal{AS}, p_{nz})} C$ is the set of states belonging to some Type III EC in Γ .

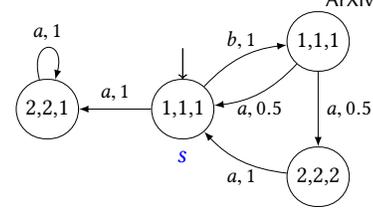


Figure 9. An example of a Type III EC

In the sequel, we relate Type III ECs to the formula $\bigwedge(\mathcal{A}, \mathcal{AS}) \wedge NZ(p_{nz})$. In particular, from all states belonging to a Type III ECs there exists a strategy satisfying the formula $\bigwedge(\mathcal{A}, \mathcal{AS}) \wedge NZ(p_{nz})$. We illustrate this by the following example.

Example 5.1. Consider the example in Figure 9. We show a strategy satisfying $A(p_1) \wedge AS(p_2) \wedge NZ(p_3)$ in an MDP that is also a Type III EC: All the states satisfy condition **(III₁)** when action a is chosen from state s , and all the states satisfy condition **(III₂)** when action b is chosen from state s .

Now the strategy from s to satisfy $A(p_1) \wedge AS(p_2) \wedge NZ(p_3)$ is the following. In *round 1*, action b is chosen some i_0 times. If the state with parity value $(2, 2, 2)$ is visited in round 1, then proceed to round 2. Otherwise action a is chosen from state s at the end of round 1. In round 2, action b is chosen $i_1 = i_0 + 1$ times and so on, resulting in action b being chosen $i_j = i_0 + j$ times in round j . From every round, we proceed to the next round if the state with parity value $(2, 2, 2)$ is visited in the current round, or otherwise we switch to playing action a from state s . Now we compute the probability of never switching to action a in state s . The probability of not choosing action a from s after the first round is $1 - 2^{-i_0}$. The probability of not choosing a from s after the first round and as well as after the second round is $(1 - 2^{-i_0}) \cdot (1 - 2^{-(i_0+1)})$, and thus the probability of not choosing a from s after each of the first n rounds is $p(n) = \prod_{j=0}^{n-1} (1 - 2^{-(i_0+j)})$. In [4], it has been shown that $\lim_{n \rightarrow \infty} p(n) > 0$, implying that with non-zero probability, action a will never be played in s . Hence with non-zero probability p_3 holds.

Lemma 5.2. *Given an MDP Γ , and a Type III EC C in Γ , for all states $s \in C$, we have $s \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS}) \wedge NZ(p_{nz})$.*

Proof. Consider some $\varepsilon > 0$, and $f: \mathbb{N}_0 \rightarrow \mathbb{Q} \cap (0, 1]$ a series of probabilities such that the infinite product $\prod_{i \in \mathbb{N}_0} f(i) > 1 - \varepsilon$. Let σ_1 be a strategy satisfying **(III₁)**, and σ_2 be a strategy satisfying **(III₂)**. Using Proposition 4.5, we associate with σ_2 a sequence of numbers $g: \mathbb{N}_0 \rightarrow \mathbb{N}_0$ such that if σ_2 is played for $g(i)$ steps then for all $a \in \mathcal{A}$ we have that $C_{\text{even}}^{\max}(p_a)$ is visited with probability at least $f(i)$. We consider the following infinite-memory strategy σ by combining strategies σ_1 and σ_2 as follows:

- a) Let $i = 0$
- b) Play σ_2 for $g(i)$ steps. Let $i = i + 1$.
- c) if for all $a \in \mathcal{A}$ we have $C_{\text{even}}^{\max}(p_a)$ was visited, then go to b), else play σ_1 forever.

When following σ , in each round i , we have probability at least $f(i)$ of continuing to play σ_2 . The probability of playing σ_2 forever is thus the same as the probability of visiting all $C_{\text{even}}^{\max}(p_a)$ in each round, that is, at least $1 - \varepsilon$, and thus satisfying the parity conditions p_{as} for $as \in \mathcal{AS}$ and p_{nz} with probability $1 - \varepsilon$. In all the paths where σ_2 keeps being played, for all $a \in \mathcal{A}$ we have that p_a is satisfied since $C_{\text{even}}^{\max}(p_a)$ is visited infinitely often. For the plays switching to σ_1 at some point (we have probability ε to switch to one of these plays at some point), we have $\bigwedge(\mathcal{A}, \mathcal{AS})$. This implies that considering all possibilities, we have that all conditions p_{as} are satisfied with probability $1 - \varepsilon + \varepsilon = 1$, and that all possible plays satisfy all p_a . Finally, p_{nz} is satisfied with probability $1 - \varepsilon$, and hence with probability greater than 0. Thus for $s \in C$, we have $s, \sigma \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS}) \wedge \text{NZ}(p_{nz})$. \square

We now relate the existence of a Type III end component with the $\bigwedge_{a \in \mathcal{A}} A(p_a)$ objective. We begin with the following observation.

