Acta Informatica manuscript No.
(will be inserted by the editor)

Gilles Geeraerts · **Jean-François Raskin** ·
Laurent Van Begin

Well-Structured Languages

the date of receipt and acceptance should be inserted later

Abstract This paper introduces the notion of *well-structured language*. A well-structured language can be defined by a *labelled well-structured transition system*, equipped with an *upward-closed set* of accepting states. That peculiar class of transition systems has been extensively studied in the field of *computer-aided verification*, where it has direct an important applications. Petri nets, and their monotonic extensions (like Petri nets with non-blocking arcs or Petri nets with transfer arcs), for instance, are special subclasses of well-structured transition systems.

We show that the class of well-structured languages enjoy several important closure properties. We propose several pumping lemmata that are applicable respectively to the whole class of well-structured languages and to the classes of languages recognized by Petri nets or Petri nets with non-blocking arcs. These pumping lemmata allow us to characterize the limits in the expressiveness of these classes of language. Furthermore, we exploit the pumping lemmata to strictly separate the expressive power of Petri nets, Petri nets with non-blocking arcs and Petri nets with transfer arcs.

1 Introduction

In this paper, we study the family of languages defined by *well-structured (labelled) transition systems* (WSTS for short). WSTS [10] are transition systems whose state space is infinite but equipped with a well-quasi ordering (wqo for

Laurent Van Begin has been supported by a "First Europe" grant EPH3310300R0012 of the Walloon Region.

This work has been partially supported by a FRFC grant number 2.4530.02 of the belgian National Fund for Scientific Research (FNRS).

Université Libre de Bruxelles – Département d'Informatique – CPI 212 – Boulevard du Triomphe, B-1050 Bruxelles BELGIUM. E-mail: {gigeerae, jraskin, lvbegin}@ulb.ac.be

short) and whose transition relation is monotonic w.r.t. this wqo. WSTS have recently attracted a large interest in the community of *model-checking* because they enjoy nice decidability results and are useful to model important classes of systems (like parametric systems [8] and communication protocols [2]). In particular, the *coverability problem* (a variation of the reachability problem) has been shown decidable for the whole class of WSTS [1,10]. A large number of popular models define WSTS: Petri nets [16], monotonic extensions of Petri nets (e.g., Petri nets with transfer arcs [4]), lossy channel systems [2], broadcast protocols [8].

While the decidability properties of those models have been studied extensively (see, for example [10]), there are few known results about their expressive power in term of *recognized languages*. For example, several extensions of Petri nets have been proposed but their expressive power has not been studied and compared[1] so far.

In a previous paper [9], we have started to study the expressive power of monotonic extensions of Petri nets w.r.t. their ability to define sets of infinite words (omega languages). Unfortunately, the techniques that we had developed in that work were only applicable to omega languages. In the present paper, we generalize those techniques to make them applicable to the study of the expressive power of WSTS measured in term of definable sets of finite words. This classical measure allows us to compare the expressive power of WSTS with other well-studied formalisms like finite automata (defining regular languages), push-down automata (defining context free languages) or Turing machines (defining recursively enumerable languages). We propose proof techniques that intensively use basic properties of wqo. We believe that those proof techniques are interesting on their own.

The main contributions of our paper can be summarized as follows: (i) we define a natural class of languages recognized by WSTS for which the emptiness problem is decidable, (ii) we show that this class has important closure properties and forms an *Abstract Family of Languages* (AFL for short), (iii) to show the limits of the expressive power of WSTS, we introduce a general pumping lemma and show some examples of its possible applications, (iv) we study the relative expressive power of Petri nets and two important monotonic extensions of theirs. This study is made possible by two stronger pumping lemmata for these models.

The rest of this paper is structured as follows. In section 2, we recall some preliminaries about wqo, WSTS and (monotonic extensions of) Petri nets. In section 3, by considering different kinds of accepting conditions, we define three classes of languages recognized by WSTS, and we show that one of them has several interesting properties. That class is called the *well-structured languages* (WSL for short). In section 4, we propose a general pumping lemma applicable to any formalism that defines WSL. Two stronger versions of this lemma are defined and shown applicable to monotonic extensions of Petri nets. In section 5, we use the pumping lemmata to show the limits of WSL, some non-closure properties, and a strict hierarchy of expressive power among the monotonic extensions of Petri nets that we have considered.

[1] Some partial results are known about Petri nets, see for example [16].

2 Preliminaries

In this first section, we recall the main basic results that will be useful in the sequel. More precisely, we recall the classical notions of *languages* and *Abstract Family of Languages* [12, 18]. Then, we define *well-quasi orderings* and *well-structured transitions systems* that form the basis of our definition of well-structured languages. We close the section by recalling the (monotonic extensions of) *Petri nets*, whose languages are actually well-structured.

Throughout this paper, we denote by $\mathbb{N}$ the set $\{0,1,2,\ldots\}$ of *natural numbers* (0 included), and by $\mathbb{Z}^+$ the set $\{1,2,\ldots\}$ of *strictly positive natural numbers*.

Languages and abstract family of languages Given a (finite) alphabet Σ, a (finite) word on Σ is either the empty word ε (we assume that $\varepsilon \notin \Sigma$) or a finite concatenation of symbols in Σ. Given a word w on the alphabet Σ, the *length* of w, denoted $\|w\|$ is defined as follows. If $w = \varepsilon$, then $\|w\| = 0$. Otherwise, $w = a_1 a_2 \cdots a_n$, where $\{a_1, \ldots a_n\} \subseteq \Sigma$, and $\|w\| = n$. A language on Σ is a (possibly infinite) set of words on Σ.

Let $\cdot$ denote the word concatenation. As usual $w \cdot \varepsilon = \varepsilon \cdot w = w$. The concatenation of two languages L_1 and L_2 is the language $L_1 \cdot L_2 = \{w_1 \cdot w_2 \mid w_1 \in L_1, w_2 \in L_2\}$. The iteration of a language L is the language $L^+ = \{w_1 \cdot \ldots \cdot w_n \mid n \geq 1 \wedge \forall 1 \leq i \leq n : w_i \in L\}$. Given a finite alphabet Σ, a *homomorphism* is a function $h : \Sigma^* \mapsto \Sigma^*$ s.t. $\forall w_1, w_2 \in \Sigma^* : h(w_1 \cdot w_2) = h(w_1) \cdot h(w_2)$. The inverse of h is the function $h^{-1} : \Sigma^* \mapsto 2^{\Sigma^*}$ such that $h^{-1}(w) = \{w' \mid h(w') = w\}$. If L is a language on Σ, then $h(L) = \{h(w) \mid w \in L\}$ and $h^-1(L) = \cup_{w \in L} h^-1(w)$.

Definition 1 ([12,18]) A full *abstract family of languages* (full AFL for short) is a set of languages closed under (i) union, (ii) concatenation, (iii) intersection with regular languages, (iv) iteration, (v) homomorphism and (vi) inverse homomorphism. ■

Well-quasi orderings Well-quasi orderings are special cases of quasi orders that are the cornerstone of the definition of WSTS.

Definition 2 A *well quasi ordering* $\leq$ on C (wqo for short) is a *reflexive* and *transitive* relation s.t. for any infinite sequence $c_0, c_1, \ldots$ of elements in C, there are $i, j \in \mathbb{N}$, with $i < j$ and $c_i \leq c_j$. ■

In the sequel, we write $c_i < c_j$ iff $c_i \leq c_j$ but $c_j \not\leq c_i$. When a set C of elements is equipped with an ordering $\leq$, one can define the notion of *upward-closed set*. That notion will be useful in the sequel to define *accepting conditions* of languages of WSTS.

Definition 3 $\mathscr{U} \subseteq C$ is a $\leq$*-upward-closed set* if and only if: for any $c \in \mathscr{U}$, for any $c' \in C$ such that $c \leq c'$: $c' \in \mathscr{U}$. ■

Given a $\leq$-upward closed set $\mathscr{U}$, let $\mathsf{min}(\mathscr{U})$ be a maximal set such that:

- for all $c, c' \in \mathsf{min}(\mathscr{U}) : c \neq c'$ implies $c \not\leq c'$ and $c' \not\leq c$, i.e., all the elements of $\mathsf{min}(\mathscr{U})$ are incomparable to each other;
- $\forall c \in \mathsf{min}(\mathscr{U}) : \neg\exists c' \in \mathscr{U} : c' < c$, i.e. all the elements in $\mathsf{min}(\mathscr{U})$ are $\leq$-minimal in $\mathscr{U}$.

The following lemma is well-known and is a direct consequence of the definitions of min and of a wqo:

Lemma 1 *Given a set C and a wqo $\leq \subseteq C \times C$: for any $\leq$-upward-closed set $\mathscr{U} \subseteq C$: the set $\mathsf{min}(\mathscr{U})$ is finite and $\mathscr{U} = \{c \mid \exists c' \in \mathsf{min}(\mathscr{U}) : c' \leq c\}$.*

Well-structured transition systems These transition systems have the characteristic that their set of configurations is ordered by a wqo $\leq$, and their transition relation is $\leq$-monotonic, as stated by the following definition:

Definition 4 A *(labelled) well-structured transition system* (WSTS for short) is a tuple $\langle C, c_0, \Sigma, \Rightarrow, \leq \rangle$ where:

- C is a (possibly infinite) set of configurations;
- $c_0 \in C$ is the initial configuration;
- Σ is a finite alphabet;
- $\Rightarrow \subseteq C \times \Sigma \cup \{\varepsilon\} \times C$ is the transition relation;
- $\leq$ is a wqo for the elements of C.

Moreover, $\Rightarrow$ is *monotonic* w.r.t. to $\leq$, that is, for any c_1, c_2 and c_3 in C: for any $a \in \Sigma \cup \{\varepsilon\}$: if $(c_1, a, c_2) \in \Rightarrow$ and $c_1 \leq c_3$, then, there exist a finite sequence $\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_k \in C$ (with $k \geq 2$) and $1 \leq \ell < k$, such that:

- $\overline{c}_1 = c_3$;
- for any $1 \leq i < \ell$: $(\overline{c}_i, \varepsilon, \overline{c}_{i+1}) \in \Rightarrow$;
- $(\overline{c}_\ell, a, \overline{c}_{\ell+1}) \in \Rightarrow$;
- for any $\ell + 1 \leq i < k$: $(\overline{c}_i, \varepsilon, \overline{c}_{i+1}) \in \Rightarrow$;
- $c_2 \leq \overline{c}_k$. ■

In the sequel we often write $c_1 \overset{a}{\Rightarrow} c_2$ instead of $(c_1, a, c_2) \in \Rightarrow$. When the character labelling the transition is not relevant, we might omit it and write $c_1 \Rightarrow c_2$ to mean that there exists $a \in \Sigma \cup \{\varepsilon\}$ s.t. $c_1 \overset{a}{\Rightarrow} c_2$.

