On the ω-language Expressive Power of Extended Petri Nets

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1 Introduction

Reactive systems are non-terminating systems that interact with an environment. Those systems are often embedded in environments which are *safety critical*, making their correctness a crucial issue.

To formally reason about the correctness of such systems, we need formal models of their behaviours. At some abstract level, the behaviour of a non-terminating reactive system within its environment can be seen as an *infinite sequence of events* (usually taken within a finite set of events). The semantics of those systems is thus a (usually infinite) set of those infinite behaviours. Sets of infinite sequences of events have been studied intensively in automata theory where infinite sequences of events are called *infinite words*, and sets of such sequences are called *omega languages* (ω-languages).

If the global system (the reactive system and its environment) has a meaningful *finite state abstraction* then there are well-studied formalisms that can be used. For example, *finite state automata* allow us to specify any *omega regular language* [Tho90]. Furthermore, as the global system is naturally composed of several components (at least two: the reactive system and its environment), it is convenient to model the reactive system and its environment compositionally by several (at least two) automata. This is possible using simple synchronization mechanisms. In the case of finite state machines, synchronizations on *common events* allow to model naturally most of the interesting communication mechanisms between processes.

Recently, a lot of research works have tried to generalize the *computer aided verification methods* that have been proposed for finite state systems toward infinite state systems. In particular, interesting positive (decidability) results have been obtained for a class of *parametric systems*. New methods have been proposed for automatically verifying temporal properties of concurrent systems containing an *arbitrary number* (parametric number) of finite-state processes that communicate. Contrary to the finite state case, three primitives of communication have been proposed:

- in [GS92], German *et al.* introduce a model where an arbitrary number of processes communicate via *rendez-vous* (synchronization on common events);
- in [EN98, EFM99], Emerson *et al.*, and Esparza *et al.* study the automatic analysis of models where an arbitrary number of processes communicate through rendez-vous and *broadcasts*. A broadcast is a *non-blocking* synchronization mechanism where the emitter sends a signal to all the possibly awaiting processes, and continue its execution without waiting (whether there are receivers or not). In [Del00], Delzanno uses broadcast protocols to model and verify cache coherency protocols [Han93];
- In the model introduced in [DRV02, RV03] by Delzanno *et al.*, an arbitrary number of processes can communicate thanks to *non-blocking rendez-vous* (in addition to rendez-vous and broadcasts). In a non-blocking rendez-vous synchronization, the sender emits an event, and if there are automata waiting for that event, one of those automata is chosen non-deterministically and synchronizes with the sender. As for broadcast, this synchronization mechanism is non-blocking. This model is useful to model multi-threaded programs written in JAVA where instructions like `NotifyAll` are modeled by using broadcasts and `Notify` are modeled by using non-blocking rendez-vous.

In all those works, the identity of individual processes is irrelevant. Hence, we can apply to all those models the so-called *counting abstraction* [GS92, Van03] and equivalently see all those models as extended Petri nets. In has been shown in previous works that rendez-vous can be modeled by *Petri nets* [Pet81], broadcasts can be modeled by *Petri nets extended with transfer arcs*, and non-blocking rendez-vous can be modeled by *Petri nets extended with non-blocking arcs* [DRV02, Van03].

These two Petri nets extensions (and others like reset Petri nets, lossy Petri nets,...) are monotonic and well-structured [RV03]. Those models have attracted a lot of attention recently [DFS98, DJS99, DFS98] [Sut00, FS00a, FS00b] [EFM99] [DRV02, RV03]. These paper study the main decidability problems for these models: even if the *general reachability problem* is undecidable, interesting subproblems, like *control state reachability* and *termination*, are decidable for all those models, and the *boundedness* problem is decidable for Petri nets, Petri nets with non-blocking arcs and transfer nets. However, the expressiveness of

those formalisms have not been studied carefully presumably because the finite word languages definable in those formalisms are all equal to the recursively enumerable languages. Nevertheless, as recalled above, those formalisms are usually used to model non-terminating systems and so their expressive power should be measured in terms of definable omega languages.

There is currently no proof that the *expressive power* of Petri nets with transfer arcs or Petri nets with non-blocking arcs, measured in terms of *definable omega languages*, are strictly greater than the expressive power of Petri nets. In this paper, we solve this open problem. Our results are as follows. First (section 3), we show that all the omega-languages definable by Petri nets with non-blocking arcs can be recognized by Petri nets with transfer arcs, but that some languages which are definable by Petri nets with transfer arcs are not recognizable by Petri nets with non-blocking arcs (even if we allow τ-transitions). Second (section 4, we show that there exist omega languages that can be defined with Petri nets extended with non-blocking arcs and can not be defined with Petri nets (even if we allow τ-transitions). The separation of expressive power over *definable omega languages* is surprising as the expressive power of those two extended Petri net models equals, as mentioned above, the expressive power of Turing Machines when measured on *finite word languages* defined with the help of a finite accepting set of markings. We also study the expressiveness of Petri nets with reset arcs in section 5.

The techniques that we use to separate the expressive power of extended Petri nets on omega languages are based on properties of *well-quasi orderings* and monotonicity. They are, to the best of our knowledge, original in the context of (extended) Petri nets.

In this version of the paper, several technical proofs have been omitted owing to lack of space. A full version of this paper is available as Technical Report number 519 of the Computer Science Department of Brussels University[1]

2 Preliminaries

In this section, we introduce the preliminaries of the discussion. In 2.1, we introduce two Petri nets extensions (non-blocking arcs and transfer arcs) and define the notion of ω-language *accepted* by these models. In 2.2 we recall and prove a basic result on well-quasi orderings, which is the cornerstone of the proofs of sections 3 and 4.

2.1 Extended Petri nets

Definition 1 An **Extended Petri Net** (EPN for short) $\mathcal{N}$ is a tuple $\langle \mathcal{P}, \mathcal{T}, \Sigma, \mathbf{m}_0 \rangle$, where $\mathcal{P} = \{p_1, p_2, \ldots p_n\}$ is a finite set of places, $\mathcal{T}$ is finite set of transitions and Σ is a finite alphabet containing a special silent symbol τ. A **marking** of the places is a function $\mathbf{m} : \mathcal{P} \mapsto \mathbb{N}$. A marking can also be seen as a vector

[1] which can be downloaded at: http://www.ulb.ac.be/di/ssd/ggeeraer/papers/express.pdf

v such that $v^T = [\mathbf{m}(p_1), \mathbf{m}(p_2), \ldots, \mathbf{m}(p_n)]$. $\mathbf{m}_0 : \mathcal{P} \mapsto \mathbb{N}$ is the initial marking. Each transition is of the form $\langle I, O, s, d, b, \lambda \rangle$, where I and $O : \mathcal{P} \mapsto \mathbb{N}$ are multi-sets of input and output places respectively. By convention, $O(p)$ (resp. $I(p)$) denotes the number of occurrences of p in O (resp. I). $s, d \in \mathcal{P} \cup \{\perp\}$ are the source and the destination places respectively, $b \in \mathbb{N} \cup \{+\infty\}$ is the bound and $\lambda \in \Sigma$ is the label of the transition.

Let us divide $\mathcal{T}$ into $\mathcal{T}_r$ and $\mathcal{T}_e$ such that $\mathcal{T} = \mathcal{T}_r \cup \mathcal{T}_e$ and $\mathcal{T}_e \cap \mathcal{T}_r = \emptyset$. Without loss of generality, we assume that for each transition $\langle I, O, s, d, b, \lambda \rangle \in \mathcal{T}$, either $b = 0$ and $s = \perp = d$ (regular Petri transitions, grouped into $\mathcal{T}_r$); or $b > 0$, $s \neq d$, $s \neq \perp$ and $d \neq \perp$ (extended transitions, grouped into $\mathcal{T}_e$). We identify several non-disjoint classes of Extended Petri Nets, depending on $\mathcal{T}_e$:

- **Petri net** (PN for short): $\mathcal{T}_e = \emptyset$.
- **Petri net with non-blocking arcs** (PN+NBA): $\forall t = \langle I, O, s, d, b, \lambda \rangle \in \mathcal{T}_e : b = 1$.
- **Petri net with transfer arcs** (PN+T): $\forall t = \langle I, O, s, d, b, \lambda \rangle \in \mathcal{T}_e : b = +\infty$. ■

As usual, places are graphically depicted by circles; transitions by filled rectangles. For any transition $t = \langle I, O, s, d, b, \lambda \rangle$, we draw an arrow from any place $p \in I$ to transition t and from t to any place $p \in O$. For a PN+NBA (resp. PN+T), we draw a dotted (grey) arrow from s to t and from t to d (provided that $s, d \neq \perp$).