Lemma 5.3. *Given an end-component C in an MDP Γ , a set of parity conditions $\{p_x \mid x \in X\}$, for all states $s \in C$, we have $s \models_{\Gamma|C} \text{AS}(\bigwedge_{x \in X} p_x)$ iff $s \models_{\Gamma|C} \text{NZ}(\bigwedge_{x \in X} p_x)$.*

Proof. The left to right implication is obvious. For the right to left implication, consider σ such that $s, \sigma \models_{\Gamma|C} \text{NZ}(\bigwedge_{x \in X} p_x)$.

By Proposition 2.1, when playing strategy σ from s there is probability 1 that the set of states visited infinitely often is an end-component. We consider sets of states, such that every set forms an end-component with non-zero probability when playing σ from s . We call this set of sets of states C . Formally $C = \{D \mid \Pr(\{\pi \in \text{Paths}^{\Gamma[\sigma]}(s) \mid \inf(\text{proj}_S(\pi)) = D\}) > 0\}$. There exists at least one sub-EC $D_X \in C$ of C such that for all $x \in X$, we have $\max(\{p_x(s) \mid s \in D_X\})$ is even. We define a strategy σ' as follows: since for reachability objective in an MDP a deterministic memoryless strategy suffices, we use such a deterministic memoryless strategy inside C until we reach a state $s \in D_X$; then we play a uniform randomized strategy that has probability 1 of visiting all states of D_X while staying inside D_X . We have $s, \sigma' \models_{\Gamma|C} \text{AS}(\bigwedge_{x \in X} p_x)$, and the result follows. \square

Now we state the main result of this section that relates the existence of a Type III end component to the objective $\bigwedge(\mathcal{A}, \mathcal{AS}) \wedge \text{NZ}(p_{nz})$.

Lemma 5.4. *Given an MDP Γ , two sets of parity conditions $\{p_a \mid a \in \mathcal{A}\}$, $\{p_{as} \mid as \in \mathcal{AS}\}$, and another parity condition p_{nz} , for all states s , we have $s \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS}) \wedge \text{NZ}(p_{nz})$ iff $s \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS}) \wedge \text{NZ}(\diamond \mathcal{T}_{\text{III}, \mathcal{A}, \mathcal{AS}, p_{nz}})$.*

Proof. We begin with the right to left implication. Consider a strategy σ_s such that $s, \sigma_s \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS}) \wedge \text{NZ}(\diamond \mathcal{T}_{\text{III}, \mathcal{A}, \mathcal{AS}, p_{nz}})$. We show below that $s \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS}) \wedge \text{NZ}(p_{nz})$. First note

that for all states $q \in \mathcal{T}_{\text{III}, \mathcal{A}, \mathcal{AS}, p_{nz}}$, we consider a strategy σ_q such that $q, \sigma_q \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS}) \wedge \text{NZ}(p_{nz})$ (we know that such a strategy always exists in a Type III EC, thanks to Lemma 5.2). Now we construct a strategy σ , such that $s, \sigma \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS}) \wedge \text{NZ}(p_{nz})$. Strategy σ is defined from σ_s and σ_q as follows: We play σ_s from s . If we reach a state q belonging to a Type III EC, we play σ_q from q forever. Since such a state q can be reached with non-zero probability, and strategy σ_q satisfies p_{nz} with non-zero probability, we have that σ satisfies p_{nz} with non-zero probability. As both σ_s and σ_q satisfy $\bigwedge(\mathcal{A}, \mathcal{AS})$, and thus, while following σ , since every path in the corresponding MC ends up either following σ_s forever or following σ_q forever, we have that σ also satisfies $\bigwedge(\mathcal{A}, \mathcal{AS})$.