We also write $c \overset{w}{\Rightarrow}{}^* c'$ to mean that there exists a (finite) sequence of configurations $c_1, c_2, \ldots, c_n$ such that (i) $c \overset{a_0}{\Rightarrow} c_1 \overset{a_1}{\Rightarrow} c_2 \cdots c_n \overset{a_n}{\Rightarrow} c'$ and (ii) $w = a_0 \cdot a_1 \cdots a_n$ (thus, some of the a_i's may be ε). Remark that, for any pair of configurations c_1 and c_2, and any character a, $c_1 \overset{a}{\Rightarrow} c_2$ implies $c_1 \overset{a}{\Rightarrow}{}^* c_2$, but that the reverse implication does not hold. When two configurations c_1 and c_2 are such that $c_1 \overset{w}{\Rightarrow}{}^*_c {}_2$ for some word w, se say that c_2 is *reachable* from c_1.

For any configuration $c \in C$, let $\mathsf{PreUp}(c)$ be the set of all configurations whose one-step successors by $\Rightarrow$ are larger (w.r.t. $\leq$) than c i.e., $\mathsf{PreUp}(c) = \{c' \mid c' \Rightarrow c'', c \leq c''\}$. When both $\Rightarrow$ and $\leq$ are decidable, and when we can effectively compute $\mathsf{PreUp}(c)$, for any $c \in C$, the WSTS is called an *effective* WSTS (EWSTS for short).

Remark 1 We assume that, for any EWSTS $S = \langle C, c_0, \Sigma, \Rightarrow, \leq \rangle$, we are provided with the procedures that allow us to compute $\mathsf{PreUp}(c)$ for any $c \in C$, and to decide whether $c_1 \leq c_2$ and $c_1 \overset{a}{\Rightarrow} c_2$, for any pair of configurations c_1 and c_2 in C, and any $a \in \Sigma \cup \{\varepsilon\}$. This also implies that there is an effective representation for any configuration $c \in C$.

Remark 2 Several well-studied models of computation such as Extended Petri Nets (defined hereunder) and Lossy Channel Systems (see [3]) are EWSTS.

The following lemma is a direct consequence of the definition of wqo:

Lemma 2 *Given a set C with the well-quasi ordering $\leq \subseteq C \times C$ and an infinite sequence $S = c_1, c_2, \ldots$ with $\forall i \geq 1 : c_i \in C$, there exists an infinite subsequence $c_{\rho(1)}, c_{\rho(2)}, \ldots$ of S such that $\rho : \mathbb{N} \mapsto \mathbb{N}$ is a striclty monotonic function and $\forall j \geq 1 : c_{\rho(j)} \leq c_{\rho(j+1)}$.*

Extended Petri nets In the sequel, we study in particular a subclass of EWSTS defined by Extended Petri Nets. Intuitively, an Extended Petri net is a Petri net model where transitions are extended with a special arc that connects a source place to a destination place, and whose semantics is different from the semantics of classical Petri net arcs. We distinguish three subclasses of Extended Petri nets: the (regular) Petri nets, the Petri nets with non-blocking arcs and the Petri nets with transfer arcs. Those models are classically used to model parameterized systems [20, 11].

A (labelled) *Extended Petri Net* (EPN) $\mathscr{N}$ is a tuple $\langle \mathscr{P}, \mathscr{T}, \Sigma, \mathbf{m}_0 \rangle$, where $\mathscr{P}$ is a finite set $\{p_1, p_2, \ldots, p_n\}$ of places, $\mathscr{T}$ is a finite set of transitions and Σ is a finite alphabet. A *marking* of the places is a function $\mathbf{m} : \mathscr{P} \mapsto \mathbb{N}$. A marking can also be seen as a vector v such that $v^T = [\mathbf{m}(p_1), \mathbf{m}(p_2), \ldots, \mathbf{m}(p_n)]$. $\mathbf{m}_0 : \mathscr{P} \mapsto \mathbb{N}$ is the initial marking. Given a set of places $\{p_1, p_2, \ldots, p_k\}$, we denote by $\mathbf{m}\big(\{p_1, p_2, \ldots, p_k\}\big)$ the value $\sum_{1 \leq i \leq k} \mathbf{m}(p_i)$. Each transition is of the form $\langle I, O, s, d, b, \lambda \rangle$, where I and $O : \mathscr{P} \mapsto \mathbb{N}$ are multi-sets of input and output places respectively. By convention, $O(p)$ (resp. $I(p)$) denotes the number of occurrences of p in O (resp. I). $s, d \in \mathscr{P} \cup \{\perp\}$ are the source and the destination places respectively of a *special arc*, $b \in \mathbb{N} \cup \{+\infty\}$ is the bound associated to the special arc and $\lambda \in \Sigma \cup \{\varepsilon\}$ is the label of the transition. Let us partition $\mathscr{T}$ into $\mathscr{T}_r$ and $\mathscr{T}_e$ such that $\mathscr{T} = \mathscr{T}_r \cup \mathscr{T}_e$ and $\mathscr{T}_r \cap \mathscr{T}_e = \emptyset$. Without loss of generality, we assume that for each transition $\langle I, O, s, d, b, \lambda \rangle \in \mathscr{T}$, either $b = 0$ and $s = \perp = d$ (regular Petri transitions, grouped into $\mathscr{T}_r$); or $b > 0$, $s \neq d$, $s \neq \perp$ and $d \neq \perp$ (extended transitions, grouped into $\mathscr{T}_e$). We identify several non-disjoint classes of EPN, depending on $\mathscr{T}_e$:

1. *Petri nets* (PN for short): an EPN is a PN iff $\mathscr{T}_e = \emptyset$;
2. *Petri nets with non-blocking arcs* (PN+NBA): an EPN is a PN+NBA iff for any $t = \langle I, O, s, d, b, \lambda \rangle$ in $\mathscr{T}_e$: $b = 1$;
3. *Petri nets with transfer arcs* (PN+T): an EPN is a PN+T iff for any $t = \langle I, O, s, d, b, \lambda \rangle$ in $\mathscr{T}_e$: $b = +\infty$.

Places are graphically depicted by circles; transitions by filled rectangles. For any transition $t = \langle I, O, s, d, b, \lambda \rangle$, we draw an arrow from any place $p \in I$ to transition t and from t to any place $p \in O$. When $I(p)$ (resp. $O(p)$) is strictly greater than 1, we label the corresponding arrow by $I(p)$ ($O(p)$). For a PN+NBA (resp. PN+T), we draw a dotted (grey) arrow from s to t and from t to d (provided that $s, d \neq \perp$).

Given an extended Petri net $\mathscr{N} = \langle \mathscr{P}, \mathscr{T}, \Sigma, \mathbf{m}_0 \rangle$, and a marking $\mathbf{m}$ of $\mathscr{N}$, a transition $t = \langle I, O, s, d, b, \lambda \rangle$ is said to be *enabled in* $\mathbf{m}$ (notation: $\mathbf{m} \xrightarrow{t}$) iff $\forall p \in \mathscr{P} : \mathbf{m}(p) \geq I(p)$. An enabled transition $t = \langle I, O, s, d, b, \lambda \rangle$ can *occur*, which deterministically transforms the marking $\mathbf{m}$ into a new marking $\mathbf{m}'$ (we denote this by $\mathbf{m} \xrightarrow{t} \mathbf{m}'$). $\mathbf{m}'$ is computed as follows:

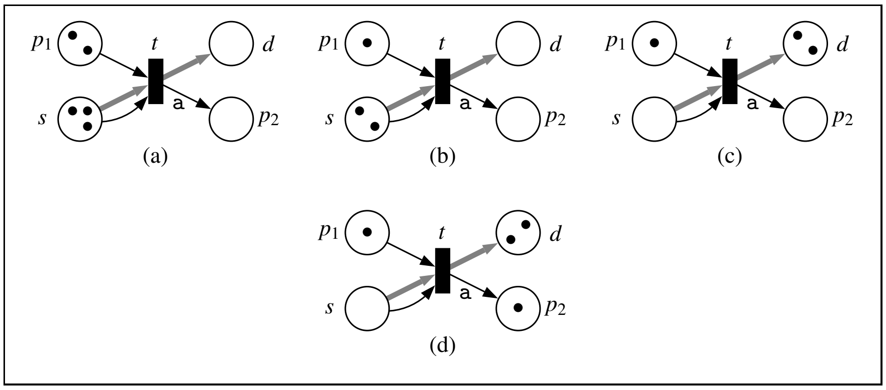

Fig. 1 The four steps to compute the effect of a transfer arc

1. First compute $\mathbf{m}_1$ such that: $\forall p \in \mathscr{P} : \mathbf{m}_1(p) = \mathbf{m}(p) - I(p)$.
2. Then compute $\mathbf{m}_2$ as follows. If $s = d = \perp$, then $\mathbf{m}_2 = \mathbf{m}_1$. Otherwise:

$$\mathbf{m}_2(s) = \begin{cases} 0 & \text{if } \mathbf{m}_1(s) \leq b \\ \mathbf{m}_1(s) - b & \text{otherwise} \end{cases} \quad \mathbf{m}_2(d) = \begin{cases} \mathbf{m}_1(d) + \mathbf{m}_1(s) & \text{if } \mathbf{m}_1(s) \leq b \\ \mathbf{m}_1(d) + b & \text{otherwise} \end{cases}$$
$$\forall p \in \mathscr{P} \setminus \{d, s\} : \mathbf{m}_2(p) = \mathbf{m}_1(p)$$

3. Finally, compute $\mathbf{m}'$, such that $\forall p \in O : \mathbf{m}'(p) = \mathbf{m}_2(p) + O(p)$.

Let $\sigma = t_1 t_2 \ldots t_n$ be a sequence of transitions. We write $\mathbf{m} \xrightarrow{\sigma} \mathbf{m}'$ to mean that there exist $\mathbf{m}_1, \ldots, \mathbf{m}_{n-1}$ such that $\mathbf{m} \xrightarrow{t_1} \mathbf{m}_1 \xrightarrow{t_2} \ldots \xrightarrow{t_{n-1}} \mathbf{m}_{n-1} \xrightarrow{t_n} \mathbf{m}'$. Moreover, we let $\Lambda(\sigma) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$, where $\forall 1 \leq i \leq n$: λ_i is the label of t_i. We sometimes write $\mathbf{m} \xrightarrow{*} \mathbf{m}'$ to mean that there exists a sequence of transitions σ such that $\mathbf{m} \xrightarrow{\sigma} \mathbf{m}'$.