Definition 2 Given an extended Petri Net $\mathcal{N} = \langle \mathcal{P}, \mathcal{T}, \Sigma, \mathbf{m}_0 \rangle$, and a marking $\mathbf{m}$ of $\mathcal{N}$, a transition $t = \langle I, O, s, d, b, \lambda \rangle$ is said to be *enabled in* $\mathbf{m}$ (notation: $\mathbf{m} \xrightarrow{t}$) iff $\forall p \in \mathcal{P} : \mathbf{m}(p) \geq I(p)$. An enabled transition $t = \langle I, O, s, d, b, \lambda \rangle$ can *occur*, which deterministically transforms the marking $\mathbf{m}$ into a new marking $\mathbf{m}'$ (we denote this by $\mathbf{m} \xrightarrow{t} \mathbf{m}'$). $\mathbf{m}'$ is computed as follows:

1. First compute $\mathbf{m}_1$ such that: $\forall p \in \mathcal{P} : \mathbf{m}_1(p) = \mathbf{m}(p) - I(p)$.
2. Then compute $\mathbf{m}_2$ as follows. If $s = d = \perp$, then $\mathbf{m}_2 = \mathbf{m}_1$. Otherwise:

$$\mathbf{m}_2(s) = \begin{cases} 0 & \text{if } \mathbf{m}_1(s) \leq b \\ \mathbf{m}_1(s) - b & \text{otherwise} \end{cases} \qquad \mathbf{m}_2(d) = \begin{cases} \mathbf{m}_2(d) + \mathbf{m}_1(s) & \text{if } \mathbf{m}_1(s) \leq b \\ \mathbf{m}_2(d) + b & \text{otherwise} \end{cases}$$

$$\forall p \in \mathcal{P} \setminus \{d, s\} : \mathbf{m}_2(p) = \mathbf{m}_1(p)$$

3. Finally, compute $\mathbf{m}'$, such that $\forall p \in O : \mathbf{m}'(p) = \mathbf{m}_2(p) + O(p)$.

Let $\sigma = t_1 t_2 \ldots t_n$ be a sequence of transition. We write $\mathbf{m} \xrightarrow{\sigma} \mathbf{m}'$ to mean that there exist $\mathbf{m}_1, \ldots, \mathbf{m}_{n-1}$ such that $\mathbf{m} \xrightarrow{t_1} \mathbf{m}_1 \xrightarrow{t_1} \ldots \xrightarrow{t_{n-1}} \mathbf{m}_{n-1} \xrightarrow{t_n} \mathbf{m}'$. ■

Definition 3 Let σ be a sequence of transitions. $\Lambda(\sigma)$ is defined inductively as follows (where λ_i denotes the label of t_i). If $\sigma = t_1$, then $\Lambda(\sigma) = \varepsilon$ if $\lambda_1 = \tau$; $\Lambda(\sigma) = \lambda_1$ otherwise. In the case where $\sigma = t_1 t_2 \ldots$, then $\Lambda(\sigma) = \Lambda(t_2 \ldots)$ if $\lambda_1 = \tau$; $\Lambda(\sigma) = \lambda_1 \cdot \Lambda(t_2 \ldots)$ otherwise. ■

Remark that this definition is sound even in the case where σ is infinite since Λ associates one and only one word to any infinite sequence of transitions.

Definition 4 Let $\mathcal{N} = \langle \mathcal{P}, \mathcal{T}, \Sigma, \mathbf{m}_0 \rangle$. An infinite word x on Σ is said to be *accepted* by $\mathcal{N}$ if there exists an infinite sequence of transitions $\sigma = t_1 t_2 \ldots$ and an infinite set of markings $\{\mathbf{m}_1, \mathbf{m}_2 \ldots\}$ such that $\mathbf{m}_1 \xrightarrow{t_1} \mathbf{m}_2 \xrightarrow{t_2} \mathbf{m}_3 \ldots$, $\mathbf{m}_1 = \mathbf{m}_0$ and $\Lambda(\sigma) = x$. The *language* $L^\omega(\mathcal{N})$ is defined as the set of all the infinite words accepted by $\mathcal{N}$. The language $L^\omega_{\not\tau}(\mathcal{N})$ is the set of infinite words accepted by sequences of transitions of $\mathcal{N}$ that *do not contain* τ-transitions. ■

By abuse of notation we also write $\mathbf{m} \xrightarrow{x} \mathbf{m}'$ to mean that there exists a *finite* sequence of transitions σ such that $\Lambda(\sigma) = x$ and $\mathbf{m} \xrightarrow{\sigma} \mathbf{m}'$, and $\mathbf{m} \xrightarrow{x'}$ to mean that we can fire the *infinite* sequence of transitions σ' (with $\Lambda(\sigma') = x'$) from $\mathbf{m}$.

In the following, $L^\omega(\mathsf{PN})$ (respectively $L^\omega(\mathsf{PN{+}T})$, $L^\omega(\mathsf{PN{+}NBA})$) denotes the set of all the ω-languages that can be recognised by a PN (respectively PN+T, and PN+NBA). $L^\omega_{\not\tau}(\mathsf{PN})$, $L^\omega_{\not\tau}(\mathsf{PN{+}NBA})$ and $L^\omega_{\not\tau}(\mathsf{PN{+}T})$ are defined similarly in the case where we disallow τ-transitions.

In the sequel a notion of ordering on the markings will appear to be useful. Let $\preccurlyeq$ denote the quasi ordering on markings, defined as follows: let $\mathbf{m}$ and $\mathbf{m}'$ be two markings on the set of places $\mathcal{P}$, then $\mathbf{m} \preccurlyeq \mathbf{m}'$ if $\forall p \in \mathcal{P} : \mathbf{m}(p) \leq \mathbf{m}'(p)$. We come back on important properties of $\preccurlyeq$ in 2.2.

An important property of sequences of transitions of PN is their *constant effect* (it is well-known that the effect of such a sequence, when it is enabled, can be expressed by a vector of integers stating how many tokens are removed and put in each place). In the case of PN+NBA or PN+T, the effect is not constant anymore, since it is dependant on the marking at the time of the firing. However, the effect of a sequence of transitions with non-blocking arcs can be *bounded*, as stated by the following Lemma.

Lemma 1 *Let $\mathcal{N} = \langle \mathcal{P}, \mathcal{T}, \Sigma, \mathbf{m}_0 \rangle$ be a* PN+NBA*, and let σ be a finite sequence of transitions of $\mathcal{N}$ that contains n occurrences of transitions in $\mathcal{T}_e$. Let $\mathbf{m}_1$, $\mathbf{m}'_1$, $\mathbf{m}_2$ and $\mathbf{m}'_2$ be four makings such that (i) $\mathbf{m}_1 \xrightarrow{\sigma} \mathbf{m}'_1$, (ii) $\mathbf{m}_2 \xrightarrow{\sigma} \mathbf{m}'_2$ and (iii) $\mathbf{m}_2 \succcurlyeq \mathbf{m}_1$. Then, for every place $p \in \mathcal{P}$: $\mathbf{m}'_2(p) - \mathbf{m}'_1(p) \geq \mathbf{m}_2(p) - \mathbf{m}_1(p) - n$.*

Proof. Let us consider a place $p \in \mathcal{P}$. First, we remark that when we fire σ from $\mathbf{m}_2$ instead of $\mathbf{m}_1$, its Petri net arcs will have the same effect on p. On the other hand, since we want to find a lower bound on $\mathbf{m}'_2(p) - \mathbf{m}'_1(p)$, we consider the situation where no non-blocking arcs affect p when σ is fired from $\mathbf{m}_1$, but they all remove one token from p when σ is fired from $\mathbf{m}_2$. In the latter case, the effect of σ on p is $\mathbf{m}'_1(p) - \mathbf{m}_1(p) - n$. We obtain thus: $\mathbf{m}'_2(p) \geq \max\{\mathbf{m}_2(p) + \mathbf{m}'_1(p) - \mathbf{m}_1(p) - n, 0\}$. Hence $\mathbf{m}'_2(p) \geq \mathbf{m}'_1(p) + \mathbf{m}_2(p) - \mathbf{m}_1(p) - n$, and thus: $\mathbf{m}'_2(p) - \mathbf{m}'_1(p) \geq \mathbf{m}_2(p) - \mathbf{m}_1(p) - n$. □

2.2 Properties of infinite sequences on well-quasi ordered elements

Following [Fin90, ACJT96], $\preccurlyeq$ is a *well-quasi ordering* (wqo for short). This means that $\preccurlyeq$ is a reflexive and transitive relation such that for any infinite sequence $\mathbf{m}_1, \mathbf{m}_2, \ldots$ there is $i < j$ such that $\mathbf{m}_i \preccurlyeq \mathbf{m}_j$. Hence, we get this property on $\preccurlyeq$:

Lemma 2 *Given an infinite sequence of markings $\mathbf{m}_1, \mathbf{m}_2, \ldots$ we can always extract an infinite sub-sequence $\mathbf{m}_{i_1}, \mathbf{m}_{i_2}, \ldots$ $(\forall j : i_j < i_{j+1})$ such that for all place p, either $\mathbf{m}_{i_j}(p) < \mathbf{m}_{i_{j+1}}(p)$ for all $j \geq 1$ or $\mathbf{m}_{i_j}(p) = \mathbf{m}_{i_{j+1}}(p)$ for all $j \geq 1$.*

The following Lemma is easy to prove [RV03].

Lemma 3 (Monotonicity) *Let $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}'_1$ be three markings of an EPN, such that $\mathbf{m}_1 \preccurlyeq \mathbf{m}_2$ and $\mathbf{m}_1 \xrightarrow{t} \mathbf{m}'_1$ for some transition t of the EPN. Then, there exists $\mathbf{m}'_2$ such that $\mathbf{m}_2 \xrightarrow{t} \mathbf{m}'_2$ and $\mathbf{m}'_1 \preccurlyeq \mathbf{m}'_2$.*

3 PN+T are more expressive than PN+NBA

In this section, one will find the first important result of the paper (as stated by Theorem 2): PN+T are strictly more expressive, on ω-languages, than PN+NBA. We prove this in two steps. First, we show that any ω-language accepted by a PN+NBA can be accepted by a PN+T (this is the purpose of Lemma 4 and Theorem 1). Then, we prove the strictness of the inclusion thanks to the PN+T $\mathcal{N}_1$ of Fig. 2 (a). Namely, we show that $L^\omega(\mathcal{N}_1)$ contains at least the words $(\mathtt{a}^k\mathtt{b}^k)^\omega$, for any $k \geq 1$ (Lemma 5). On the other hand we show that $\mathcal{N}_1$ rejects the words whose prefix belongs to $(\mathtt{a}^{n_3}\mathtt{b}^{n_3})^*\mathtt{a}^{n_3}(\mathtt{b}^{n_1}\mathtt{a}^{n_1})^+\mathtt{b}^{n_2}$ with $n_1 < n_2 < n_3$ (Lemma 6). We finally show that any PN+NBA accepting words of the form $(\mathtt{a}^k\mathtt{b}^k)^\omega$ also has to accept words whose prefix belongs to $(\mathtt{a}^{n_3}\mathtt{b}^{n_3})^*\mathtt{a}^{n_3}(\mathtt{b}^{n_1}\mathtt{a}^{n_1})^+\mathtt{b}^{n_2}$ with $n_1 < n_2 < n_3$. Since $\mathcal{N}_1$ rejects the latter, we conclude that no PN+NBA can accept $L^\omega(\mathcal{N}_1)$.