Now for the left to right implication, let σ be a strategy such that $s, \sigma \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS}) \wedge \text{NZ}(p_{nz})$. It is easy to see that $s, \sigma \models_{\Gamma} \text{NZ}(\bigwedge_{a \in \mathcal{A}} (p_a) \wedge \bigwedge_{as \in \mathcal{AS}} (p_{as}) \wedge (p_{nz}))$. From Proposition 2.1, there is probability 1 that an infinite path ends up in an end-component. Hence in the Markov chain $\Gamma^{[\sigma]}$ there is a non-zero probability that an infinite path will reach an end-component C such that for all states $s' \in C$, we have $s', \sigma \models_{\Gamma|C} \text{NZ}(\bigwedge_{a \in \mathcal{A}} (p_a) \wedge \bigwedge_{as \in \mathcal{AS}} (p_{as}) \wedge (p_{nz}))$. From Lemma 5.3, we thus have that for all $s' \in C$, there exists σ' such that $s', \sigma' \models_{\Gamma|C} \text{AS}(\bigwedge_{a \in \mathcal{A}} (p_a) \wedge \bigwedge_{as \in \mathcal{AS}} (p_{as}) \wedge (p_{nz}))$. Thus condition (III₂) is satisfied. As we consider a pruned MDP thanks to Assumption 4.1, for all $s' \in C$ we have that $s' \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS})$.

Thus C is a Type III EC that can be reached from s with non-zero probability, and thus $s, \sigma \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS}) \wedge \text{NZ}(\diamond \mathcal{T}_{\text{III}, \mathcal{A}, \mathcal{AS}, p_{nz}})$. \square

6 Formulas with multiple Non-Zero and multiple Exists

In this section, we discuss how to compute strategies for formulas that consist of several sure parity objectives, several almost-sure parity objectives, several non-zero parity objectives, and several existential parity objectives. We show in Lemma 6.1 that such a formula can be split into sub-formulas having a single non-zero or a single existential parity objective. Further, we show in Lemma 6.3 that a single non-zero parity objective can be transformed into an existential parity objective. We finally show in Lemma 6.5 how to check the satisfiability of a formula that consists of several sure parity objectives, several almost-sure parity objectives, and one existential parity objective.

Lemma 6.1. *Given an MDP Γ , a state s , four sets of parity conditions $\{p_a \mid a \in \mathcal{A}\}$, $\{p_{as} \mid as \in \mathcal{AS}\}$, $\{p_{nz} \mid nz \in \mathcal{NZ}\}$, and $\{p_e \mid e \in \mathcal{E}\}$, the following holds: $s \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS}) \wedge$*

$\bigwedge_{nz \in \mathcal{NZ}} \text{NZ}(p_{nz}) \wedge \bigwedge_{e \in \mathcal{E}} E(p_e)$ iff for all $nz \in \mathcal{NZ}$ we have $s \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS}) \wedge \text{NZ}(p_{nz})$, and for all $e \in \mathcal{E}$, we have $s \models_{\Gamma} \bigwedge(\mathcal{A}, \mathcal{AS}) \wedge E(p_e)$

Proof. As the left to right implication is obvious, we prove here the other direction. For $i \in \mathcal{NZ} \cup \mathcal{E}$, we consider a strategy σ_i such that $s, \sigma_i \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge Q_i(p_i)$ for the appropriate $Q_i \in \{\mathcal{NZ}, \mathcal{E}\}$. Now we construct a randomized strategy σ given all σ_i in which each σ_i is chosen uniformly, that is with equal probability. Clearly, $s \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \bigwedge_{nz \in \mathcal{NZ}} \mathcal{NZ}(p_{nz}) \wedge \bigwedge_{e \in \mathcal{E}} \mathcal{E}(p_e)$, and hence the result. \square

We now show that a non-zero objective can be replaced with an existential parity objective. Towards this, we first observe the following.

Proposition 6.2. *Every reachability condition can be translated to a parity condition.*

Proof. Given an MDP $\Gamma = (S, E, Act, Pr)$, we construct an MDP $\Gamma' = (S', E', Act, Pr')$ such that $S' = S \times \{1, 2\}$ where intuitively Γ' consists of two copies of Γ with state space $S \times \{1\}$ and $S \times \{2\}$ respectively, and the reachability condition being satisfied corresponds to moving from the first copy to the second copy, and staying there forever. Formally, we consider the parity condition p such that for $s' \in S'$, with $s' = (s, i)$ for $s \in S$ and $i \in \{1, 2\}$, we have $p(s') = i$. \square

We use this Γ' in the following two lemmas.