Example 1 Fig. 1 presents a transition $t = \langle I, O, s, d, +\infty, \mathtt{a} \rangle$ equipped with a transfer arc. I and O are such that : $I(p_1) = I(s) = 1$, $I(p_2) = I(d) = 0$, $O(p_2) = 1$ and $O(p_1) = O(s) = O(d) = 0$.

The successive steps to compute the effect of the firing of t are shown. Namely, (a) presents a marking $\mathbf{m}$ before the firing of t; (b) presents the marking $\mathbf{m}_1$ obtained by removing $I(p)$ tokens in every place p; (c) presents $\mathbf{m}_2$ obtained from $\mathbf{m}_1$ by transferring to d the two tokens present in s; and (d) presents the resulting marking $\mathbf{m}'$ obtained after producing $O(p)$ tokens in every place p.

If t had been equipped with a non-blocking arc (hence $t = \langle I, O, s, d, 1, \mathtt{a} \rangle$), only one token would have been transfered from s to d at step (c). In both cases, t would have been firable even if $\mathbf{m}_1(s)$ had been 1. $\diamond$

Let $\preccurlyeq$ denote the wqo on markings, defined as follows: let $\mathbf{m}$ and $\mathbf{m}'$ be two markings on the set of places $\mathscr{P}$, then $\mathbf{m} \preccurlyeq \mathbf{m}'$ iff $\forall p \in \mathscr{P} : \mathbf{m}(p) \leq \mathbf{m}'(p)$. By Dickson's Lemma [7] we know that $\preccurlyeq$ is a wqo. Hence, we obtain the following property, which can be regarded as a consequence of Lemma 2:

Lemma 3 *Given an infinite sequence of markings (ranging on the set of places $\mathscr{P}$) $\mathbf{m}_1, \mathbf{m}_2, \ldots$ we can always extract an infinite subsequence $\mathbf{m}_{\rho(1)}, \mathbf{m}_{\rho(2)}, \ldots$ (ρ :*

$\mathbb{N} \mapsto \mathbb{N}$ *is strictly monotonic function) s.t. for any place* $p \in \mathscr{P}$*, either* $\mathbf{m}_{\rho(j)}(p) < \mathbf{m}_{\rho(j+1)}(p)$ *for all* $j \geq 1$ *or* $\mathbf{m}_{\rho(j)}(p) = \mathbf{m}_{\rho(j+1)}(p)$ *for all* $j \geq 1$.

An EPN $\langle \mathscr{P}, \mathscr{T}, \Sigma, \mathbf{m}_0 \rangle$, defines an EWSTS $S = \langle \mathbb{N}^{|\mathscr{P}|}, \mathbf{m}_0, \Sigma, \Rightarrow, \preccurlyeq \rangle$; where $\Rightarrow$ is such that $\mathbf{m}_1 \stackrel{a}{\Rightarrow} \mathbf{m}_2$ iff there is a transition $t \in \mathscr{T}$ with label a and $\mathbf{m}_1 \xrightarrow{t} \mathbf{m}_2$.

3 Well-structured languages

This section is mainly devoted to the definitions of languages of WSTS (and the motivations of these definitions). In accordance to previous classical works on the expressive power of Petri nets, we distinguish several classes of languages of WSTS, depending on the form of the set of *accepting states*. Then, we study several properties of these different classes of languages. As we will see, the class one obtains when considering $\leq$*-upward-closed* sets of accepting states enjoys nice properties (the emptiness is decidable, that class forms a full AFL, closed under intersection) that do not hold if we choose, for instance, a finite set of accepting states. This will motivate our choice for the definition of *well-structured languages*. Unfortunately, the universality problem is undecidable for EWSTS, even when the accepting set of configuration is $\leq$-upward-closed. That result is proved by reducing the place boundedness problem for PN+NBA (which has been proved undecidable in [17]) to the universality problem for PN+NBA.

3.1 Languages of WSTS

We first define the notion of language of a WSTS:

Definition 5 Given a WSTS $S = \langle C, c_0, \Sigma, \Rightarrow, \leq \rangle$, and a set $C' \subseteq C$ of *accepting configurations*, the *language* of S, noted $L(S, C')$ is the set of all the finite words w such that $c_0 \stackrel{w}{\Rightarrow}^* c$ for some $c \in C'$. ■

By imposing some well-chosen restrictions about the set of accepting configurations, one can obtain different classes of languages. In the restricted case of PN, this approach has already been followed in classical works of the literature such as [16], [19] or [14]. Namely, if $\mathscr{S}$ is a set of WSTS, then $L^L(\mathscr{S})$, $L^T(\mathscr{S})$ and $L^G(\mathscr{S})$ are the classes of languages defined by a WSTS in $\mathscr{S}$, and where the set of accepting configurations is (resp.) a *finite set* of configurations; the set of every *deadlock* configuration or; a $\leq$*-upward-closed set* of configurations.

Remark 3 It is worth recalling that a fourth kind of accepting condition has been routinely studied in the literature. In our context, it is the class $L^P(\mathsf{WSTS})$ of prefix languages one obtains by taking the whole set of configurations as accepting set. By definition, such a set is upward-closed. Since a language that contains no words of length < 2 cannot be in $L^P(\mathsf{WSTS})$, we have: $L^P(\mathsf{WSTS}) \subset L^G(\mathsf{WSTS})$. Most of the results about the classes L^G we are about to present can easily be adapted to their corresponding classes L^P.

Not surprisingly, these different classes of languages enjoy different properties, as shown by the following propositions. Proposition 1 states that $L^L(\mathsf{EWSTS})$ and $L^T(\mathsf{EWSTS})$ are both equal to the set of recursively enumerable languages (R.E.). This proposition stems from the fact that $L^L(\mathsf{PN{+}T}) = R.E.$ (see [4]).

Proposition 1 ([4]) $L^L(\mathsf{EWSTS}) = L^T(\mathsf{EWSTS}) = R.E.$

Since many problems are undecidable on the class R.E.[2], this result is a strong indication that other accepting conditions should be considered to obtain positive decidability results. As a matter of fact, the emptiness is decidable for EWSTS with $\leq$-upward-closed accepting sets. That result stems from the fact that the *coverability problem* is decidable on that class:

Problem 1 Given an EWSTS S and an upward-closed set $\mathscr{U}$ of configurations of S, the *coverability problem* asks whether there exists a configuration c that is reachable in S and that belongs to $\mathscr{U}$.

Theorem 1 ([10]) *The coverability problem for* EWSTS *is decidable.*

From the definition of the problem, it is not difficult to see that, given an EWSTS S and an upward-closed set $\mathscr{U}$ of configurations of S, we have $L(S,\mathscr{U}) = \emptyset$ iff the answer to the coverability problem is *negative* on S and $\mathscr{U}$. This provides us with an effective procedure to test the emptiness of the language of an EWSTS when an upward-closed set of accepting configurations is considered. Hence, the Corollary to Theorem 1:

Corollary 1 *The emptiness problem is decidable for the class of* EWSTS*, when we consider $\leq$-upward-closed accepting sets.*

We will prove in section 5.1 (see Proposition 5) that some Context Free Languages (C.F.L.) are not in $L^G(\mathsf{EWSTS})$. This implies that $L^G(\mathsf{EWSTS}) \neq$ R.E., which is not surprising since the emptiness problem is decidable.

Finally, one can prove that $L^G(\mathsf{WSTS})$ is a full AFL closed under intersection, which is a strong indication that it is a class worth of attention. The proof consists in showing that, given two languages L_1 and L_2 in $L^G(\mathsf{WSTS})$, there are WSTS that accept respectively $L_1 \cap L_2$, $L_1 \cup L_2$, $L_1 \cdot L_2$, L_1^+, $L_1 \cap L_R$ (where L_R is any regular language), $h(L_1)$ and $h^{-1}(L_1)$ (where h is any arbitrary homomorphism). Remark that we only prove the *existence* of these WSTS, and these constructions are thus *not effective* in general, since we have not fixed any formalism to describe WSTS. However, we will present in section 5.4 effective constructions for these operations when the WSTS considered are PN+T.

In order to show that $L^G(\mathsf{WSTS})$ is a full AFL closed under intersection, we first introduce a construction that turns any WSTS S into another WSTS S_s that accepts the same language as S does (for any set of accepting configurations) and that is *simply monotonic*:

Definition 6 A labelled WSTS $S = \langle C, c_0, \Rightarrow, \leq, \Sigma\rangle$ is *simply monotonic* iff for any $c_1, c_2, c_3 \in C$, for any $a \in \Sigma \cup \{\varepsilon\}$: $c_1 \overset{a}{\Rightarrow} c_2$ and $c_1 \leq c_3$ implies that there exists $c_4 \in C$ s.t.t $c_3 \overset{a}{\Rightarrow} c_4$ and $c_2 \leq c_4$.

[2] Remark that the aforementioned proof works by translating a two-counter machine [15], which are as expressive as Turing machines, to a PN+T that accepts the same language.

The construction works as follows. First, given a WSTS $S = \langle C, c_0, \Rightarrow, \leq, \Sigma \rangle$, and a configuration $c \in C$, we let $\varepsilon{-}\mathsf{closure}^{\Rightarrow}(c) = \{c' \mid c \overset{\varepsilon}{\Rightarrow}{}^* c'\}$. Remark that, for any $c \in C$: $c \in \varepsilon{-}\mathsf{closure}^{\Rightarrow}(c)$. Then, for any WSTS $S = \langle C, c_0, \Rightarrow, \leq, \Sigma \rangle$, we build the WSTS $S_s = \langle C, c_0, \Rightarrow_s, \leq, \Sigma \rangle$ s.t.:

$$\Rightarrow_s = \left\{ (c, a, c') \middle| \exists c_1, c_2 \in C : \begin{array}{c} c_1 \in \varepsilon{-}\mathsf{closure}^{\Rightarrow}(c) \wedge \\ c_1 \overset{a}{\Rightarrow} c_2 \wedge \\ c' \in \varepsilon{-}\mathsf{closure}^{\Rightarrow}(c_2) \end{array} \right\} \cup \{(c, \varepsilon, c) \mid c \in C\}$$

We can now show that this new transition relation enjoys the desired monotonicity property:

Lemma 4 *Let $S = \langle C, c_0, \Rightarrow, \leq, \Sigma \rangle$ be a* WSTS *and let $S_s = \langle C, c_0, \Rightarrow_s, \leq, \Sigma \rangle$ be obtained from S by the above construction. Then, for any $c_1, c_2, c_3 \in C$, for any $a \in \Sigma \cup \{\varepsilon\}$: $c_1 \leq c_3$ and $c_1 \overset{a}{\Rightarrow}_s c_2$ implies that there exists c_4 s.t. $c_2 \leq c_4$ and $c_3 \overset{a}{\Rightarrow}_s c_4$.*

Proof Let c_1, c_2, c_3 be three configurations of C and let $a \in \Sigma \cup \{\varepsilon\}$ be a letter s.t. $c_1 \overset{a}{\Rightarrow}_s c_2$ and $c_1 \leq c_3$. Remark that, by definition, $c_1 \overset{a}{\Rightarrow}_s c_2$ implies that $c_1 \overset{a}{\Rightarrow}{}^* c_2$. Hence, by monotonicity of $\Rightarrow$, there exists c_4 s.t. $c_3 \overset{a}{\Rightarrow}{}^* c_4$ and $c_2 \leq c_4$. By definition of $\Rightarrow^*$, and since a is a single character, this means either that $c_3 \overset{a}{\Rightarrow}_s c_4$, or that there are two configurations c and c' s.t. $c_3 \overset{\varepsilon}{\Rightarrow}{}^* c \overset{a}{\Rightarrow} c' \overset{\varepsilon}{\Rightarrow}{}^* c_4$. Hence, $c \in \varepsilon{-}\mathsf{closure}^{\Rightarrow}(c_3)$, $c_4 \in \varepsilon{-}\mathsf{closure}^{\Rightarrow}(c')$, and we conclude that $c_3 \overset{a}{\Rightarrow}_s c_4$. $\square$

Thus, S_s is indeed a simply monotonic WSTS. Let us show that for any set of accepting configurations C', both S and S_s accept the same language.