3.1 PN+NBA are not more expressive than PN+T.

Let us consider a PN+NBA $\mathcal{N} = \langle \mathcal{P}, \mathcal{T}, \Sigma, \mathbf{m}_0 \rangle$, and let us show how to transform it into a PN+T $\mathcal{N}'$ such that $L^\omega(\mathcal{N}) = L^\omega(\mathcal{N}')$.

Let us consider the partition of $\mathcal{T}$ into $\mathcal{T}_e$ and $\mathcal{T}_r$ as defined in Definition 1, and a new place p_{Tr} (the trash place). We now show how to build $\mathcal{N}' = \langle \mathcal{P}', \mathcal{T}', \Sigma, \mathbf{m}'_0 \rangle$. First, $\mathcal{P}' = \mathcal{P} \cup \{p_{Tr}\}$. For each transition $t = \langle I, O, s, d, 1, \lambda \rangle$ in $\mathcal{T}_e$, we put in $\mathcal{T}'$ two new transitions $t_l = \langle I, O, s, p_{Tr}, +\infty, \lambda \rangle$ and $t_e = \langle I_e, O_e, \bot, \bot, +\infty, \lambda \rangle$, such that: $\forall p \in \mathcal{P} : \big(p \neq s \Rightarrow I_e(p) = I(p) \wedge p \neq d \Rightarrow O_e(p) = O(p)\big)$, $I_e(s) = I(s) + 1$ and $O_e(d) = O(d) + 1$. We also add into $\mathcal{T}'$ all the transitions of $\mathcal{T}_r$. Finally, $\forall p \in \mathcal{P} = \mathbf{m}'_0(p) = \mathbf{m}_0(p)$ and $\mathbf{m}'_0(p_{Tr}) = 0$. Fig. 1 illustrates the construction.

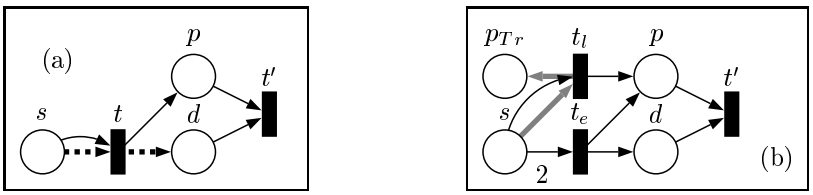

Figure 1: A PN+NBA $\mathcal{N}$ (a) and the corresponding PN+T $\mathcal{N}'$ (b)

Lemma 4 $L^\omega(\mathcal{N}) = L^\omega(\mathcal{N}')$.

Proof. $[L^\omega(\mathcal{N}) \subseteq L^\omega(\mathcal{N}')]$ We show that, for every infinite sequence of transitions σ of $\mathcal{N}$, we can find a sequence of transitions σ' of $\mathcal{N}'$ such that $\Lambda(\sigma) = \Lambda(\sigma')$.

Let us define the function $f : \mathcal{T} \times \mathbb{N}^{|\mathcal{P}|} \to \mathcal{T}'$ such that $\forall t \in \mathcal{T}_r : f(t, \mathbf{m}) = t$ and $\forall t = \langle O, I, s, d, 1, \lambda\rangle \in \mathcal{T}_e : f(t, \mathbf{m}) = t_e$, if $\mathbf{m}(s) > I(s)$ (the non-blocking arc still has an effect after the firing of the Petri part of the transition); and $f(t, \mathbf{m}) = t_l$, otherwise.

Let $\sigma = \mathbf{m}_0 \xrightarrow{t_1} \mathbf{m}_1 \xrightarrow{t_2} \ldots \xrightarrow{t_n} \mathbf{m}_n \xrightarrow{t_{n+1}} \ldots$ be a sequence of $\mathcal{N}$. Then we may see that $\sigma' = \mathbf{m}'_0 \xrightarrow{f(t_1,\mathbf{m}'_0)} \mathbf{m}'_1 \xrightarrow{f(t_2,\mathbf{m}'_1)} \ldots \xrightarrow{f(t_n,\mathbf{m}'_{n-1})} \mathbf{m}'_n \xrightarrow{f(t_{n+1},\mathbf{m}'_n)} \ldots$ is a sequence of $\mathcal{N}'$, where $\mathbf{m}'_i$ is such that $\mathbf{m}'_i(p) = \mathbf{m}_i(p)$ for all $p \in \mathcal{P}$ and $\mathbf{m}'_i(p_{Tr}) = 0$ for all $i \geq 1$. Since we have $\forall i \geq 1 : \Lambda(t_i) = \Lambda(f(t_i, \mathbf{m}_{i-1}))$, we conclude that $\Lambda(\sigma) = \Lambda(\sigma')$, hence $L^\omega(\mathcal{N}) \subseteq L^\omega(\mathcal{N}')$.

$[L^\omega(\mathcal{N}') \subseteq L^\omega(\mathcal{N})]$ We show that, for every infinite sequence of transitions σ' of $\mathcal{N}'$, we can find a sequence of transitions σ of $\mathcal{N}$ such that $\Lambda(\sigma') = \Lambda(\sigma)$.

We define the function $g : \mathcal{T}' \to \mathcal{T}$ such that for all $t \in \mathcal{T}_r$: $g(t) = t$ and for all $t \in \mathcal{T}_e : g(t_e) = g(t_l) = t$. Moreover, we define the relation $\preccurlyeq_{\mathcal{P}}$ that compares two markings only on the places that are in $\mathcal{P}$. Thus, if $\mathbf{m}$ is defined on $\mathcal{P}$ and $\mathbf{m}'$ on $\mathcal{P}$ with $\mathcal{P}' \subseteq \mathcal{P}$, $\mathbf{m}' \preccurlyeq_{\mathcal{P}'} \mathbf{m}$ iff $\forall p \in \mathcal{P}' : \mathbf{m}'(p) \leq \mathbf{m}(p)$. Moreover, $\mathbf{m}' \prec_{\mathcal{P}'} \mathbf{m}$ iff $\mathbf{m}' \preccurlyeq_{\mathcal{P}'} \mathbf{m}$ but $\mathbf{m} \not\preccurlyeq_{\mathcal{P}'} \mathbf{m}'$.

Let $\sigma' = \mathbf{m}'_0 \xrightarrow{t_1} \mathbf{m}'_1 \xrightarrow{t_2} \ldots \xrightarrow{t_n} \mathbf{m}'_n \xrightarrow{t_{n+1}} \ldots$ be a sequence of $\mathcal{N}'$. Then, there exist $\mathbf{m}_1, \mathbf{m}_2, \ldots$ in $\mathcal{N}$ such that we have $\mathbf{m}_0 \xrightarrow{g(t_1)} \mathbf{m}_1 \xrightarrow{g(t_2)} \ldots \xrightarrow{g(t_n)} \mathbf{m}_n \xrightarrow{g(t_{n+1})} \ldots$ We prove that this sequence exists by contradiction. Suppose that it is not the case, i.e. there exists $i \geq 0$ such that $g(t_{i+1})$ is not firable from $\mathbf{m}_i$. Let us show by induction on the indexes, that $\mathbf{m}'_j \preccurlyeq_{\mathcal{P}} \mathbf{m}_j$ for all j such that $0 \leq j \leq i$.

Base case: $j = 0$. The base case is trivially verified.

Induction step: $j = k$. By induction hypothesis, we have: $\forall 0 \leq j \leq k-1 : \mathbf{m}'_j \preccurlyeq_{\mathcal{P}} \mathbf{m}_j$. In the case where $t_k = \langle I, O, s, d, b, \lambda\rangle$ (from $\mathbf{m}'_{k-1}$) has the same effect on $\mathcal{P}$ than $g(t_k)$ (from $\mathbf{m}_{k-1}$), we directly have that $\mathbf{m}'_k \preccurlyeq_{\mathcal{P}} \mathbf{m}_k$. This happens if t_k is a regular Petri transition or if $\mathbf{m}_{k-1}(s) = \mathbf{m}'_{k-1}(s) = I(s)$.