Lemma 6.3. *Given an MDP Γ , two sets of parity conditions $\{p_a \mid a \in \mathcal{A}\}$ and $\{p_{as} \mid as \in \mathcal{AS}\}$, and a reachability set R , there exists a state s' in Γ' , and a parity condition denoted $p_{\diamond R}$ such that for all states s , we have $s \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \mathcal{NZ}(\diamond R)$ iff $s' \models_{\Gamma'} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \mathcal{E}(p_{\diamond R})$.*

Proof. Let $s \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \mathcal{NZ}(\diamond R)$. As a reachability condition is satisfied for \mathcal{NZ} iff it is satisfied for \mathcal{E} , we want to satisfy $s \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \mathcal{E}(\diamond R)$. We now use Proposition 6.2 and denote $p_{\diamond R}$ the parity condition associated to $\diamond R$, we get that it is necessary and sufficient to find a strategy for $s' \models_{\Gamma'} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \mathcal{E}(p_{\diamond R})$. \square

We get from Lemma 5.4 and Lemma 6.3 the following:

Lemma 6.4. *Given an MDP Γ , and two sets of parity conditions $\{p_a \mid a \in \mathcal{A}\}$ and $\{p_{as} \mid as \in \mathcal{AS}\}$, and a parity condition p_{nz} , for all states s , we have $s \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \mathcal{NZ}(p_{nz})$ iff there exists s' in Γ' such that $s' \models_{\Gamma'} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \mathcal{E}(p_{\diamond \mathcal{T}_{III, nz}})$, where $\mathcal{T}_{III, nz}$ is defined w.r.t. Γ' .*

Now, since by Assumption 4.1 we have removed all the states that do not satisfy $\bigwedge (\mathcal{A}, \mathcal{AS})$, we have the following:

Lemma 6.5. *Given an MDP Γ , a state s , two sets of parity conditions $\{p_a \mid a \in \mathcal{A}\}$ and $\{p_{as} \mid as \in \mathcal{AS}\}$, and another parity condition p , we have that $s \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \mathcal{E}(p)$ iff $s \models_{\Gamma} \mathcal{E}(\bigwedge_{a \in \mathcal{A}} p_a \wedge p)$.*

Proof. The left to right result is obvious as $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \mathcal{E}(p)$ implies $\mathcal{E}(\bigwedge_{a \in \mathcal{A}} p_a \wedge p)$.

For the other direction, consider a strategy $\widehat{\sigma}$ such that $s, \widehat{\sigma} \models_{\Gamma} \mathcal{E}(\bigwedge_{a \in \mathcal{A}} p_a \wedge p)$. Now, a conjunction of parity conditions is a Streett condition [14], and non-emptiness problem of Streett automaton is decidable.

We noted in Section 2 that satisfying existentially is the same as finding a satisfying path in the nondeterministic automaton. This means that if the Streett automaton is non-empty, then there exists a finite-memory strategy σ (linear in the indices of p_a and p) in Γ such that there exists a path π in the MC $\Gamma^{[\sigma]}$ satisfying both p and all p_a for $a \in \mathcal{A}$.

Now by assumption, there exists a strategy σ_{\wedge} such that $s, \sigma_{\wedge} \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS})$. A strategy σ such that $s, \sigma \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \mathcal{E}(p)$ is obtained below by combining $\widehat{\sigma}$ and σ_{\wedge} as follows. At each step a coin is tossed. If it gives head, then we play σ_{\wedge} forever. Otherwise, we play this step as specified by the strategy $\widehat{\sigma}$. If this results in deviating from the path π , then we play σ_{\wedge} forever, else repeat the same process.

Strategy σ has a path ensuring p : the one where we always follow $\widehat{\sigma}$, that happens when all the coin tosses give tails, and the state randomly taken in the MDP is always in π . The probability of switching to σ_{\wedge} some time is 1, thus satisfying $\bigwedge_{as \in \mathcal{AS}} \mathcal{AS}(p_{as})$. If we follow the path π , then for all $a \in \mathcal{A}$ we have that p_a is satisfied, otherwise we switch to σ_{\wedge} at some point, and for all $a \in \mathcal{A}$ we again have that p_a is satisfied, thus ensuring $\bigwedge_{a \in \mathcal{A}} A(p_a)$. \square

This concludes the decidability proof of the realizability of the negation and disjunction-free fragment of QPL. Indeed, given a formula $s \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \bigwedge_{nz \in \mathcal{NZ}} \mathcal{NZ}(p_{nz}) \wedge \bigwedge_{e \in \mathcal{E}} \mathcal{E}(p_e)$, we use Lemma 6.1 to split it into formulas of the form $s \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \mathcal{NZ}(p_{nz})$, and of the form $s \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \mathcal{E}(p_e)$. We then apply Lemma 5.4 on formulas of the form $s \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \mathcal{NZ}(p_{nz})$, and then use Lemma 6.4 to transform the non-zero objective into an existential objective. For formulas of the form $s \models_{\Gamma} \bigwedge (\mathcal{A}, \mathcal{AS}) \wedge \mathcal{E}(p_e)$, we use Lemma 6.5.