Proposition 2 *Let $S = \langle C, c_0, \Rightarrow, \leq, \Sigma \rangle$ be a* WSTS *and let $S_s = \langle C, c_0, \Rightarrow_s, \leq, \Sigma \rangle$ be the simply monotonic* WSTS *obtained from S. Then, for any $C' \subseteq C$: $L(S, C') = L(S_s, C')$.*

Proof First remark that $\Rightarrow \subseteq \Rightarrow_s$. Hence, $L(S, C') \subseteq L(S_s, C')$. Let us show that $L(S_s, C') \subseteq L(S, C')$. Let us consider $w \in L(S_s, C')$ and let us show that $w \in L(S, C')$.

By definition of L, there is $c \in C'$ s.t. $c_0 \overset{w}{\Rightarrow}{}_s^* c$. Hence, by definition of $\Rightarrow_s^*$ there is $k \geq 1$ s.t. there are $c_1, c_2, \ldots, c_k \in C$ and $b_1, b_2, \ldots b_{k-1} \in \Sigma \cup \{\varepsilon\}$ with $c_1 = c_0$, $c_k = c$, $b_1 \cdot b_2 \cdots b_{k-1} = w$ and $c_1 \overset{b_1}{\Rightarrow}_s c_2 \overset{b_2}{\Rightarrow}_s \cdots \overset{b_{k-1}}{\Rightarrow}_s c_k$. Without loss of generality, we assume that there is no $1 \leq i \leq k-1$ s.t. $c_i = c_{i+1}$ and $b_i = \varepsilon$. Indeed, if such transitions appear in the sequence, they can be removed because they do not add any character to the words, and are not necessary to reach C'.

By definition of $\Rightarrow_s$, there are $\overline{c}_1, \overline{c}_2, \ldots \overline{c}_{k-1}$ and $\widehat{c}_1, \widehat{c}_2, \ldots, \widehat{c}_{k-1}$ in C s.t.:

$$c_1 \overset{\varepsilon}{\Rightarrow}{}^* \overline{c}_1 \overset{b_1}{\Rightarrow} \widehat{c}_1 \overset{\varepsilon}{\Rightarrow}{}^* c_2 \overset{\varepsilon}{\Rightarrow}{}^* \overline{c}_2 \overset{b_2}{\Rightarrow} \widehat{c}_2 \overset{\varepsilon}{\Rightarrow}{}^* \cdots \overset{\varepsilon}{\Rightarrow}{}^* \overline{c}_{k-1} \overset{b_{k-1}}{\Rightarrow} \widehat{c}_{k-1} \overset{\varepsilon}{\Rightarrow}{}^* c_k$$

Since $c_k \in C'$, this implies that

$$\varepsilon \cdot b_1 \cdot \varepsilon \cdot \varepsilon \cdot b_2 \cdot \varepsilon \cdots \varepsilon \cdot b_{k-1} \cdot \varepsilon = b_1 \cdot b_2 \cdots b_{k-1} = w \in L(S, C')$$

$\square$

Theorem 2 *$L^G(\mathsf{WSTS})$ is a full AFL, closed under intersection.*

Proof According to Definition 1, one has to show seven closure properties (the six properties that define an AFL, plus the closure under intersection) in order to establish this result. In the sequel, we assume that $S_1 = \langle C_1, i_1, \Sigma_1, \Rightarrow_1, \leq_1\rangle$ and $S_2 = \langle C_2, i_2, \Sigma_2, \Rightarrow_2, \leq_2\rangle$ are two WSTS (with $C_1 \cap C_2 = \emptyset$), and that $\mathscr{U}_1$ and $\mathscr{U}_2$ are their associated upward-closed sets of accepting states. In order to make the proofs easier, we further assume that both S_1 and S_2 are simply monotonic. According to Proposition 2, this is not restrictive since, for any labelled WSTS S, there exists a simply monotonic WSTS S_s that accepts the same language. We finally assume that $h : \Sigma_1 \mapsto \Sigma_1^*$ is a homomorphism s.t. $h(\varepsilon) = \varepsilon$, according to the definition from [12, 18]. We prove the closure of the seven operations by showing the existence of a WSTS $S = \langle C, i, \Sigma, \Rightarrow, \leq\rangle$ and a set of accepting states $\mathscr{U}$, s.t. $L(S, \mathscr{U})$ is the result of the operation in question. We ensure that $L(S, \mathscr{U}) \in L^G(\mathsf{WSTS})$ by proving that $\leq$ is a wqo, $\Rightarrow$ is $\leq$-monotonic and $\mathscr{U}$ is upward-closed.

Intersection Let us show that there are S and $\mathscr{U}$ s.t. $L(S, \mathscr{U}) = L(S_1, \mathscr{U}_1) \cap L(S_2, \mathscr{U}_2)$. S is built as follows: $C = C_1 \times C_2$; $i = (i_1, i_2)$; $\Sigma = \Sigma_1 \cap \Sigma_2$. The wqo is obtained as follows: $\leq = \{\big((c_1, c_2), (c_1', c_2')\big) \mid c_1 \leq_1 c_1' \wedge c_2 \leq_2 c_2'\}$. The transition relation $\Rightarrow$ is defined as:

$$\begin{aligned}\Rightarrow\, = \{&\big((c_1, c_2), a, (c_1', c_2')\big) \mid c_1 \xRightarrow{a}_1 c_1' \wedge\ c_2 \xRightarrow{a}_2 c_2' \wedge a \in \Sigma\} \cup \\ \{&\big((c_1, c_2), \varepsilon, (c_1', c_2')\big) \mid \big(c_1 \xRightarrow{\varepsilon}_1 c_1' \wedge c_2 = c_2'\big) \text{ or } \big(c_1 = c_1' \wedge c_2 \xRightarrow{\varepsilon}_2 c_2'\big)\}\end{aligned}$$

Finally, $\mathscr{U} = \{(c_1, c_2) \mid c_1 \in \mathscr{U}_1 \wedge c_2 \in \mathscr{U}_2\}$.

Clearly, $L(S, \mathscr{U}) = L(S_1, \mathscr{U}_1) \cap L(S_2, \mathscr{U}_2)$. Let us prove that $\leq$, $\Rightarrow$ and $\mathscr{U}$ have the desired properties:

- **$\leq$ is a wqo** Let $\varsigma = (c_1^1, c_1^2), (c_2^1, c_2^2), \ldots, (c_n^1, c_n^2), \ldots$ be an infinite sequence of elements of C. Since $\leq_1$ is a wqo on C_1, following Lemma 2, one can extract from ς an infinite subsequence

  $$\varsigma' = (c_{\rho(1)}^1, c_{\rho(1)}^2), (c_{\rho(2)}^1, c_{\rho(2)}^2), \ldots, (c_{\rho(n)}^1, c_{\rho(n)}^2), \ldots$$

  such that for any $j \geq 1$: $c_{\rho(j)}^1 \leq_1 c_{\rho(j+1)}^1$. Since $\leq_2$ is a wqo on the elements of C_2, there are, in ς', two positions k and ℓ s.t. $k < \ell$ and $c_{\rho(k)}^2 \leq_2 c_{\rho(\ell)}^2$. Hence, $(c_{\rho(k)}^1, c_{\rho(k)}^2) \leq (c_{\rho(\ell)}^1, c_{\rho(\ell)}^2)$, which proves that $\leq$ is a wqo, according to Definition 2.
- **$\Rightarrow$ is $\leq$-monotonic** Let (c_1^1, c_1^2), (c_2^1, c_2^2), and (c_3^1, c_3^2) be three configurations of C. We consider two cases. Either there is $a \in \Sigma$ s.t. $(c_1^1, c_1^2) \xRightarrow{a} (c_2^1, c_2^2)$ and $(c_1^1, c_1^2) \leq (c_3^1, c_3^2)$. By definition of $\Rightarrow$ and $\leq$, this implies that $c_1^1 \xRightarrow{a}_1 c_2^1$, $c_1^2 \xRightarrow{a}_2 c_2^2$, $c_1^1 \leq_1 c_3^1$ and $c_1^2 \leq_2 c_3^2$. Since $\Rightarrow_1$ and $\Rightarrow_2$ are resp. $\leq_1$- and $\leq_2$- simply monotonic, there are $c \in C_1$ and $c' \in C_2$ s.t.: $c_3^1 \xRightarrow{a}_1 c$, $c_3^2 \xRightarrow{a}_2 c'$, $c_2^1 \leq_1 c$ and $c_2^2 \leq_2 c'$. The first two point imply that $(c_3^1, c_3^2) \xRightarrow{a} (c, c')$. The last two points imply that $(c_2^1, c_2^2) \leq (c, c')$.

  On the other hand, if $(c_1^1, c_1^2) \xRightarrow{\varepsilon} (c_2^1, c_2^2)$ then either (i) $c_1^1 \xRightarrow{\varepsilon}_1 c_2^1$ and $c_1^2 = c_2^2$ or (ii) $c_1^2 \xRightarrow{\varepsilon}_1 c_2^2$ and $c_1^1 = c_2^1$. In the first case, since $\Rightarrow_1$ is simply monotonic, and since $c_1^1 \leq_1 c_3^1$, there exists c_4^1 s.t. $c_3^1 \xRightarrow{\varepsilon}_1 c_4^1$ and $c_2^1 \leq c_4^1$. Thus, $(c_4^1, c_1^2) \leq (c_3^1 c_1^2)$

by definition of $\leq$ and $(c_3^1, c_1^2) \overset{\varepsilon}{\Rightarrow} (c_4^1, c_1^2)$ by definition of $\Rightarrow$. The second case is similar.