Otherwise t_k has a transfer arc and we must consider two cases:

- The transfer of t_k has no effect and the non-blocking arc of $g(t_k)$ moves one token from the source s to the target d, hence $I(s) = \mathbf{m}'_{k-1}(s) < \mathbf{m}_{k-1}(s)$. Since t_k and $g(t_k)$ have the same effect except that $g(t_k)$ removes one more token from s and adds one more token in d, and since $\mathbf{m}'_{k-1} \preccurlyeq_{\mathcal{P}} \mathbf{m}_{k-1}$ with $\mathbf{m}'_{k-1}(s) < \mathbf{m}_{k-1}(s)$, we conclude that $\mathbf{m}'_k \preccurlyeq_{\mathcal{P}} \mathbf{m}_k$.
- The transfer of t_k moves at least one token from the source s to p_{Tr} and the non-blocking arc of $g(t_k)$ moves one token from s to d. Since t_k and $g(t_k)$ have the same effect on the places in $\mathcal{P}$ except that $g(t_k)$ adds one more token in d and t_k may remove more tokens from s, and since $\mathbf{m}'_{k-1} \preccurlyeq_{\mathcal{P}} \mathbf{m}_{k-1}$, we conclude that $\mathbf{m}'_k \preccurlyeq_{\mathcal{P}} \mathbf{m}_k$.

Thus $\mathbf{m}'_i \preccurlyeq_{\mathcal{P}} \mathbf{m}_i$. Since t_{i+1} is firable from $\mathbf{m}'_i$, we conclude that $g(t_{i+1})$ is firable from $\mathbf{m}_i$ because $g(t_k)$ consumes no more tokens in any place p than t_k does. Hence the contradiction.

Thus, there exists $\mathbf{m}_1, \mathbf{m}_2 \ldots$ such that we have $\mathbf{m}_0 \xrightarrow{g(t_1)} \mathbf{m}_1 \xrightarrow{g(t_2)} \ldots \xrightarrow{g(M_n)} \mathbf{m}_n \xrightarrow{g(t_{n+1})} \ldots$ in $\mathcal{N}$. Since $\Lambda(t_i) = \Lambda(g(t_i))$ for all $i \geq 1$, we conclude that $\Lambda(\sigma') = \Lambda(\sigma)$, hence $L^\omega(\mathcal{N}') \subseteq L^\omega(\mathcal{N})$. □

Thus, we immediately obtain:

Theorem 1 *For every ω-language L that is accepted by a* PN+NBA*, there exists a* PN+T *that accepts L.*

3.2 PN+T are more expressive than PN+NBA

Let us now prove that L^ω(PN+NBA) is *strictly* included in L^ω(PN+T). We consider the PN+T $\mathcal{N}_1$ presented in Fig.2 (a) with the initial marking $\mathbf{m}_0(p_1) = 1$ and $\mathbf{m}_0(p) = 0$ for $p \in \{p_2, p_3, p_4\}$. The two following Lemmata allow us to better understand the behaviour of $\mathcal{N}_1$.

Lemma 5 *For any $k \geq 1$, the word $\left(\mathtt{a}^k\mathtt{b}^k\right)^\omega$ is accepted by $\mathcal{N}_1$.*

Lemma 6 *Let n_1, n_2, n_3 and m be four natural numbers such that $0 < n_1 < n_2 < n_3$ and $m > 0$. Then, for any $k \geq 0$ the words $(\mathtt{a}^{n_3}\mathtt{b}^{n_3})^k\mathtt{a}^{n_3}(\mathtt{b}^{n_1}\mathtt{a}^{n_1})^m(\mathtt{b}^{n_2}\mathtt{a}^{n_2})^\omega$ are not accepted by $\mathcal{N}_1$.*

These two proofs are quite straightforward and can be found in the appendix.

We can now show that no PN+NBA can accept $L^\omega(\mathcal{N}_1)$. Remark that the proof technique used hereafter relies on Lemmata 2 and 3, and is somewhat similar to a *pumping Lemma*. To the best of our knowledge, it is the first time such a technique is applied in the context of Petri nets (and their extensions).

Lemma 7 *No* PN+NBA *accepts $L^\omega(\mathcal{N}_1)$.*

Proof. Let $\mathcal{N}$ be a PN+NBA such that $L^\omega(\mathcal{N}_1) \subseteq L^\omega(\mathcal{N})$. We will show that this implies that $L^\omega(\mathcal{N}_1) \subsetneq L^\omega(\mathcal{N})$. As $L^\omega(\mathcal{N}_1) \subseteq L^\omega(\mathcal{N})$, by Lemma 5 we know that, for all $k \geq 1$, the word $(\mathtt{a}^k\mathtt{b}^k)^\omega$ belongs to $L^\omega(\mathcal{N})$. Suppose that $\mathbf{m}_{init}$

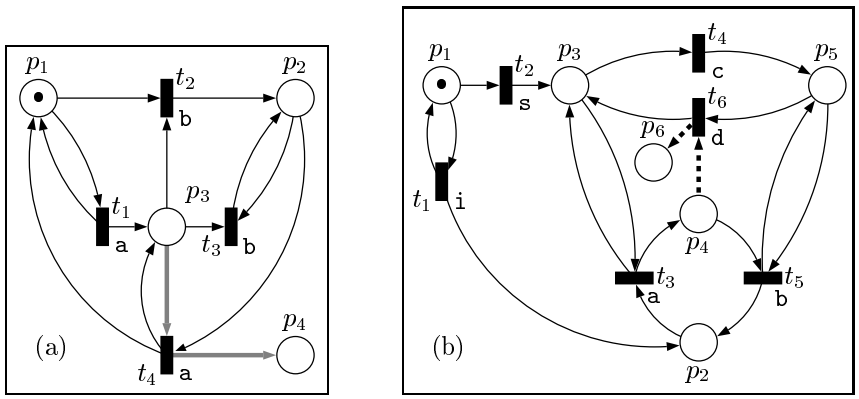

Figure 2: (a) The PN+T $\mathcal{N}_1$ and (b) the PN+NBA $\mathcal{N}_2$.

is the initial marking of $\mathcal{N}$. Thus, for all $k \geq 1$, there exists a marking $\widehat{\mathbf{m}}_k$, a finite sequence of transitions σ_k and a natural number ℓ_k such that:

$$\mathbf{m}_{init} \xrightarrow{(\mathtt{a}^k\mathtt{b}^k)^{\ell_k}\mathtt{a}^k} \widehat{\mathbf{m}}_k \xrightarrow{\Lambda(\sigma_k)} \widehat{\mathbf{m}}'_k \text{ and } \widehat{\mathbf{m}}'_k \succcurlyeq \widehat{\mathbf{m}}_k \text{ and } \Lambda(\sigma_k) = (\mathtt{b}^k\mathtt{a}^k)^{n_k} \text{ with } n_k \geq 1$$

Indeed, suppose that it is not the case, we would have $\mathbf{m}_{init} \xrightarrow{\mathtt{a}^k} \mathbf{m}_1 \xrightarrow{\mathtt{b}^k\mathtt{a}^k} \mathbf{m}_2 \ldots \xrightarrow{\mathtt{b}^k\mathtt{a}^k} \mathbf{m}_i \xrightarrow{\mathtt{b}^k\mathtt{a}^k} \ldots$ such that there does not exist $1 \leq i < j$ with $\mathbf{m}_i \preccurlyeq \mathbf{m}_j$. But from Lemma 2, this never occurs.

Let us consider the infinite sequence $\widehat{\mathbf{m}}_1, \widehat{\mathbf{m}}_2, \ldots, \widehat{\mathbf{m}}_i, \ldots$ Following Lemma 2 again, we extract a sub-sequence from it $\widehat{\mathbf{m}}_{\rho(1)}, \widehat{\mathbf{m}}_{\rho(2)}, \ldots, \widehat{\mathbf{m}}_{\rho(n)}, \ldots$ such that: $\forall p \in \mathcal{P}$: either $\forall i \geq 1 : \widehat{\mathbf{m}}_{\rho(i)}(p) = \widehat{\mathbf{m}}_{\rho(i+1)}(p)$ or $\forall i \geq 1 : \widehat{\mathbf{m}}_{\rho(i)}(p) < \widehat{\mathbf{m}}_{\rho(i+1)}(p)$. Let us denote by $\mathcal{P}'$ the set of places that strictly increase in that sequence.

Let n be the number of occurrences of transitions of $\mathcal{T}_e$ in $\sigma_{\rho(1)}$ and let us consider $\widehat{\mathbf{m}}_{\rho(1)}$, $\widehat{\mathbf{m}}_{\rho(2)}$, $\widehat{\mathbf{m}}_{\rho(n+3)}$, and $\mathbf{m}$ such that: $\widehat{\mathbf{m}}_{\rho(n+3)} \xrightarrow{\sigma_{\rho(1)}} \mathbf{m}$ (from Lemma 3, the sequence $\sigma_{\rho(1)}$ is firable since $\widehat{\mathbf{m}}_{\rho(1)} \preccurlyeq \widehat{\mathbf{m}}_{\rho(n+3)}$ and $\sigma_{\rho(1)}$ is firable from $\widehat{\mathbf{m}}_{\rho(1)}$). We first prove that $\mathbf{m} \succcurlyeq \widehat{\mathbf{m}}_{\rho(2)}$.