By Remark 2.1 it shows the decidability of the QPL-realizability problem. We state the complexity of this realizability problem in the next Section.

7 Complexity results

In this section, we analyze the complexity of deciding the existence of strategies to satisfy QPL formulas. Recall from the results of Sections 5 and 6, that in order to find strategies for formulas of the form $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} \mathcal{AS}(p_{as}) \wedge$

$\bigwedge_{nz \in \mathcal{NZ}} \mathcal{NZ}(p_{nz}) \wedge \bigwedge_{e \in \mathcal{E}} \mathcal{E}(p_e)$, we need to compute the set of maximal Type III ECs. In particular, we need these ECs for subformulas of the form $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} \mathcal{AS}(p_{as}) \wedge \bigwedge_{nz \in \mathcal{NZ}} \mathcal{NZ}(p_{nz})$.

This in turn requires solving several Streett games in general (Lemma 3.5). The procedure is described in Algorithm

Algorithm 5

Input : An MDP Γ , s a state of Γ , parity conditions $\{p_a \mid a \in \mathcal{A}\}$, $\{p_{as} \mid as \in \mathcal{AS}\}$, $\{p_{nz} \mid nz \in \mathcal{NZ}\}$, and $\{p_e \mid e \in \mathcal{E}\}$.

Output : true if

$s \models_{\Gamma} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as}) \wedge \bigwedge_{nz \in \mathcal{NZ}} NZ(p_{nz}) \wedge \bigwedge_{e \in \mathcal{E}} E(p_e)$
else false.

- 1: Compute the set of maximal Type I $(\mathcal{A}, \mathcal{A})$ ECs C such that $\forall s' \in C, s' \models_{\Gamma|C} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(\diamond C_{\text{even}}^{\max}(p_a))$. //

By Lemma 3.5.

- 2: Compute the set of maximal Type II $(\mathcal{A}, \mathcal{AS})$ ECs C : C is a maximal Type I $(\mathcal{A}, \mathcal{AS})$ EC and $\forall s' \in C, s' \models_{\Gamma|C} \bigwedge_{a \in \mathcal{A}} AS(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$.

- 3: Compute the set \mathcal{S}_1 of states s' such that $s' \models_{\Gamma} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\diamond \mathcal{T}_{II, \Gamma, \mathcal{A}, \mathcal{AS}})$. // Correct by Lemma 4.2.
If $s \notin \mathcal{S}_1$ then return false.

- 4: Compute $\Gamma \upharpoonright \mathcal{S}_1$ where all the states that do not satisfy $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\diamond \mathcal{T}_{II, \Gamma, \mathcal{A}, \mathcal{AS}})$ have been pruned. // Correct by Lemma 4.11.

- 5: **for all** $nz \in \mathcal{NZ}$ **do**

- 6: Compute the set of maximal Type III $(\mathcal{A}, \mathcal{AS}, p_{nz})$ ECs C of $\Gamma \upharpoonright \mathcal{S}_1$: $\forall s \in C$, we have that $s \models_{(\Gamma \upharpoonright \mathcal{S}_1) \upharpoonright C} \bigwedge_{a \in \mathcal{A}} AS(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as}) \wedge AS(p_{nz})$. // By Lemma 5.4.

- 7: Compute the parity condition $p_{\diamond \mathcal{T}_{III, \mathcal{A}, \mathcal{AS}, p_{nz}}}$ and the MDP Γ' with set \mathcal{S}'_1 of states // Γ' and \mathcal{S}'_1 are defined in Proposition 6.2.

- 8: Check if $(s, 1) \models_{\mathcal{S}'_1} E(\bigwedge_{a \in \mathcal{A}} p_a \wedge p_{\diamond \mathcal{T}_{III, \mathcal{A}, \mathcal{AS}, p_{nz}}})$. // By lemmas 6.4 and 6.5; $(s, 1)$ is defined in Proposition 6.2.

- 9: **for all** $e \in \mathcal{E}$ **do**

- 10: Check if $s \models_{\mathcal{S}_1} E(\bigwedge_{a \in \mathcal{A}} p_a \wedge p_e)$. // By Lemma 6.5.

- 11: If any of the checks in Steps 8 and 10 fails then return false, else return true.

5. We show that the algorithm runs in time $\Sigma_2^P = \text{P}^{\text{NP}}$ (Theorem 7.1). We also show that we have a polynomial algorithm for the special case where the set \mathcal{A} is empty (Theorem 7.2), and that randomization and finite memory are required (Theorem 7.3). The problem is in $\text{NP} \cap \text{CoNP}$ when \mathcal{A} is singleton (Theorem 7.4). Finally, we show that finding a strategy for an arbitrary QPL formula that may have combination of conjunctions and disjunctions is in $\Sigma_2^P (= \text{NP}^{\text{NP}})$ (Theorem 7.5), and it is both NP-hard and coNP-hard (Theorem 7.6).