– $\mathscr{U}$ **is $\leq$-upward-closed** Let (c_1^1, c_1^2) and (c_2^1, c_2^2), both in C, be s.t. $(c_1^1, c_1^2) \leq (c_2^1, c_2^2)$ and $(c_1^1, c_1^2) \in \mathscr{U}$. Let us show that $(c_2^1, c_2^2) \in \mathscr{U}$ too. Since $(c_1^1, c_1^2) \in \mathscr{U}$, we have $c_1^1 \in \mathscr{U}_1$ and $c_1^2 \in \mathscr{U}_2$, by definition of $\mathscr{U}$. Since $(c_1^1, c_1^2) \leq (c_2^1, c_2^2)$, $c_1^1 \leq_1 c_2^1$ and $c_1^2 \leq_2 c_2^2$, by definition of $\leq$. But $\mathscr{U}_1$ and $\mathscr{U}_2$ are resp. $\leq_1$- and $\leq_2$-upward-closed, which implies that $c_2^1 \in \mathscr{U}_1$ and $c_2^2 \in \mathscr{U}_2$. Hence $(c_2^1, c_2^2) \in \mathscr{U}$.

Union Let us show that there are S and $\mathscr{U}$ such that $L(S,\mathscr{U}) = L(S_1,\mathscr{U}_1) \cup L(S_2,\mathscr{U}_2)$. We let $C = \{i\} \uplus C_1 \uplus C_2$; $\Sigma = \Sigma_1 \cup \Sigma_2$; $\leq = \leq_1 \cup \leq_2 \cup \{(i,i)\}$; $\mathscr{U} = \mathscr{U}_1 \cup \mathscr{U}_2$ and $\Rightarrow = \{(i,\varepsilon,i_1),(i,\varepsilon,i_2)\} \cup \Rightarrow_1 \cup \Rightarrow_2$.

Clearly, $L(S,\mathscr{U}) = L(S_1,\mathscr{U}_1) \cup L(S_2,\mathscr{U}_2)$. Let us show that S has the desired properties. By definition, $\Rightarrow$ is $\leq$-monotonic (remark that i is $\leq$-incomparable to any other element of C). Thus, it remains to prove that:

– $\leq$ **is a wqo** Let $\varsigma = c_0, c_2, \ldots, c_n, \ldots$ be an infinite sequence of elements of C. Because it is infinite, one can extract, from that sequence, an infinite subsequence $\varsigma' = c_{j_1}, c_{j_2}, c_{j_3}, \ldots$, s.t. either $\forall k \geq 1 : c_{j_k} \in C_1$ or $\forall k \geq 1 : c_{j_k} \in C_2$ or $\forall k \geq 1 : c_{j_k} = i$. In the case where $\forall k \geq 1 : c_{j_k} = i$, there are clearly two positions $k < \ell$ s.t. $c_{j_k} \leq c_{j_\ell}$, since $i \leq i$. Otherwise, since $\leq_1$ and $\leq_2$ are both wqo, there exist two positions k and ℓ s.t. $k < \ell$ and either $c_{j_k} \leq_1 c_{j_\ell}$ or $c_{j_k} \leq_2 c_{j_\ell}$. In either cases, this implies that $c_{j_k} \leq c_{j_\ell}$, which proves that $\leq$ is a wqo following Definition 2.

– $\mathscr{U}$ **is $\leq$-upward-closed** Let c_1, c_2 be two configurations in C s.t. $c_1 \in \mathscr{U}$ and $c_1 \leq c_2$. Let us show that $c_2 \in \mathscr{U}$. We consider two cases: either $c_1 \in \mathscr{U}_1$ or $c_1 \in \mathscr{U}_2$. In the former case, since c_1 and c_2 are $\leq$-comparable, we deduce that $c_2 \in C_1$ and thus, $c_1 \leq_1 c_2$, by definition of $\leq$. Hence, $c_2 \in \mathscr{U}_1$, since $\mathscr{U}_1$ is $\leq_1$-upward-closed. This implies that $c_2 \in \mathscr{U}$. The same reasoning can be applied to the latter case.

Concatenation Let us show that there are S and $\mathscr{U}$ such that $L(S,\mathscr{U}) = L(S_1,\mathscr{U}_1) \cdot L(S_2,\mathscr{U}_2)$. We let $C = C_1 \cup C_2$; $i = i_1$; $\Sigma = \Sigma_1 \cup \Sigma_2$; $\Rightarrow = \{(c,\varepsilon,i_2) \mid c \in \mathscr{U}_1\} \cup \Rightarrow_2 \cup \Rightarrow_1$; $\leq = \leq_1 \cup \leq_2$ and $\mathscr{U} = \mathscr{U}_2$.

Clearly, $L(S,\mathscr{U})$ is the concatenation of $L(S_1,\mathscr{U}_1)$ and $L(S_2,\mathscr{U}_2)$. The transition relation $\Rightarrow$ is $\leq$-monotonic from its definition. Indeed, let c_1, c_2 and c_3 be three configurations from C and $a \in \Sigma \cup \{\varepsilon\}$ be a character s.t. $c_1 \overset{a}{\Rightarrow} c_2$ and $c_1 \leq c_3$. In the case where $\{c_1,c_2,c_3\} \subseteq C_1$ or $\{c_1,c_2,c_3\} \subseteq C_2$, there exists $c_4 \geq c_2$ in $C = C_1 \cup C_2$ s.t. $c_3 \overset{a}{\Rightarrow} c_4$, by monotonicity of $\Rightarrow_1$ and $\Rightarrow_2$. In the case where $c_1 \in C_1$ and $c_2 \in C_2$, we have $c_1 \in \mathscr{U}_1$, $c_2 = i_2$ and $a = \varepsilon$, by construction. Hence, $c_3 \in \mathscr{U}$ and $c_3 \overset{\varepsilon}{\Rightarrow} i_2$ by construction again. Remark that it is not possible that $c_1 \in C_2$ and $c_2 \in C_1$. Since $\mathscr{U}_2 = \mathscr{U}$ is $\leq_2$-upward-closed, $\leq = \leq_1 \cup \leq_2$ and $C_1 \cap C_2 = \emptyset$, we conclude that $\mathscr{U}$ is $\leq$-upward-closed. Finally, one can show that $\leq$ is a wqo by reusing the same reasoning as for the union.

Iteration Let us show that there are S such that $L(S,\mathscr{U}) = L(S_1,\mathscr{U}_1)^+$. We consider a new configuration $i_0 \notin C_1$ and let $C = C_1 \cup \{i_0\}$; $i = i_0$; $\leq = \leq_1 \cup \{(i_0,i_0)\}$; $\Rightarrow = \{(i_0,\varepsilon,i_1)\} \cup \{(c,\varepsilon,i_0) \mid c \in \mathscr{U}_1\} \cup \Rightarrow_1$ and $\mathscr{U} = \mathscr{U}_1$.

From these definitions, it is trivial to see that $L(S,\mathscr{U}) = L(S_1,\mathscr{U}_1)^+$, $\leq$ is a wqo, $\Rightarrow$ is $\leq$-monotonic, and $\mathscr{U}$ is $\leq$-upward-closed.

Intersection with regular languages It is not difficult to see that any deterministic finite-state automaton is a WSTS, when we choose the equality between states as wqo. Hence, any regular language is a WSL. Since WSL are closed under intersection (see above), the closure with regular languages holds too.

Arbitrary homomorphism Let us show that there are S and $\mathscr{U}$ with $L(S,\mathscr{U}) = h\big(L(S_1,\mathscr{U}_1)\big)$. We extend the set of states C_1 with elements from $C_1 \times \Sigma \times \mathbb{N}$ in the following way: $C = C_1 \uplus \{(c,a,j) \mid c \in C_1 \wedge a \in \Sigma \cup \{\varepsilon\} \wedge 0 \leq j \leq |h(a)| \wedge \exists c' : c \stackrel{a}{\Rightarrow}_1 c'\}$. Intuitively, these extra states are the intermediate states that have to appear along the path from c to c' when reading $h(a)$. More precisely, (c,a,j) is the state reached after having read the j first characters of $h(a)$ from c. We also let $i = i_1$ and $\leq = \leq_1 \cup \{\big((c_1,a,j),(c_2,a,j)\big) \mid (c_1,a,j),(c_2,a,j) \in C \wedge c_1 \leq_1 c_2\}$. The transition relation is built according to the intuition we have sketched when introducing C:

$$\Rightarrow = \left\{ \begin{array}{l|c} \big(c,\varepsilon,(c,a,0)\big), & \\ \big((c,a,0),w_1,(c,a,1)\big), & a \in \Sigma \cup \{\varepsilon\} : \\ \quad\vdots & c \stackrel{a}{\Rightarrow}_1 c' \text{ and} \\ \big((c,a,|h(a)|-1),w_{|h(a)|},(c,a,|h(a)|)\big) & h(a) = w_1 w_2 \ldots w_{|h(a)|} \\ \big((c,a,|h(a)|),\varepsilon,c'\big) & \end{array} \right\}$$

Finally, $\mathscr{U} = \mathscr{U}_1$.