We know that:

$$\widehat{\mathbf{m}}_{\rho(1)} \xrightarrow{\sigma_{\rho(1)}} \widehat{\mathbf{m}}'_{\rho(1)} \quad \wedge \quad \widehat{\mathbf{m}}'_{\rho(1)} \succcurlyeq \widehat{\mathbf{m}}_{\rho(1)} \tag{1}$$

$$\forall p \in \mathcal{P}' \quad : \quad \widehat{\mathbf{m}}_{\rho(n+3)}(p) \geq \widehat{\mathbf{m}}_{\rho(2)}(p) + n + 1 \tag{2}$$

$$\forall p \in \mathcal{P} \setminus \mathcal{P}' \quad : \quad \widehat{\mathbf{m}}_{\rho(1)}(p) = \widehat{\mathbf{m}}_{\rho(2)}(p) = \widehat{\mathbf{m}}_{\rho(n+3)}(p) \tag{3}$$

Thus:

$$\begin{array}{lllll}
\textbf{(a)} & \forall p \in \mathcal{P}' & : & \mathbf{m}(p) \geq \widehat{\mathbf{m}}'_{\rho(1)}(p) + \big(\widehat{\mathbf{m}}_{\rho(n+3)}(p) - \widehat{\mathbf{m}}_{\rho(1)}(p)\big) - n & \text{by Lemma 1} \\
\Rightarrow & \forall p \in \mathcal{P}' & : & \mathbf{m}(p) \geq \widehat{\mathbf{m}}_{\rho(1)}(p) + \big(\widehat{\mathbf{m}}_{\rho(n+3)}(p) - \widehat{\mathbf{m}}_{\rho(1)}(p)\big) - n & \text{by (1)} \\
\Rightarrow & \forall p \in \mathcal{P}' & : & \mathbf{m}(p) \geq \widehat{\mathbf{m}}_{\rho(n+3)}(p) - n & \\
\Rightarrow & \forall p \in \mathcal{P}' & : & \mathbf{m}(p) \geq \widehat{\mathbf{m}}_{\rho(2)}(p) + 1 & \text{by (2)} \\
\Rightarrow & \forall p \in \mathcal{P}' & : & \mathbf{m}(p) > \widehat{\mathbf{m}}_{\rho(2)}(p) &
\end{array}$$

(b) By monotonicity of PN+NBA, we have that $\mathbf{m} \succcurlyeq \widehat{\mathbf{m}}'_{\rho(1)}$. Moreover, by (1), we have that $\widehat{\mathbf{m}}'_{\rho(1)} \succcurlyeq \widehat{\mathbf{m}}_{\rho(1)}$. Hence, $\forall p \in \mathcal{P} : \mathbf{m}(p) \geq \widehat{\mathbf{m}}_{\rho(1)}(p)$. As a consequence, $\forall p \in \mathcal{P} \setminus \mathcal{P}' : \mathbf{m}(p) \geq \widehat{\mathbf{m}}_{\rho(2)}(p)$ from (3).

From (a) and (b), we obtain $\mathbf{m} \succcurlyeq \widehat{\mathbf{m}}_{\rho(2)}$, hence $\sigma_{\rho(2)}$ is firable from $\mathbf{m}$. So, we have

$$\mathbf{m}_{init} \xrightarrow{\left(\mathtt{a}^{\rho(3+n)}\mathtt{b}^{\rho(3+n)}\right)^{\ell_{\rho(3+n)}}\mathtt{a}^{\rho(3+n)}} \widehat{\mathbf{m}}_{\rho(3+n)} \xrightarrow{\overbrace{\left(\mathtt{b}^{\rho(1)}\mathtt{a}^{\rho(1)}\right)^{n_{\rho(1)}}}^{\Lambda(\sigma_{\rho(1)})}} \mathbf{m} \xrightarrow{\overbrace{\left(\mathtt{b}^{\rho(2)}\mathtt{a}^{\rho(2)}\right)^{n_{\rho(2)}}}^{\Lambda(\sigma_{\rho(2)})}} \mathbf{m}'$$

Finally, let us prove that we can fire $\sigma_{\rho(2)}$ infinitely often from $\mathbf{m}'$. Since $\mathbf{m} \succcurlyeq \widehat{\mathbf{m}}_{\rho(2)}$ and $\widehat{\mathbf{m}}_{\rho(2)} \xrightarrow{\sigma_{\rho(2)}} \widehat{\mathbf{m}}'_{\rho(2)}$, we have by monotonicity that $\mathbf{m}' \succcurlyeq \widehat{\mathbf{m}}'_{\rho(2)} \succcurlyeq \widehat{\mathbf{m}}_{\rho(2)}$, hence $\mathbf{m}' \xrightarrow{\sigma_{\rho(2)}} \mathbf{m}''$ for some marking $\mathbf{m}'' \succcurlyeq \widehat{\mathbf{m}}'_{\rho(2)} \succcurlyeq \widehat{\mathbf{m}}_{\rho(2)}$. Since we can repeat the reasoning infinitely often from $\mathbf{m}''$, we conclude that $\sigma_{\rho(2)}$ can be fired infinitely often from $\mathbf{m}''$ and $\left(\mathtt{a}^{\rho(3+n)}\mathtt{b}^{\rho(3+n)}\right)^{\ell_{\rho(3+n)}}\mathtt{a}^{\rho(3+n)}\left(\mathtt{b}^{\rho(1)}\mathtt{a}^{\rho(1)}\right)^{n_{\rho(1)}}\left(\mathtt{b}^{\rho(2)}\mathtt{a}^{\rho(2)}\right)^{\omega}$ is a word of $L^\omega(\mathcal{N})$ (with $\rho(3+n) > \rho(2) > \rho(1) > 0$ and $n_{\rho(1)} > 0$). But, following Lemma 6, this word is not in $L^\omega(\mathcal{N}_1)$. We conclude that $L^\omega(\mathcal{N}_1) \subsetneq L^\omega(\mathcal{N})$. □

We can now state the main Theorem of this section:

Theorem 2 PN+T *are more expressive, on infinite words, than* PN+NBA*, i.e.:* $L^\omega(\text{PN+NBA}) \subsetneq L^\omega(\text{PN+T})$.

Proof. The theorem stems from Theorem 1 and Lemma 7. □

Since our construction proposed to build a PN+T $\mathcal{N}'$ from a PN+NBA $\mathcal{N}$ that has the same ω-language than $\mathcal{N}$ does not use τ-transitions, $\mathcal{N}_1$ contains no τ-transitions and since we have made no assumptions regarding the τ-transitions in the previous proofs, we obtain:

Corollary 1 PN+T *without τ-transitions are more expressive on infinite words than* PN+NBA*, i.e.:* $L^\omega_{\not\tau}(\text{PN+NBA}) \subsetneq L^\omega_{\not\tau}(\text{PN+T})$.

4 PN+NBA are more expressive than PN

In this section we prove the second main result of the paper: the class of ω-languages accepted by any PN+NBA strictly contains the class of ω-languages accepted by any PN.

The strategy adopted in the proof is similar to the one we have used in section 3. We look into the PN+NBA $\mathcal{N}_2$ of Fig. 2 (b), and prove it accepts every words of the form $\mathtt{i}^k\mathtt{s}\left(\mathtt{a}^k\mathtt{c}\mathtt{b}^k\mathtt{d}\right)^\omega$, for $k \geq 1$ (Lemma 8), but rejects words of the form $\mathtt{i}^{n_3}\mathtt{s}\left(\mathtt{a}^{n_3}\mathtt{c}\mathtt{b}^{n_3}\mathtt{d}\right)^m\mathtt{a}^{n_3}\mathtt{c}\left(\mathtt{b}^{n_1}\mathtt{d}\mathtt{a}^{n_1}\mathtt{c}\right)^k\left(\mathtt{b}^{n_2}\mathtt{d}\mathtt{a}^{n_2}\mathtt{c}\right)^\omega$, for k big enough, and $0 < n_1 < n_2 < n_3$ (Lemma 9). Then, we prove Lemma 10, stating that any PN accepting at least the words of the first form must also accept the words

of the latter form. We conclude that no PN can accept $L^\omega(\mathcal{N}_2)$. Since any PN is also a PN+NBA, the inclusion is immediate, and we obtain Theorem 3, that states the strictness of the inclusion $L^\omega(\mathsf{PN}) \subsetneq L^\omega(\mathsf{PN{+}NBA})$.

Let us consider the PN+NBA $\mathcal{N}_2$ in Figure 2 (b), with the initial marking $\mathbf{m}_0$ such that $\mathbf{m}_0(p_1) = 1$ and $\mathbf{m}_0(p) = 0$ for $p \in \{p_2, p_3, p_4, p_5, p_6\}$.

Lemma 8 *For any $k \geq 0$, the word $\mathtt{i}^k\mathtt{s}\left(\mathtt{a}^k\mathtt{cb}^k\mathtt{d}\right)^\omega$ is accepted by $\mathcal{N}_2$.*

Lemma 9 *Let n_1, n_2 and n_3 be three natural numbers such that $0 < n_1 < n_2 < n_3$. Then, for all $m > 0$, $\forall k \geq n_3 - n_1 - 1$: the words*

$$\mathtt{i}^{n_3}\mathtt{s}\left(\mathtt{a}^{n_3}\mathtt{cb}^{n_3}\mathtt{d}\right)^m\mathtt{a}^{n_3}\mathtt{c}\left(\mathtt{b}^{n_1}\mathtt{da}^{n_1}\mathtt{c}\right)^k\left(\mathtt{b}^{n_2}\mathtt{da}^{n_2}\mathtt{c}\right)^\omega$$

are not accepted by $\mathcal{N}_2$.

These two proofs are quite straightforward and can be found in the appendix.

We are now ready to prove that no PN accepts exactly the ω-language of the PN+NBA $\mathcal{N}_2$.

Lemma 10 *No* PN *accepts $L^\omega(\mathcal{N}_2)$*

Proof. Let $\mathcal{N}$ be a PN such that $L^\omega(\mathcal{N}_2) \subseteq L^\omega(\mathcal{N})$. We will show that this implies that $L^\omega(\mathcal{N}_2) \subsetneq L^\omega(\mathcal{N})$.