Theorem 7.1. *Given an MDP Γ , and a state s , Algorithm 5 decides if $s \models_{\Gamma} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as}) \wedge \bigwedge_{nz \in \mathcal{NZ}} NZ(p_{nz}) \wedge \bigwedge_{e \in \mathcal{E}} E(p_e)$, and solves the problem in P^{NP} time.*

Proof. For the correctness of Algorithm 5, consider a formula $\varphi = \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as}) \wedge \bigwedge_{nz \in \mathcal{NZ}} NZ(p_{nz}) \wedge \bigwedge_{e \in \mathcal{E}} E(p_e)$.

Given φ , we explain below how to check its satisfiability.

According to Lemma 6.1, to check if $s \models_{\Gamma} \varphi$, we need to check for all $nz \in \mathcal{NZ}$, if $s \models_{\Gamma} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(p_{nz})$ and that for all $e \in \mathcal{E}$ if $s \models_{\Gamma} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge E(p_e)$.

For the non-zero (NZ) part, recall from Section 6, that for each p_{nz} , we need to compute the set of all Type III ECs for the objective $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(p_{nz})$. Recall that each Type III EC C is such that two properties hold. First $\forall s' \in C$, we have $s' \models_{\Gamma} \bigwedge_{a \in \mathcal{A}} A(p_a)$; we do this in Step 3 of the algorithm. Then we also check that $\forall s' \in C, s' \models_{\Gamma|C} \bigwedge_{a \in \mathcal{A}} AS(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as}) \wedge AS(p_{nz})$. This is done in Step 6.

Now for an existential parity objective $E(p_e)$, by Lemma 6.5, we only need to check if: $s \models_{\Gamma \upharpoonright \mathcal{S}_1 \upharpoonright \bigwedge_{a \in \mathcal{A}} A(p_a)}$ $E(\bigwedge_{a \in \mathcal{A}} p_a \wedge p_e)$. It remains to explain how the sub-MDP in which all states satisfy the formula $\bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$ is computed. This is done in Steps 1 to 4.

To do so we first find the set of states s' such that $s' \models_{\Gamma} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$. We find the set of maximal Type II ECs.

By definition of Type I and Type II ECs, a maximal Type I EC that satisfies (II_2) is a maximal Type II EC. We now prove that if C is a maximal Type II EC, it is included in C_1 , a maximal Type I EC. Since for all $s' \in C$, we have $s' \models_{\Gamma|C} \bigwedge_{a \in \mathcal{A}} AS(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$, we also have that for all $s' \in C_1$, $s' \models_{\Gamma|C_1} \bigwedge_{a \in \mathcal{A}} AS(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as})$ (it suffices to take a strategy that has probability 1 of reaching some state $s'' \in C$ and then play the strategy ensuring $s'' \models_{\Gamma|C} \bigwedge_{a \in \mathcal{A}} AS(p_a) \wedge$

$\bigwedge_{as \in \mathcal{AS}} AS(p_{as})$). Hence C_1 is a Type II EC. By maximality of C , we have $C = C_1$. This means that to find the maximal Type II EC, it is sufficient to compute the maximal Type I EC and remove those maximal Type I ECs that do not ensure condition (II_2) . Finding the maximal Type I EC is done in Step 1 thanks to Lemma 3.5. We then check condition (II_2) in Step 2.

It then remains to find the states s' such that $s' \models_{\Gamma} \bigwedge_{a \in \mathcal{A}} A(p_a) \wedge AS(\diamond \mathcal{T}_{II, \Gamma, \mathcal{A}, \mathcal{A}})$ (Step 3). It can be done by transforming the MDP into a Streett-Büchi game that in turn can be transformed into a Streett game since a Büchi condition is a special case of a parity condition, and a conjunction of parity conditions is a Streett condition.

For the complexity, Steps 2, 4, 6, 7, 8 and 10 are polynomial. Step 3 is parity-complete if there is only one parity condition, polynomial if there is none, and is co-NP complete in general (we have to solve a Streett game). Step 1 is parity-complete if there is only one parity condition, polynomial if there is none. In the general case, Step 1 requires to iteratively solve a polynomial number of Streett games

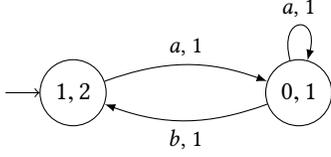


Figure 10. An MDP requiring randomized finite-memory strategy for the objective $AS(p_1) \wedge E(p_2)$.