By construction, $L(S,\mathscr{U}) = h\big(L(S_1,\mathscr{U}_1)\big)$, and $\mathscr{U}$ is a $\leq$-upward-closed set. It remains to show that:

- $\leq$ **is a wqo** Let us suppose it is not the case. Then, there exists a sequence of elements of C: $\varsigma = c_1,c_2,\ldots,c_n,\ldots$ s.t. for any $k \geq 1$, for any $1 \leq n < k$: $c_n \not\leq c_k$ (each configuration is $\leq$-incomparable to all the previous ones). Remark that, since $\leq_1$ is a wqo on the elements of C_1 and since $c \leq_1 c'$ implies $c \leq c'$ (by definition of $\leq$), one cannot find in ς, infinitely many elements from C_1. Otherwise, the infinite subsequence of ς made of all the elements $c_i \in C_1$ would be an infinite sequence of $\leq_1$-incomparable elements from C_1. But this cannot exist since $\leq_1$ is a wqo. Thus, there is, in ς, an infinite subsequence $\varsigma' = c_{j_1}, c_{j_2}, \ldots c_{j_n}, \ldots$ s.t. for any $k \geq 1$: (i) $c_{j_k} \notin C_1$ and (ii) for any $1 \leq n < k$: $c_{j_n} \not\leq c_{j_k}$.
  By definition of a homomorphism, the value $\ell = \max_{a \in \Sigma \cup \{\varepsilon\}} \{|h(a)|\}$ is a finite value. Hence, there exists $0 \leq \ell' \leq \ell$ and a character a of $\Sigma \cup \{\varepsilon\}$ s.t. the sequence $(c_{j_1},a,\ell'),(c_{j_2},a,\ell'),\ldots,(c_{j_n},a,\ell'),\ldots$ is an infinite subsequence of ς' and for any $n < k$: $(c_{j_n},a,\ell') \not\leq (c_{j_k},a,\ell')$. However, this implies that for any $n < k$: $c_{j_n} \not\leq_1 c_{j_k}$, which contradicts the fact that $\leq_1$ is a wqo.
- $\Rightarrow$ **is $\leq$-monotonic** Let us show that, for any $c_1,c_2,c_3 \in C$, and for any $a \in \Sigma$ s.t. $c_1 \stackrel{a}{\Rightarrow} c_2$ and $c_1 \leq c_3$, there exists c_4 s.t. $c_3 \stackrel{a}{\Rightarrow}^* c_4$ and $c_2 \leq c_4$. We consider two cases.
  1. Either $c_1 \in C_1$. In that case, by definition of $\Rightarrow$, we have $a = \varepsilon$ and $c_2 = (c_1,b,0)$ for some b. By construction, there is thus $c_1' \in C_1$ s.t. $c_1 \stackrel{b}{\Rightarrow}_1 c_1'$. Moreover, $c_1 \leq c_3$ implies that $c_3 \in C_1$, and thus that, $c_1 \leq_1 c_3$. Since $\Rightarrow_1$ is $\leq_1$-simply monotonic, there is c_3' s.t. $c_3 \stackrel{b}{\Rightarrow}_1 c_3'$. Hence, by construction, the configuration $c_4 = (c_3,b,0) \geq (c_1,b,0)$ is in C, and satisfies $c_3 \stackrel{\varepsilon}{\Rightarrow} c_4$.
  2. Or, $c_1 \notin C_1$. In that case $c_1 = (c',b,i)$ and $c_3 = (c'',b,i)$ with $c' \leq_1 c''$, for some b. Again, we have to consider two subcases.

(a) In the case where $i < |h(b)|$, $c_2 = (c', b, i+1)$, by construction. We can choose $c_4 = (c'', b, i+1)$, which satisfies the conditions.

(b) In the case where $i = |h(b)|$, c_2 is a configuration of C_1 s.t. $c' \stackrel{b}{\Rightarrow}_1 c_2$. By definition of $\Rightarrow$, we have: $c_1 = (c', b, |h(b)|) \stackrel{\varepsilon}{\Rightarrow} c_2$. By $\leq_1$-simple monotonicity of $\Rightarrow_1$, there exists a configuration c_4 s.t. $c'' \stackrel{b}{\Rightarrow}_1 c_4$ and $c_2 \leq_1 c_4$. Thus, $c_2 \leq c_4$, and, by definition of $\Rightarrow$, we have $c_3 = (c'', b, |h(b)|) \stackrel{\varepsilon}{\Rightarrow} c_4$. Hence, c_4 satisfies the conditions.

In any case, we conclude that $\Rightarrow$ is $\leq$-monotonic.

Inverse homomorphism Let us build S and $\mathscr{U}$ s.t. $L(S, \mathscr{U}) = h^{-1}\big(L(S_1, \mathscr{U}_1)\big)$. We let $C = C_1$; $i = i_1$; $\leq = \leq_1$; $\Rightarrow = \{(c_1, a, c_2) \mid a \in \Sigma \cup \{\varepsilon\} \wedge \exists w \in \Sigma^* : h(a) = w \wedge c_1 \stackrel{w}{\Rightarrow}{}_1^* c_2\}$ and $\mathscr{U} = \mathscr{U}_1$.

Clearly, $L(S, \mathscr{U}) = h^{-1}\big(L(S_1, \mathscr{U}_1)\big)$. By definition, $\mathscr{U}$ is $\leq$-upward-closed and $\leq$ is a wqo. It remains to show that $\Rightarrow$ is $\leq$-monotonic. Let c_1, c_2, c_3 be three configurations in C s.t. $c_1 \stackrel{a}{\Rightarrow} c_2$ for some a, and $c_1 \leq c_3$. By definition of $\Rightarrow$, there exists $w \in \Sigma^*$ s.t. $h(a) = w$ and $c_1 \stackrel{w}{\Rightarrow}{}_1^* c_2$. Moreover, $c_3 \in C_1$ and $c_1 \leq_1 c_3$, by definition. By using an inductive reasoning on the length of w, one can show that there exists $c_4 \in C_1$ s.t. $c_3 \stackrel{w}{\Rightarrow}{}_1^* c_4$ and $c_2 \leq_1 c_4$. Hence, $c_4 \in C$ and $c_3 \stackrel{a}{\Rightarrow} c_4$, by definition of $\Rightarrow$. □

Remark 4 $L^P(\mathsf{WSTS})$ is not a full AFL. Indeed, let us consider the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$. Clearly, the language $\mathscr{L}_{\mathtt{a}} = \{\mathtt{a}, \varepsilon\}$ is in $L^P(\mathsf{WSTS})$. Let $h : \Sigma \mapsto \Sigma^*$ be an homomorphism s.t. $h(\mathtt{a}) = \mathtt{bb}$. Then, $h(\mathscr{L}_{\mathtt{a}}) = \{(\mathtt{bb}), \varepsilon\}$ is not in $L^P(\mathsf{WSTS})$ because it is not prefix-closed (the word $\mathtt{b}$ is missing).

It should now be clear that the class $L^G(\mathsf{WSTS})$ enjoys interesting properties: the emptiness is decidable on this class, under reasonable effectiveness assumptions (Theorem 1), and it forms a full AFL closed under intersection (Theorem 2). Moreover, the transition relation of WSTS is, by definition, $\leq$-monotonic. Thus, $\leq$-upward-closed sets are perfectly suited accepting conditions for these systems. For all these reasons, we will henceforth restrict ourselves to the study of $L^G(\mathsf{WSTS})$. The languages in this class are called *well-structured languages*:

Definition 7 A language L is a *well-structured language* (WSL for short) iff $L \in L^G(\mathsf{WSTS})$. ■

3.2 Undecidability of universality

Unfortunately, the universality problem is undecidable on EWSTS. This problem is defined as follows:

Problem 2 Given an EWSTS $S = \langle C, c_0, \Rightarrow, \leq, \Sigma \rangle$, and an upward-closed set of accepting markings $\mathscr{U}_f$, the *universality problem* asks whether $L(S, \mathscr{U}_f) = \Sigma^*$.

The proof consists in showing that the universality problem is undecidable on $\mathsf{PN{+}NBA}$. In order to prove the undecidability of universality for $\mathsf{PN{+}NBA}$, we reduce the place boundedness problem for $\mathsf{PN{+}NBA}$ (which is know to be undecidable – see[17]) to the universality problem for $\mathsf{PN{+}NBA}$. The place-boundedness problem for PN+NBA asks whether there is *bound* on the number of tokens that any reachable marking assigns to a given place p. More precisely, it is defined as follows:

Problem 3 Given a PN+NBA $\langle \mathscr{P}, \mathscr{T}, \Sigma, \mathbf{m}_0 \rangle$ and $p \in \mathscr{P}$, the *place-boundedness problem* asks whether there exists $k \in \mathbb{N}$ such that for any $\mathbf{m}_{\in}\mathbb{N}^{|\mathscr{P}|}$: if $\mathbf{m}_0 \xrightarrow{*} \mathbf{m}$ then $\mathbf{m}(p) \leq k$.

Given a PN+NBA $\mathscr{N} = \langle \mathscr{P}, \mathscr{T}, \Sigma, \mathbf{m}_0 \rangle$ and a place $p \in \mathscr{P}$, the reduction consists in building a new PN+NBA $\mathscr{N}_p = \langle \mathscr{P}', \mathscr{T}', \{\mathtt{a}\}, \mathbf{m}'_0 \rangle$ s.t. $\mathscr{N}'$ accepts (with $\mathbb{N}^{|\mathscr{P}'|}$ as accepting set) the universal language (i.e., $\mathtt{a}^*$) if and only if the place p is unbounded in $\mathscr{N}$. The construction works as follows:

- $\mathscr{P}' = \mathscr{P} \cup \{run, stop\}$, provided that $\mathscr{P} \cap \{run, stop\} = \emptyset$;
- $\mathscr{T}'$ is the smallest set of transitions that contains the transitions t_a and t_f with $t_a = \langle \{stop, p\}, \{stop\}, \bot, \bot, 0, \mathtt{a} \rangle$; $t_f = \langle \{run\}, \{stop\}, \bot, \bot, 0, \varepsilon \rangle$; and such that if $\langle I, O, s, d, b, \lambda \rangle \in \mathscr{T}$ then $\langle I \cup \{run\}, O \cup \{run\}, s, d, b, \varepsilon \rangle \in \mathscr{T}'$;
- $\mathbf{m}'_0(run) = 1$, $\mathbf{m}'_0(stop) = 0$ and $\forall p' \in \mathscr{P} : \mathbf{m}'_0(p') = \mathbf{m}_0(p')$.

In other words, $\mathscr{N}_p$ is similar to $\mathscr{N}$ except that its transitions (apart from t_a) may fire only if the place *run* is marked. Besides this, the transitions of $\mathscr{N}'$ that have been adapted from transitions of $\mathscr{N}$ have the same effect in $\mathscr{N}'$ than in $\mathscr{N}$. Remark that all the transitions in $\mathscr{T}' \setminus \{t_a\}$ are labelled by ε. The transition t_f moves the unique token from *run* to *stop*. This has the effect to prevent the transitions in $\mathscr{T}' \setminus \{t_a\}$ from firing. Hence, t_a only (labelled by $\mathtt{a}$) can be fired after t_f has been fired. Since t_a consumes one token from place p, that transition can be fired at most k times where k is the number of tokens in p when firing t_f.

The following lemma states that the the construction we have just introduced is correct:

Lemma 5 *Given a* PN+NBA $\mathscr{N} = \langle \mathscr{P}, \mathscr{T}, \Sigma, \mathbf{m}_0 \rangle$ *with* $p \in \mathscr{P}$*, the place boundedness problem for* $\mathscr{N}$ *and* p *has a negative answer iff* $L^G(\mathscr{N}_p, \mathbb{N}^{|\mathscr{P}|}) = \mathtt{a}^*$.