Suppose that $\mathbf{m}_{init}$ is the initial marking of $\mathcal{N}$. Following Lemma 8, since $L^\omega(\mathcal{N}_2) \subseteq L^\omega(\mathcal{N})$, we have $\forall k \geq 1 : \mathtt{i}^k\mathtt{s}\left(\mathtt{a}^k\mathtt{cb}^k\mathtt{d}\right)^\omega \in L(\mathcal{N})$. Thus, for all $k \geq 1$, there exists a marking $\widehat{\mathbf{m}}_k$, a sequence of transitions σ_k and a natural ℓ_k such that:

$$\mathbf{m}_{init} \xrightarrow{\mathtt{i}^k\mathtt{s}\left(\mathtt{a}^k\mathtt{cb}^k\mathtt{d}\right)^{\ell_k}\mathtt{a}^k\mathtt{c}} \widehat{\mathbf{m}}_k \xrightarrow{\Lambda(\sigma_k)} \widehat{\mathbf{m}}'_k \text{ with } \widehat{\mathbf{m}}_k \preccurlyeq \widehat{\mathbf{m}}'_k \text{ and } \Lambda(\sigma_k) \in \left(\mathtt{b}^k\mathtt{da}^k\mathtt{c}\right)^+$$

Indeed, suppose that it is not the case, we would have $\mathbf{m}_{init} \xrightarrow{\mathtt{i}^k\mathtt{sa}^k\mathtt{c}} \mathbf{m}_1 \xrightarrow{\mathtt{b}^k\mathtt{da}^k\mathtt{c}} \ldots \xrightarrow{\mathtt{b}^k\mathtt{da}^k\mathtt{c}} \mathbf{m}_i \xrightarrow{\mathtt{b}^k\mathtt{da}^k\mathtt{c}} \ldots$ such that there do not exist $1 \leq i < j$ with $\mathbf{m}_i \preccurlyeq \mathbf{m}_j$. But, from Lemma 2, this never occurs.

Let us consider the sequence $\widehat{\mathbf{m}}_1, \widehat{\mathbf{m}}_2, \widehat{\mathbf{m}}_3, \ldots$ Following Lemma 2, we extract an infinite sub-sequence $\widehat{\mathbf{m}}_{\rho(1)}, \widehat{\mathbf{m}}_{\rho(2)}, \widehat{\mathbf{m}}_{\rho(3)}, \ldots$ such that $\forall p \in P$: either $\forall i \geq 1 : \widehat{\mathbf{m}}_{\rho(i)}(p) = \widehat{\mathbf{m}}_{\rho(i+1)}(p)$ or $\forall i \geq 1 : \widehat{\mathbf{m}}_{\rho(i)}(p) < \widehat{\mathbf{m}}_{\rho(i+1)}(p)$ (each place stays constant or strictly increases along the sequence).

Since $\widehat{\mathbf{m}}_{\rho(3)} \succcurlyeq \widehat{\mathbf{m}}_{\rho(1)}$ and $\sigma_{\rho(1)}$ has a non-negative and constant effect on each place, i.e. the effect is characterized by a tuple of naturals, we can fire $\sigma_{\rho(1)}$ any number of time from $\widehat{\mathbf{m}}_{\rho(3)}$, i.e. for all $k' \geq 0$ we have $\widehat{\mathbf{m}}_{\rho(3)} \xrightarrow{\left(\sigma_{\rho(1)}\right)^{k'}} \mathbf{m}^{k'}$ with $\mathbf{m}^{k'} \succcurlyeq \widehat{\mathbf{m}}_{\rho(3)}$. Since $\widehat{\mathbf{m}}_{\rho(3)} \succcurlyeq \widehat{\mathbf{m}}_{\rho(2)}$ and $\sigma_{\rho(2)}$ has a constant non-negative effect on each place, $\sigma_{\rho(2)}$ can be fired infinitely often from $\mathbf{m}^{k'}$ for any $k' \geq 1$. Thus, we have

$$\mathbf{m}_{init} \xrightarrow{\mathtt{i}^{\rho(3)}\mathtt{s}\left(\mathtt{a}^{\rho(3)}\mathtt{cb}^{\rho(3)}\mathtt{d}\right)^{\ell_{\rho(3)}}\mathtt{a}^{\rho(3)}\mathtt{c}} \widehat{\mathbf{m}}_{\rho(3)} \xrightarrow{(\sigma_{\rho(1)})^{k'}} \mathbf{m}^{k'} \xrightarrow{(\sigma_{\rho(2)})^\omega}$$

Following Lemma 9, if we choose k' large enough (that is, $k' \geq \rho(3)-\rho(1)-1$), the word accepted by the previous sequence is not in $L^\omega(\mathcal{N}_2)$. We conclude that $L^\omega(\mathcal{N}_2) \subsetneq L^\omega(\mathcal{N})$. □

This result allows us to state the second important theorem of the paper:

Theorem 3 PN+NBA *are more expressive, on infinite words, than* PN*, i.e.:* $L^\omega(\mathsf{PN}) \subsetneq L^\omega(\mathsf{PN{+}NBA})$.

Proof. As the PN class is a syntactic subclass of the PN+NBA, each PN-language is also a PN+NBA-language. On the other hand, some PN+NBA-languages are not PN-languages, by Lemma 10. Hence the Theorem. □

Again, since PN is a syntactic subclass of PN+NBA and we have made no assumptions about the τ-transitions in the previous proofs, and since $\mathcal{N}_2$ contains no τ-transition, we obtain:

Corollary 2 PN+NBA *are more expressive, on infinite words and without* τ*-transitions than* PN*, i.e.:* $L^\omega_\mathcal{T}(\mathsf{PN}) \subsetneq L^\omega_\mathcal{T}(\mathsf{PN{+}NBA})$.

5 Reset nets

In this section we show how Petri nets with reset arcs – another widely studied class of Petri nets [Pet81, DFS98] – fit into our classification. We first recall the definition of this class, then show that it is as expressive, on ω-languages, than PN+T. It is important to remark here that our construction requires τ-transitions.

An extended Petri net $\mathcal{N} = \langle \mathcal{P}, \mathcal{T}, \Sigma, \mathbf{m}_0 \rangle$ is a **Petri net with reset arcs** (PN+R for short) if it is a PN+T, with the following additional restrictions: (i) there exists a place $p_{Tr} \in \mathcal{P}$ that is not an input or source place of any transition of $\mathcal{T}$ and (ii) for any extended transition $t = \langle I, O, s, d, +\infty, \lambda \rangle \in \mathcal{T}_e$, $d = p_{Tr}$. The special place p_{Tr} is called the *trashcan*. Intuitively, we see the reset of a place as a transfer where the consumed tokens are sent to the trashcan, from which they can never escape.

Let us now exhibit a construction to prove that any ω-language accepted by a PN+T can also be accepted by a PN+R. We consider the PN+T $\mathcal{N}_t = \langle \mathcal{P}, \mathcal{T}, \Sigma, \mathbf{m}_0 \rangle$, and build the reset $\mathcal{N}_r = \langle \mathcal{P}', \mathcal{T}', \Sigma, \mathbf{m}'_0 \rangle$ as follows. Let $\mathcal{P}' = \mathcal{P} \cup \{p_b, p_{Tr}, \} \cup \{p_t | t \in \mathcal{T}_e\}$. Then for each transition $t = \langle I, O, s, d, +\infty, \lambda \rangle \in \mathcal{T}_e$, we put three transitions in $\mathcal{T}'$: $t^s = \langle I \uplus \{p_b\}, \{p_t\}, \bot, \bot, 0, \lambda \rangle$; $t^c = \langle \{p_t, s\}, \{p_t, d\}, \bot, \bot, 0, \tau \rangle$; and $t^e = \langle \{p_t\}, O \uplus \{p_b\}, s, p_{Tr}, +\infty, \tau \rangle$. For any $t = \langle I, O, \bot, \bot, 0, \lambda \rangle \in \mathcal{T}_r$, we add $t' = \langle I \uplus \{p_b\}, O \cup \{p_b\}, \bot, \bot, 0, \lambda \rangle$ in $\mathcal{T}'$. Finally, $\forall p \in \mathcal{P} : \mathbf{m}'_0(p) = \mathbf{m}_0(p)$, $\mathbf{m}'_0(p_b) = 1$, $\mathbf{m}'_0(p_{Tr}) = 0$ and $\forall t \in \mathcal{T}_e : \mathbf{m}'_0(p_t) = 0$. Fig. 3 shows the construction.

Let us now prove that the PN+R obtained thanks to this construction has the same ω-language as the PN+T it corresponds to.

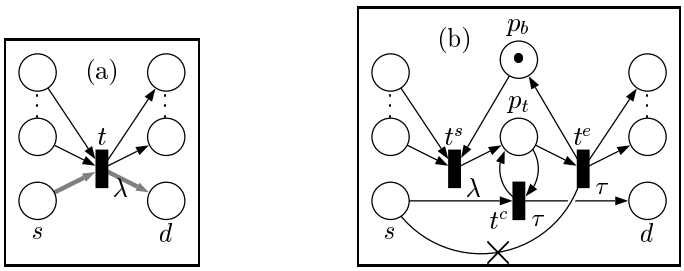

Figure 3: How to transform a PN+T (left) into a PN+R (right). The edge bearing a × links the (source) place to be reset to the extended transition. p_{Tr} is not shown.

Lemma 11 $L^\omega(\mathcal{N}_r) = L^\omega(\mathcal{N}_t)$.

Proof. $[L^\omega(\mathcal{N}_t) \subseteq L^\omega(\mathcal{N}_r)]$ Let $\sigma = t_1 t_2 \ldots$ be a sequence of transitions of $\mathcal{N}_t$. Then, $\mathcal{N}_r$ accepts $\Lambda(\sigma)$ thanks to σ' built as follows. We simply replace in σ each regular Petri transition t by t' and each extended transition $t = \langle I, O, s, d, b, \lambda \rangle$ by $\sigma_t = t^s (t^c)^{(k-I(s))} t^e$, where k is the marking of place s that is reached in $\mathcal{N}_r$ before the firing of σ_t. Clearly $\Lambda(\sigma_t) = \Lambda(t)$ an their respective effects are equal on the places in $\mathcal{P}$.