(which is in coNP), and use the result of this computation to remove some of the states, resulting in a P^{NP} complexity. Further details of the procedure are given in algorithms 1 to 4 in the proof of Lemma 3.5. This leads us to a P^{NP} complexity for Algorithm 5. \square

Theorem 7.2. *Given an MDP Γ , and a state s , it is decidable in polynomial time if $s \models_{\Gamma} \bigwedge_{as \in \mathcal{AS}} AS(p_{as}) \wedge \bigwedge_{nz \in \mathcal{NZ}} NZ(p_{nz}) \wedge \bigwedge_{e \in \mathcal{E}} E(p_e)$.*

Proof. The proof is similar to that of Theorem 7.1, but as $\mathcal{A} = \emptyset$ we compute Type I $(\mathcal{A}, \mathcal{A})$ that is Type I (\emptyset, \emptyset) ECs, that is any EC, and so Step 1 returns the maximal ECs. In the same way, Step 3 polynomially computes one almost-sure reachability problem. As we consider Type I (\emptyset, \emptyset) , the checks in Steps 2,3,6,8 and 10 can be done with a randomized finite-memory strategy. Further, a strategy for each of these steps can be computed in polynomial time [16]. \square

In the above proof, we show that randomized finite-memory strategies are sufficient for formulas of the form $s \models_{\Gamma} \bigwedge_{as \in \mathcal{AS}} AS(p_{as}) \wedge \bigwedge_{nz \in \mathcal{NZ}} NZ(p_{nz}) \wedge \bigwedge_{e \in \mathcal{E}} E(p_e)$. Below, we show that such strategies are indeed necessary.

Theorem 7.3. *Given an MDP Γ , and a state in Γ , to solve the realizability problem for formulas of the form $\bigwedge_{as \in \mathcal{AS}} AS(p_{as}) \wedge \bigwedge_{nz \in \mathcal{NZ}} NZ(p_{nz}) \wedge \bigwedge_{e \in \mathcal{E}} E(p_e)$, a randomized finite-memory winning strategy may be necessary.*

Proof. We show that the lemma already holds for formulas of the form $AS(p_1) \wedge E(p_2)$. Consider the MDP in Figure 10. We want a strategy σ such that in $\Gamma^{[\sigma]}$, we have a subset of $(ab)^*a^\omega$ with measure 1 to satisfy $AS(p_1)$, and we want at least one path in $(a^+b)^\omega$ to satisfy $E(p_2)$. We can argue that no randomized memoryless strategy, as well as no deterministic memoryful strategy suffices here. However, we can use the following randomized strategy with one bit of memory:

Toss a coin every time the token is in state $(0, 1)$. If the coin gives heads, play ba , otherwise if the coin gives tails once, play a forever from the state $(0, 1)$ (one bit of memory is used to store if the coin has given tails). \square

Theorem 7.4. *Given an MDP Γ , and a state s , we can decide in $\text{NP} \cap \text{coNP}$ if $s \models_{\Gamma} A(p_a) \wedge \bigwedge_{as \in \mathcal{AS}} AS(p_{as}) \wedge \bigwedge_{nz \in \mathcal{NZ}} NZ(p_{nz}) \wedge \bigwedge_{e \in \mathcal{E}} E(p_e)$.*

This is done by solving a polynomial number of parity games.

Proof. The proof is similar to that of Theorem 7.1, but Step 1 computes maximal Type I $(\{a\}, \{a\})$ ECs which can be done in $\text{P}^{\text{NP} \cap \text{coNP}}$ [1]. Also the set of states that ensure the objective in Step 3 with only one sure parity condition can be computed by solving a polynomial number of parity games [1, 4], and by pruning states after solving each parity game. The result follows since $\text{P}^{\text{NP} \cap \text{coNP}} = \text{NP} \cap \text{coNP}$ [8]. \square

We now turn to the case of Boolean combinations of $A(p)$, $AS(p)$, $NZ(p)$, and $E(p)$ atoms.

Theorem 7.5. *QPL realizability is in NP^{NP} .*

Proof. Assume that we are given an MDP Γ , a state s , and a QPL formula φ . We assume that φ is in negation-free normal form. A negation-free equivalent formula can be found in polynomial time, it suffices to take the negation normal form (by using De Morgan's law to push negations inwards), and to take the dual of the negated atomic parity conditions.