Proof If the place-boundedness problem has a negative answer for $\mathscr{N}$ and p, then, for any $k \in \mathbb{N}$, there is a sequence of transitions σ s.t. $\mathbf{m}_0 \xrightarrow{\sigma} \mathbf{m}$ with $\mathbf{m}(p) \geq k$. Let σ' be the sequence of transitions of $\mathscr{N}_p$ obtained by replacing in σ each sequence of transitions by its corresponding transition in $\mathscr{N}_p$. Let $\mathbf{m}'$ be the marking s.t. $\mathbf{m}'_0 \xrightarrow{\sigma'} \mathbf{m}'$. By construction, we have: $\mathbf{m}'(run) = 1$ and for any $p \in \mathscr{P}$: $\mathbf{m}'(p) = \mathbf{m}(p)$. In particular, this implies that $\mathbf{m}'(p) \geq k$. Hence, the sequence $t_f t_a^k$ is firable from $\mathbf{m}'$. Since the accepting upward-closed set is $\mathbb{N}^{|\mathscr{P}|}$, the sequence$\sigma' t_f t_a^k$ is accepting, with $\Lambda(\sigma' t_f t_a^k) = \mathtt{a}^k$. This holds for any $k \in \mathbb{N}$, and we conclude that $\mathscr{N}_p$ accepts $\mathtt{a}^*$, and is thus universal because the alphabet of $\mathscr{N}_p$ is $\{\mathtt{a}\}$.

On the other hand, if $\mathscr{N}_p$ accepts $\mathtt{a}^*$, then, for any $k \in \mathbb{N}$, there exists a sequence of transitions σ' s.t. $\sigma' t_f t_a^k$ is firable from $\mathbf{m}'_0$. This holds because t_a is the only transition of $\mathscr{N}_p$ that is labelled by $\mathtt{a}$, and because t_f has to be fired before t_a can fire. Moreover, no ε-labelled transition can be fired once t_f has fired because it removes the token from place *run*. Let $\mathbf{m}'$ be the marking s.t. $\mathbf{m}'_0 \xrightarrow{\sigma'} \mathbf{m}'$. Clearly, $\mathbf{m}'(run) = 1$ and $\mathbf{m}'(p) \geq k$. Hence, the sequence σ' contains transitions from $\mathscr{T}' \setminus \{t_f, t_a\}$ only. Thus, by construction of $\mathscr{N}_p$, there exists a sequence of transitions σ

of $\mathscr{N}$ s.t. $\mathbf{m}_0 \xrightarrow{\sigma} \mathbf{m}$ with $\mathbf{m}(p) \geq k$. Since this is true for any $k \in \mathbb{N}$, it implies that p is *unbounded* in $\mathscr{N}$. □

This allows us to obtain the following proposition:

Proposition 3 *The universality problem for* PN+NBA *is undecidable.*

Proof From [17], we know that the place boundedness problem is undecidable for PN+NBA. Lemma 5 reduces this problem to the universality problem. Hence, the latter is undecidable. □

Since PN+NBA form a syntactic subclass of EWSTS, we immediately obtain:

Theorem 3 *The universality problem for* EWSTS *is undecidable.*

4 Pumping lemmata

This section presents three lemmata that show the limitations in the expressiveness of WSTS (for the first one), PN (for the second one), and PN+NBA (for the third one). All these lemmata have a similar statement: if a given WSTS (resp. PN, PN+NBA) accepts an infinite set of words $\{w_1, w_2, \ldots\}$ with a given structure, then it must also accept other words that are built upon the words $w_1, w_2, \ldots$ In some sense, these lemmata allow to "inflate" the set of accepted words. For that reason, we have chosen to call them *pumping lemmata*, owing to their similarities to the classical pumping lemmata for regular and context-free languages (see for instance [13]).

The proof techniques rely on properties of infinite sequences of configurations (equipped with a wqo), and monotonicity properties. The usefulness of these pumping lemmata will be demonstrated in Section 5, where we apply them to obtain several results about WSL.

4.1 A pumping lemma for WSL

Our first pumping lemma deals with WSL, and is very easy to prove:

Lemma 6 *Let L be a* WSL*, and let $w_1, w_2, \ldots$ be an infinite sequence of words s.t. $\forall k \geq 1 : w_k \in L$ and $w_k = B_k \cdot E_k$. Then, there exist $i < j$ s.t. $B_j \cdot E_i \in L$.*

Proof Let $S = \langle C, c_0, \Sigma, \Rightarrow, \leq \rangle$ be a WSTS s.t. $L(S, \mathscr{U}) = L$ for some $\leq$-upward-closed set $\mathscr{U}$. For any $k \geq 1$, let $c_k \in C$ be a configuration s.t. $c_0 \overset{B_k}{\Rightarrow}{}^* c_k \overset{E_k}{\Rightarrow}{}^* c'_k$, with $c'_k \in \mathscr{U}$. Since $\leq$ is a wqo, there is $i < j$ s.t. $c_i \leq c_j$. Hence, $c_0 \overset{B_j}{\Rightarrow}{}^* c_j \overset{E_i}{\Rightarrow}{}^* c'$, with $c'_i \leq c'$ by monotonicity. Thus, $c' \in \mathscr{U}$ and $B_j \cdot E_i \in L$. □

4.2 A pumping lemma for PN

Our second pumping lemma states properties of languages of Petri nets (more precisely, languages in the class $L^G(\mathsf{PN})$). This lemma will be exploited mainly in section 5.2, to strictly separate the expressive power of PN and PN+NBA. Other results of interest that one can obtain thanks to this lemma are mentioned in section 5.5.

The proof of the pumping lemma on WSL (see Lemma 6 above) exploited the properties of wqo and the monotonicity property in a rather straightforward fashion: from a well-chosen infinite sequence of configurations, we have extracted two comparable elements (property of wqo). Thanks to these two comparable elements, and by the monotonicity property, we have devised a new execution of the WSTS that allows to prove the lemma.

We follow the same pattern in the proof of the present pumping lemma for PN. Thus, starting from some well-chosen executions of the PN, we build *infinite sequences* of comparable markings that are reached along these sequences. This construction exploits the properties of wqo. However, it is much more intricate in the present case than in the case of Lemma 6 and deserves some attention. This is the purpose of lemma 7, that we introduce now.

Intuitively, Lemma 7 shows that, given a matrix $\mathscr{M}$ with infinitely many lines and columns containing tuples of natural numbers and given a natural number n, it is possible to build n infinite increasing sequences of elements of $\mathscr{M}$ that enjoy some properties which are necessary to prove the pumping lemma. These n sequences are obtained by the means of n functions $f_1, f_2, \dots, f_n$ which take their values in $\mathbb{Z}^+ \times \mathbb{Z}^+$, and are thus meant to *select* elements from $\mathscr{M}$. Thus the first infinite sequence to consider will be $\mathscr{M}(f_1(1)), \mathscr{M}(f_1(2)), \dots$; the second $\mathscr{M}(f_2(1)), \mathscr{M}(f_2(2)), \dots$ and so forth. The lemma is as follows:

Lemma 7 *Let $\mathscr{M}$ be a matrix with an infinite number of lines and columns, and whose elements are numbered by pairs in $\mathbb{Z}^+ \times \mathbb{Z}^+$ and take their values in $\mathbb{N}^k$ (for $k \geq 1$).*

For any $n \geq 1$, there are n functions $\mathbb{Z}^+ \mapsto \mathbb{Z}^+ \times \mathbb{Z}^+$, denoted by $f_1, f_2, \dots, f_n$ such that the following holds (where $f_i^l(x)$ and $f_i^c(x)$ denote respectively the first and second coordinate of $f_i(x)$):

1. *For any $1 \leq i \leq n$, for any $x \geq 1$: $f_i^c(x) \leq i \cdot f_i^l(x)$;*
2. *For any $1 \leq i \leq n$ and $1 \leq j \leq n$, for any $x \geq 1$: $f_i^l(x) = f_j^l(x)$;*
3. *For any $1 \leq i \leq n$, for any $1 \leq p \leq k$: either for any $x \geq 1$, $\mathscr{M}(f_i(x))(p) < \mathscr{M}(f_i(x+1))(p)$ or, for any $x \geq 1$, $\mathscr{M}(f_i(x))(p) = \mathscr{M}(f_i(x+1))(p)$;*
4. *For any $1 \leq i < j \leq n$, for any $x \geq 1$: $0 < f_j^c(x) - f_i^c(x) < f_j^c(x+1) - f_i^c(x+1)$;*
5. *For any $1 \leq i \leq n$, for any $x \geq 1$: $f_i^l(x) < f_i^l(x+1)$.*

Proof The proof is constructive and by induction on n.
Base case: $n = 1$**.** Let us consider the sequence:

$$S = \mathscr{M}(1,1), \mathscr{M}(2,1), \mathscr{M}(3,1), \dots$$

By lemma 3, the exists a strictly increasing function $\rho : \mathbb{Z}^+ \mapsto \mathbb{Z}^+$ s.t. the following is a subsequence of S:

$$\mathscr{M}(\rho(1),1), \mathscr{M}(\rho(2),1), \mathscr{M}(\rho(3),1), \dots$$

with the following property: for any $1 \leq p \leq k$: either for any $i \geq 1$: $\mathscr{M}(\rho(i),1)(p) < \mathscr{M}(\rho(i+1),1)(p)$ or, for any $i \geq 1$: $\mathscr{M}(\rho(i),1)(p) = \mathscr{M}(\rho(i+1),1)(p)$. We define f_1 as follows:

$$\text{for any } x \geq 1 : f_1(x) = (\rho(x), 1) \tag{1}$$

Let us check that the lemma holds on this function:

1. We have to show that for any $x \geq 1$: $f_1^c(x) \leq 1 \cdot f_1^l(x)$. By (1), this is equivalent to $\forall x \geq 1 : 1 \leq \rho(x)$, which is true by definition of ρ.
2. Trivial for $n = 1$.
3. This holds by (1) and definition of ρ.
4. Trivial for $n = 1$.
5. We have to show that for any $x \geq 1$: $f_1^l(x) < f_1^l(x+1)$. By (1), this is equivalent to $\forall x \geq 1 : \rho(x) < \rho(x+1)$, which is true by definition of ρ.

Inductive case: $n > 1$ Let us suppose there are $n-1$ functions $g_1, g_2, \ldots, g_{n-1}$ that respect the lemma and let us show how to build n functions $f_1, f_2, \ldots, f_n$ that respect the lemma.