$[L^\omega(\mathcal{N}_r) \subseteq L^\omega(\mathcal{N}_t)]$ Let $\sigma' = t'_1 t'_2 \ldots$ be an infinite sequence of transitions of $\mathcal{N}_r$ such that $\mathbf{m}'_0 \xrightarrow{t'_1} \mathbf{m}'_1 \xrightarrow{t'_2} \mathbf{m}'_2 \ldots$. We first extract from σ' the subsequences $t^s (t^c)^n t^e (n \in \mathbb{N})$ that correspond to a given extended transition t in $\mathcal{N}_t$. Thus, we obtain $\mathbf{m}'_0 = \mathbf{m}'_{k_0} \xrightarrow{\sigma_1} \mathbf{m}'_{k_1} \xrightarrow{\sigma_2} \mathbf{m}'_{k_2} \ldots$, where σ_i is either a single regular Petri transition t'_{k_i} corresponding to the simple regular Petri transition t_{k_i} or a sequence σ_t corresponding to the extended transition t. This is possible since the firing of t^s will remove the token from p_b and block the whole net. Hence no transitions can interleave with $t^s (t^c)^* t^e$. Moreover, σ' cannot have a suffix of the form $t^s (t^c)^\omega$ since t^c decreases the marking of the source place of the transfer of the corresponding transition t.

Then, we replace each σ_i of length > 1 by the transition t it corresponds to in $\mathcal{N}_t$. Hence, we obtain a new sequence $\sigma = t_1 t_2 \ldots$ of $\mathcal{N}_t$. Clearly, $\Lambda(\sigma) = \Lambda(\sigma')$. Let us now prove that σ is firable, i.e. $\mathbf{m}_0 \xrightarrow{t_1} \mathbf{m}_1 \xrightarrow{t_2} \mathbf{m}_2 \xrightarrow{t_3} \ldots$, by showing that $\forall i \geq 0 : \mathbf{m}'_{k_i} \preccurlyeq_{\mathcal{P}} \mathbf{m}_i$.

Base case: $i = 0$. The base case is trivially verified.

Induction Step: $i = l$. By induction hypothesis, we have that $\forall 0 \leq i \leq l-1 : \mathbf{m}'_{k_i} \preccurlyeq_{\mathcal{P}} \mathbf{m}_i$. In the case where t_l is a regular transition, it has the same effect on the places in $\mathcal{P}$ as $\sigma_l = t'_{k_l}$ and it can occur since $\mathbf{m}'_{k_{l-1}} \preccurlyeq_{\mathcal{P}} \mathbf{m}_{l-1}$. Hence $\mathbf{m}'_{k_l} \preccurlyeq_{\mathcal{P}} \mathbf{m}_l$, by monotonicity. Otherwise t_l is an extended transition and its effect corresponds to the effect of σ_l. Let us observe the effect of σ_l: some tokens will be taken from s (the source place of the transfer) and put into d (the destination) by t^c_l. Finally, the tokens remaining in s will be removed by the reset arc of t^e_l. Hence, σ_l removes the same number of tokens from s than t_l, and cannot put more tokens in d than t_l does. Moreover, the effect of σ_l on

Figure 4: Summary of the results

the other places is the same than t_l. Thus $\mathbf{m}'_{k_l} \preccurlyeq_{\mathcal{P}} \mathbf{m}_l$. □

We can thus close the section with the following Theorem.

Theorem 4 PN+R *are as expressive as* PN+T *on infinite words with* τ*-transitions.*

Proof. As any PN+R is a special case of PN+T, we have that $L^\omega(\mathsf{PN{+}R}) \subseteq L^\omega(\mathsf{PN{+}T})$. The other direction stems from Lemma 11. □

In the case where we disallow τ-transitions, the previous construction doesn't allow to prove whether $L^\omega_{\not\tau}(\mathsf{PN{+}T}) \subseteq L^\omega_{\not\tau}(\mathsf{PN{+}R})$ or not. However, we have that $L^\omega(\mathsf{PN{+}NBA}) \subsetneq L^\omega(\mathsf{PN{+}R})$, since the PN+T $\mathcal{N}_1$ we have used in the proof of Lemma 7 satisfies our definition of PN+R (in this case, the place p_4 is the trashcan) and has no τ-transitions.

6 Conclusion

In the introduction of this paper, we have recalled how important extended Petri nets are to study the *non-terminating* behaviour of concurrent systems made up of an arbitrary number of communicating processes (once abstracted thanks to predicate- and counting- abstraction techniques [BCR01]). Our aim was thus to study and classify the expressive powers of these models, as far as ω-languages are concerned. This goal has been thoroughly fulfilled. Indeed, we have proved in section 3 that any ω-language accepted by a PN+NBA can be accepted by a PN+T, but that there exist ω-languages that are recognised by a PN+T but not by a PN+NBA. A similar result has been demonstrated for PN+NBA and PN in section 4. These results hold with or without τ-transitions. Finally, in section 5 we have drawn a link between these results and the class of PN+R. Fig. 4 summarises all these results.

Future works In [Pet81], Peterson studies different classes of *finite words* languages of PNand Ciardo [Cia94] extends the study to Petri nets *with marking-dependant arc multiplicity*, which subsume the four classes we have studied here. The paper states some relations between these classes, but keeps several questions open. To the best of out knowledge, most of them are still open, and we strive for applying the new proof techniques developed in this paper to solve those open problems.

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A Proof of the Lemmata

A.1 Proof of Lemma 2

Given an infinite sequence of markings $\mathbf{m}_1\mathbf{m}_2\ldots$*, we can always extract an infinite sub-sequence* $\mathbf{m}_{i_1}\mathbf{m}_{i_2}\ldots$ $(\forall j : i_j < i_{j+1})$ *such that for all place* p*, either* $\mathbf{m}_{i_j}(p) < \mathbf{m}_{i_{j+1}}(p)$ *for all* $j \geq 1$ *or* $\mathbf{m}_{i_j}(p) = \mathbf{m}_{i_{j+1}}(p)$ *for all* $j \geq 1$.

Proof. We exhibit an inductive reasoning on the dimension n (number of places) of the markings.

Base case: Let $n = 1$. Then $\mathbf{m}_1\mathbf{m}_2\ldots$ is a sequence of elements in $\mathbb{N}$. If one of the elements of the sequence occurs infinitely often, the property is trivially

true. Otherwise, let *min* be the minimal value in this sequence and i_1 be its last index, then we take $\mathbf{m}_{i_1}$ as the first element and we iterate the construction on $\mathbf{m}_{i_1+1}\mathbf{m}_{i_1+2}\ldots$ We then obtain the desired sequence $\mathbf{m}_{i_1}\mathbf{m}_{i_2}\ldots$

Induction step: Let $\mathbf{m}_1\mathbf{m}_2\ldots$ be a sequence of markings and $\mathbf{m}_{i_1}\mathbf{m}_{i_2}\ldots$ be an infinite sub-sequence such that each of the $n-1$ first dimensions are constant or always increase. Such a sub-sequence exists by induction hypothesis. By applying the procedure of the base case, we can extract a sub-sequence of it which is either constant or strictly increasing for the n-th dimension. □

A.2 Proof of Lemma 5

For any $k \geq 1$, the word $\left(\mathtt{a}^k\mathtt{b}^k\right)^\omega$ is accepted by $\mathcal{N}_1$. **Proof.** The following holds for any $k \geq 1$. From the initial marking of $\mathcal{N}_1$, we can fire $t_1^k t_2 t_3^{k-1}$ (which accepts $\mathtt{a}^k\mathtt{b}^k$), and reach the marking $\mathbf{m}_1$ such that $\mathbf{m}_1(p_2) = 1$ and $\forall i \in \{1,3,4\} : \mathbf{m}_1(p_i) = 0$. Thus, t_4 is firable from $\mathbf{m}_1$ and does not transfer any token, but produces a token in p_3 and moves the token from p_2 to p_1. The marking that is obtained is $\mathbf{m}_2$ such that $\mathbf{m}_2(p_1) = \mathbf{m}_2(p_3) = 1$ and $\mathbf{m}_2(p_2) = \mathbf{m}_2(p_4) = 0$. It is not difficult to see now that $\mathbf{m}_2 \xrightarrow{t_1^{k-1}t_2t_3^{k-1}} \mathbf{m}_1 \xrightarrow{t_4} \mathbf{m}_2$. We can thus fire the sequence $t_1^{k-1}t_2t_3^{k-1}t_4$ arbitrarily often from $\mathbf{m}_2$. Hence $\left(\mathtt{a}^k\mathtt{b}^k\right)^\omega$ is accepted by $\mathcal{N}_1$. □