We note that there exists a DNF formula equivalent to φ with the same set of atomic objectives. If φ is satisfiable, then there exists a conjunctive clause ψ that is a conjunction of these atomic objectives, and ψ can be guessed by an NP machine such that $s \models_{\Gamma} \psi$. Given this witness ψ , we can use an NP oracle to check that ψ implies φ where the atomic objectives are regarded as classical propositional atoms. We can then use the P^{NP} algorithm of Theorem 7.1 to get a proof that indeed $s \models_{\Gamma} \psi$ implying that φ is also satisfied by s . The result follows since the class NP^{NP} is the same as NP^{NP} . \square

Theorem 7.6. *QPL realizability is both NP-hard and coNP-hard.*

Proof. The coNP-hardness follows from the fragment of QPL made of conjunctions of $A(p)$ atoms that is powerful enough to encode Streett games, as proved in [14].

We prove NP-hardness for the fragment of QPL made of $\{\wedge, \vee, A(p)\}$. Given a SAT formula φ in negation normal form, we define a QPL formula φ_{obj} , and an MDP Γ both polynomial in φ such that there exists a state s of Γ and there exists a winning strategy for φ_{obj} from s on Γ if and only if φ is satisfiable. Let $\text{Var} = \{a_1, \dots, a_n\}$ be the set of propositional variables of φ . Let $S = \{a_i, \bar{a}_i \mid i \in [n]\}$ be both the set of states and the set of actions. We define the MDP $\Gamma = \{S, \{(a, b, b) \mid a, b \in S\}, S, Pr\}$ where for all $a, b \in S$ we have $Pr(a, b, b) = 1$. Note that the underlying graph is a complete graph. For each $i \in [n]$, we define two parity conditions p_{a_i} and $p_{\bar{a}_i}$ such that $p_{a_i}(a_i) = p_{\bar{a}_i}(\bar{a}_i) = 2$, such that $p_{a_i}(\bar{a}_i) = p_{\bar{a}_i}(a_i) = 3$ and $p_{a_i}(b) = p_{\bar{a}_i}(b) = 1$ if $b \notin \{a_i, \bar{a}_i\}$. We define $\psi = \bigwedge_{i \in [1, n]} A(p_{a_i}) \vee A(p_{\bar{a}_i})$. Given the SAT formula φ

in NNF, we define a QPL formula φ' by transforming φ in the following way: each $\neg a_i$ is replaced by $A(p_{\bar{a}_i})$ and each (non-negated) a_i is replaced by $A(p_{a_i})$. We define $\varphi_{obj} = \psi \wedge \varphi'$.

In the MDP Γ , we consider an arbitrary state s , a strategy σ , and the paths in $\text{Paths}^{\Gamma[\sigma]}(s)$. We now show that $s \models_{\Gamma} \varphi_{obj}$ iff φ is satisfiable.

We first prove the right to left implication. We assume φ is satisfiable, by some valuation $\rho: [n] \rightarrow \text{Var} \cup \{\neg a_i \mid a_i \in \text{Var}\}$ such that $\rho(i) = a_i$ or $\rho(i) = \neg a_i$. In the MDP Γ , we consider an arbitrary state s , and define a strategy σ with a transition system that loops over n modes. At mode i , it plays a_i if $\rho(i) = a_i$, and it plays \bar{a}_i if $\rho(i) = \neg a_i$. The only path in $\text{Paths}^{\Gamma[\sigma]}(s)$ clearly satisfies both ψ and φ' , hence $s \models_{\Gamma} \varphi_{obj}$.

We now prove the left to right implication. We assume that for some $s \in S$, there exists a strategy σ such that $s, \sigma \models_{\Gamma} \varphi_{obj}$. We note that since σ may be randomized, $\text{Paths}^{\Gamma[\sigma]}(s)$ may contain more than one path. We take $\pi \in \text{Paths}^{\Gamma[\sigma]}(s)$, and consider the following valuation: $\rho: [n] \rightarrow \text{Var} \cup \{\neg a_i \mid a_i \in \text{Var}\}$ such that for all $i \in [n]$, we have $\rho(i) = a_i$ if $a_i \in \text{inf}(\pi)$ and $\rho(i) = \neg a_i$ if $\bar{a}_i \in \text{inf}(\pi)$. We have that ρ is a well-defined and complete valuation: Since $\varphi_{obj} = \varphi' \wedge \psi$ we have that $s, \sigma \models_{\Gamma} \psi$, and for all $i \in [n]$ either $\rho(i) = a_i$ holds or $\rho(i) = \neg a_i$ holds, but not both. Valuation ρ satisfies φ since $s, \sigma \models_{\Gamma} \varphi'$, hence φ is satisfiable. \square

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