A.3 Proof of Lemma 6

Let n_1, n_2, n_3 and m be four natural numbers such that $0 < n_1 < n_2 < n_3$ and $m > 0$. Then, for any $k \geq 0$, the words $(\mathtt{a}^{n_3}\mathtt{b}^{n_3})^k\mathtt{a}^{n_3}(\mathtt{b}^{n_1}\mathtt{a}^{n_1})^m(\mathtt{b}^{n_2}\mathtt{a}^{n_2})^\omega$ are not accepted by $\mathcal{N}_1$. **Proof.** The following holds for any n_1, n_2, n_3 with $0 < n_1 < n_2 < n_3$ and for any $m > 0$ and $k \in \mathbb{N}$. From the initial marking of $\mathcal{N}_1$, the only sequence of transitions labelled by $\mathtt{a}^{n_3}$ is $t_1^{n_3}$. Firing this sequence leads to the marking $\mathbf{m}_1$ such that $\mathbf{m}_1(p_1) = 1, \mathbf{m}_1(p_3) = n_3$ and $\mathbf{m}_1(p) = 0$ if $p \in \{p_2, p_4\}$. From $\mathbf{m}_1$ the only firable sequence of transitions labelled by $\mathtt{b}^{n_3}$ is $t_2t_3^{n_3-1}$. This leads to the marking $\mathbf{m}_2$ such that $\mathbf{m}_2(p_2) = 1$ and $\mathbf{m}_2(p) = 0$ if $p \neq p_2$. The only sequence of transitions firable from $\mathbf{m}_2$ and labelled by $\mathtt{a}^{n_3}$ is $t_4t_1^{n_3-1}$. Since $\mathbf{m}_2(p_3) = 0$, the transfer of t_4 has no effect when fired from $\mathbf{m}_2$. Hence, we reach $\mathbf{m}_1$ again after firing $t_4t_1^{n_3-1}$. By repeating the reasoning, we conclude that the only sequence of transitions firable from the initial marking and labelled by $(\mathtt{a}^{n_3}\mathtt{b}^{n_3})^k\mathtt{a}^{n_3}$ (when $k > 0$) is $t_1^{n_3}t_2t_3^{n_3-1}(t_4t_1^{n_3-1}t_2t_3^{n_3-1})^{k-1}t_4t_1^{n_3-1}$ and leads to $\mathbf{m}_1$. In the case where $k = 0$, the sequence $t_1^{n_3}$ is firable and leads to $\mathbf{m}_1$ too. From $\mathbf{m}_1$, the only firable sequence of transitions labelled by $\mathtt{b}^{n_1}$ is $t_2t_3^{n_1-1}$. This leads to a marking similar to $\mathbf{m}_2$, noted $\mathbf{m}_2'$, except that p_3 contains $n_3 - n_1$ tokens. Then, the only firable sequence of transitions labelled by $\mathtt{a}^{n_1}$ is $t_4t_1^{n_1-1}$. In this case, the transfer of t_4 moves the $n_3 - n_1$ tokens from p_3 to p_4 and we reach a marking similar to $\mathbf{m}_1$, noted $\mathbf{m}_1'$, except that p_4 contains $n_3 - n_1$ tokens and p_3 contains

n_1 tokens. From $\mathbf{m}'_1$, the only firable sequence of transitions labelled by $\mathtt{b}^{n_1}\mathtt{a}^{n_1}$ is $t_2 t_3^{n_1-1} t_4 t_1^{n_1-1}$ and leads to $\mathbf{m}'_1$.

However, after firing $t_2 t_3^{n_1-1}$ from $\mathbf{m}'_1$, we reach a marking $\mathbf{m}''_2$ similar to $\mathbf{m}_2$ except that p_4 contains $n_3 - n_1$ tokens and from which no transition labelled by $\mathtt{b}$ is firable. Since $n_2 > n_1$, we conclude that there is no sequence of transitions labelled by $\mathtt{b}^{n_2}$ that is firable from $\mathbf{m}'_1$, hence $(\mathtt{a}^{n_3}\mathtt{b}^{n_3})^k \mathtt{a}^{n_3}(\mathtt{b}^{n_1}\mathtt{a}^{n_1})^m(\mathtt{b}^{n_2}\mathtt{a}^{n_2})^\omega$ is not accepted by $\mathcal{N}_1$. □

A.4 Proof of Lemma 8

For any $k \geq 0$, the word $\mathtt{i}^k\mathtt{s}(\mathtt{a}^k\mathtt{c}\mathtt{b}^k\mathtt{d})^\omega$ is accepted by $\mathcal{N}_2$. **Proof.** The following holds for any $k \geq 0$. After firing the transitions $t_1^k t_2$ from the initial marking of $\mathcal{N}_2$, we reach the marking $\mathbf{m}_1$ such that $\mathbf{m}_1(p_2) = k$, $\mathbf{m}_1(p_3) = 1$, and $\mathbf{m}_1(p_j) = 0$ for $j \in \{1, 4, 5, 6\}$. Then, we can fire $t_3^k t_4$ from $\mathbf{m}_1$. This leads to the marking $\mathbf{m}_2$ such that $\mathbf{m}_2(p_4) = k$, $\mathbf{m}_2(p_5) = 1$, and $\mathbf{m}_2(p_j) = 0$ for $j \in \{1, 2, 3, 6\}$. From $\mathbf{m}_2$, t_5^k can be fired. This sequence of transitions moves the k tokens from p_4 to p_2. Then, from the resulting marking, t_6 can be fired. Since, p_4 is now empty, the effect of t_6 only consists in moving the token from p_5 to p_3 (its non-blocking arc has no effect) and we reach $\mathbf{m}_1$ again. Then, by applying the same reasoning, we fire infinitely often $t_3^k t_4 t_5^k t_6$. The word corresponding to such a sequence is $\mathtt{i}^k\mathtt{s}(\mathtt{a}^k\mathtt{c}\mathtt{b}^k\mathtt{d})^\omega$. □

A.5 Proof of Lemma 9

Let n_1, n_2 and n_3 be three natural numbers such that $0 < n_1 < n_2 < n_3$. Then, for all $m \geq 0$, $\forall k \geq n_3 - n_1 - 1$: the words

$$\mathtt{i}^{n_3}\mathtt{s}(\mathtt{a}^{n_3}\mathtt{c}\mathtt{b}^{n_3}\mathtt{d})^m\mathtt{a}^{n_3}\mathtt{c}(\mathtt{b}^{n_1}\mathtt{d}\mathtt{a}^{n_1}\mathtt{c})^k(\mathtt{b}^{n_2}\mathtt{d}\mathtt{a}^{n_2}\mathtt{c})^\omega$$

are not accepted by $\mathcal{N}_2$. **Proof.** In this proof, we will identify a sequence of transitions with the word it accepts (all the transitions have different labels). Clearly (see the proof of Lemma 8), the firing of $\mathtt{i}^{n_3}\mathtt{s}(\mathtt{a}^{n_3}\mathtt{c}\mathtt{b}^{n_3}\mathtt{d})^m$ from $\mathbf{m}_0$ leads to a marking $\mathbf{m}_1$ such that $\mathbf{m}_1(p_2) = n_3$, $\mathbf{m}_1(p_3) = 1$, and $\forall i \in \{1, 4, 5, 6\}$: $\mathbf{m}_1(p_i) = 0$ (the non-blocking arc of t_6 hasn't consumed any token in p_4). By firing $\mathtt{a}^{n_3}\mathtt{c}\mathtt{b}^{n_1}\mathtt{d}$ from $\mathbf{m}_1$, we now have n_1 tokens in p_2 and $n_3 - n_1 - 1$ tokens in p_4 (this time the non-blocking arc has moved one token since $n_1 < n_3$). Clearly, at each subsequent firing of $\mathtt{a}^{n_1}\mathtt{c}\mathtt{b}^{n_1}\mathtt{d}$, the non-blocking arc of t_6 will remove one token from p_4 and its marking will strictly decrease until it becomes empty. Let $\ell = n_3 - n_1 - 1$. It is easy to see that $(\mathtt{a}^{n_1}\mathtt{c}\mathtt{b}^{n_1}\mathtt{d})^\ell$ leads to a marking $\mathbf{m}_2$ with $\mathbf{m}_2(p_2) = n_1$ $\mathbf{m}_2(p_3) = 1$ and $\forall j \in \{1, 4, 5, 6\} : \mathbf{m}_2(p_j) = 0$. This characterisation also implies that we can fire $\mathtt{a}^{n_1}\mathtt{c}\mathtt{b}^{n_1}\mathtt{d}$ an arbitrary number of times from $\mathbf{m}_2$ because $\mathbf{m}_2 \xrightarrow{\mathtt{a}^{n_1}\mathtt{c}\mathtt{b}^{n_1}\mathtt{d}} \mathbf{m}_2$. On the other hand, it is not possible to fire $\mathtt{a}^{n_1}\mathtt{c}\mathtt{b}^{n_2}\mathtt{d}$, with $n_2 > n_1$, from $\mathbf{m}_2$. Indeed $\mathbf{m}_2 \xrightarrow{\mathtt{a}^{n_1}\mathtt{c}\mathtt{b}^{n_1}} \mathbf{m}_3$, with $\mathbf{m}_3(p_5) = 1$, $\mathbf{m}_3(p_2) = n_1$ and $\forall j \in \{1, 3, 4, 6\} : \mathbf{m}_3(p_j) = 0$, which does not allow to fire the

$\mathtt{b}$-labelled transition t_5 anymore. We conclude that, $\forall k \geq \ell = n_3 - n_1 - 1$, a sequence labelled by $\mathtt{i}^{n_3}\mathtt{s}\big(\mathtt{a}^{n_3}\mathtt{cb}^{n_3}\mathtt{d}\big)^m\mathtt{a}^{n_3}\mathtt{c}\big(\mathtt{b}^{n_1}\mathtt{da}^{n_1}\mathtt{c}\big)^k\mathtt{b}^{n_2}\mathtt{da}^{n_2}\mathtt{c}$, is not firable in $\mathcal{N}_2$. Thus, we will not find in $L^\omega(\mathcal{N}_2)$ any word with this prefix, hence the Lemma. □