We first define a function g_n as follows:

$$\text{for any } x \geq 1 : g_n(x) = (g_{n-1}^l(x), g_{n-1}^c(x) + x) \tag{2}$$

Let us now consider the sequence:

$$\mathscr{M}(g_n(1)), \mathscr{M}(g_n(2)), \mathscr{M}(g_n(3)), \ldots$$

By Lemma 3, there exists a strictly increasing function $\rho : \mathbb{Z}^+ \mapsto \mathbb{Z}^+$ s.t.:

$$\mathscr{M}(g_n(\rho(1))), \mathscr{M}(g_n(\rho(2))), \mathscr{M}(g_n(\rho(3))), \ldots$$

has the following property:

$$\forall 1 \leq p \leq k : \begin{cases} \text{either } \ \forall i \geq 1 : \mathscr{M}(g_n(\rho(i)))(p) < \mathscr{M}(g_n(\rho(i+1)))(p) \\ \quad\ \text{or } \ \forall i \geq 1 : \mathscr{M}(g_n(\rho(i)))(p) = \mathscr{M}(g_n(\rho(i+1)))(p) \end{cases} \tag{3}$$

We can now define $f_1, f_2, \ldots, f_n$ as follows:

$$\text{For any } 1 \leq i \leq n : \text{for any } x \geq 1 : f_i(x) = g_i(\rho(x)) \tag{4}$$

Let us show that they satisfy the lemma. We prove each point of the lemma by considering several subcases:

1. (a) **In the case where** $1 \leq i \leq n-1$**:**
$$\begin{aligned} & \forall x \geq 1 : f_i^c(x) \leq i \cdot f_i^l(x) \\ \iff & \forall x \geq 1 : g_i^c(\rho(x)) \leq i \cdot g_i^l(\rho(x)) \text{ by (4)} \end{aligned}$$
and the latter is true by induction hypothesis (point 1).

(b) **In the case where** $i = n$**:**

$$\begin{aligned}
& \forall x \geq 1 : f_n^c(x) \leq n \cdot f_n^l(x) \\
\iff & \forall x \geq 1 : g_n^c(\rho(x)) \leq n \cdot g_n^l(\rho(x)) && \text{by (4)} \\
\iff & \forall x \geq 1 : g_{n-1}^c(\rho(x)) + \rho(x) \leq n \cdot g_{n-1}^l(\rho(x)) && \text{by (2)} \\
\iff & \forall x \geq 1 : g_{n-1}^c(\rho(x)) - (n-1) \cdot g_{n-1}^l(\rho(x)) \\
& \qquad \leq g_{n-1}^l(\rho(x)) - \rho(x)
\end{aligned}$$

We show that the last point is valid by establishing that, for any $x \geq 1$, (i) the left-hand side of the inequation $g_{n-1}^c(\rho(x)) - (n-1) \cdot g_{n-1}^l(\rho(x))$ is ≤ 0 and (ii) the right-hand side of the inequation $g_{n-1}^l(\rho(x)) - \rho(x)$ is ≥ 0. The first point stems from the induction hypothesis, point 1. The latter, holds since, by induction hypothesis (point 5): $0 < g_{n-1}^l(1) < g_{n-1}^l(2) < g_{n-1}^l(3), \ldots$ Hence, for any $x \geq 1 : g_{n-1}^l(x) \geq x$, and thus for any $x \geq 1 : g_{n-1}^l(x) - x \geq 0$.

2. Without loss of generality, we assume that $j \leq i$.

(a) **In the case where** $1 \leq j < i \leq n-1$**:**

$$\begin{aligned}
& \forall x \geq 1 : f_i^l(x) = f_j^l(x) \\
\iff & \forall x \geq 1 : g_i^l(\rho(x)) = g_j^l(\rho(x)) && \text{by (4)}
\end{aligned}$$

The last point is true by induction hypothesis (point 2).

(b) **In the case where** $i = n$ **and** $1 \leq j \leq n-1$**:**

$$\begin{aligned}
& \forall x \geq 1 : f_n^l(x) = f_j^l(x) \\
\iff & \forall x \geq 1 : g_n^l(\rho(x)) = g_j^l(\rho(x)) && \text{by (4)} \\
\iff & \forall x \geq 1 : g_{n-1}^l(\rho(x)) = g_j^l(\rho(x)) && \text{by (2)}
\end{aligned}$$

The last point is true by induction hypothesis (point 2).

(c) **In the case where** $i = j$**:** the point is trivially true.

3. First remark that:

$$\begin{aligned}
& \forall 1 \leq i \leq n : \forall 1 \leq p \leq k : \\
& \quad \begin{cases} \text{either } \ \forall x \geq 1 : \mathscr{M}(f_i(x))(p) < \mathscr{M}(f_i(x+1))(p) \\ \quad\ \text{or } \ \forall x \geq 1 : \mathscr{M}(f_i(x))(p) = \mathscr{M}(f_i(x+1))(p) \end{cases} \\
\iff & \forall 1 \leq i \leq n : \forall 1 \leq p \leq k : && \text{by (4)} \\
& \quad \begin{cases} \text{either } \ \forall x \geq 1 : \mathscr{M}(g_i(\rho(x)))(p) < \mathscr{M}(g_i(\rho(x+1)))(p) \\ \quad\ \text{or } \ \forall x \geq 1 : \mathscr{M}(g_i(\rho(x)))(p) = \mathscr{M}(g_i(\rho(x+1)))(p) \end{cases}
\end{aligned}$$

(a) **In the case where** $1 \leq i \leq n-1$, this last point is true by induction hypothesis (point 3).

(b) **In the case where** $i = n$, this last point is true by (3).

4. (a) **In the case where** $1 \leq i < j \leq n-1$**:**

$$\begin{aligned}
& \forall x \geq 1 : 0 < f_j^c(x) - f_i^c(x) < f_j^c(x+1) - f_i^c(x+1) \\
\iff & \forall x \geq 1 : 0 < g_j^c(\rho(x)) - g_i^c(\rho(x)) \\
& \qquad < g_j^c(\rho(x+1)) - g_i^c(\rho(x+1)) && \text{by (4)}
\end{aligned}$$

This last point is true by induction hypothesis (point 4) and the fact that $\rho(x) < \rho(x+1)$.

(b) **In the case where** $1 \leq i \leq n-1$ **and** $j = n$**:**

$$\begin{aligned}
& \forall x \geq 1 : 0 < f_n^c(x) - f_i^c(x) < f_n^c(x+1) - f_i^c(x+1) \\
\iff & \forall x \geq 1 : 0 < g_n^c(\rho(x)) - g_i^c(\rho(x)) \\
& \qquad < g_n^c(\rho(x+1)) - g_i^c(\rho(x+1)) && \text{by (4)} \\
\iff & \forall x \geq 1 : 0 < g_{n-1}^c(\rho(x)) + \rho(x) - g_i^c(\rho(x)) \\
& \qquad < g_{n-1}^c(\rho(x+1)) + \rho(x+1) - g_i^c(\rho(x+1)) && \text{by (2)}
\end{aligned}$$

This can be proved by showing two points. First: $\forall x \geq 1 : 0 < g^c_{n-1}(\rho(x)) + \rho(x) - g^c_i(\rho(x))$. This holds because (i) $\forall x \geq 1 : \rho(x) > 0$ (by definition of ρ) and (ii) $\forall x \geq 1 : g^c_{n-1}(\rho(x)) - g^c_i(\rho(x)) \geq 0$ (in the case where $i \neq n-1$, we have $g^c_{n-1}(\rho(x)) - g^c_i(\rho(x)) > 0$ by induction hypothesis, point 4. In the case where $i = n-1$, we have $g^c_{n-1}(\rho(x)) - g^c_i(\rho(x)) = 0$). Second:

$$\begin{aligned} & \forall x \geq 1 : g^c_{n-1}(\rho(x)) + \rho(x) - g^c_i(\rho(x)) \\ & \qquad < g^c_{n-1}(\rho(x+1)) + \rho(x+1) - g^c_i(\rho(x+1)) \\ \iff & \forall x \geq 1 : g^c_{n-1}(\rho(x)) - g^c_i(\rho(x)) - \left(g^c_{n-1}(\rho(x+1)) - g^c_i(\rho(x+1))\right) \\ & \qquad < \rho(x+1) - \rho(x) \end{aligned}$$

The last point holds because: (i) the left-hand side $g^c_{n-1}(\rho(x)) - g^c_i(\rho(x)) - (g^c_{n-1}(\rho(x+1)) - g^c_i(\rho(x+1))$ of the inequation is ≤ 0 (when $i \neq n-1$, it is < 0 by induction hypothesis (point 4), and when $i = n-1$, it is $= 0$) and (ii) the right-hand side $\rho(x+1) - \rho(x)$ is > 0, by definition of ρ.

5. (a) **In the case where** $1 \leq i \leq n-1$**:**

$$\begin{aligned} & \forall x \geq 1 : f^l_i(x) < f^l_i(x+1) \\ \iff & \forall x \geq 1 : g^l_i(\rho(x)) < g^l_i(\rho(x+1)) \text{ by (4)} \end{aligned}$$

This last point is true by induction hypothesis (point 5) and the fact that $\rho(x) < \rho(x+1)$.

(b) **In the case where** $i = n$**:**

$$\begin{aligned} & \forall x \geq 1 : f^l_n(x) < f^l_n(x+1) \\ \iff & \forall x \geq 1 : g^l_n(\rho(x)) < g^l_n(\rho(x+1)) \qquad \text{by (4)} \\ \iff & \forall x \geq 1 : g^l_{n-1}(\rho(x)) < g^l_{n-1}(\rho(x+1)) \text{ by (2)} \end{aligned}$$

This last point is true by induction hypothesis (point 5) and the fact that $\rho(x) < \rho(x+1)$. □

Equipped with this lemma, we can state and prove our pumping lemma for PN.

Lemma 8 *Let $\mathcal{N}$ be a* PN *and $\mathcal{U}$ be an $\preccurlyeq$-upward-closed set of markings of $\mathcal{N}$.* **If** *there exists an infinite sequence of words $w_1, w_2, \ldots$ such that for any $i \geq 1$, there exist two words B_i, E_i with $B_i w_i^* E_i \subseteq L(\mathcal{N}, \mathcal{U})$,* **then** *there exist $0 < n_1 < n_2 < n_3$ such that for any $K \geq 0$, there exists $K' \geq K$ and $i_1 \geq 0, i_2 \geq 0$ such that the word $B_{n_3} w_{n_3}^{i_1} w_{n_1}^{K'} w_{n_2}^{i_2} E_{n_2}$ is in $L(\mathcal{N}, \mathcal{U})$.*

The proof of the lemma is quite tedious and technical. However, we believe that the technique at work in this proof is interesting by itself, since it directly exploits the monotonicity and well-quasi ordering properties that are characteristic of WSTS. It also relies in great part on the fact that Petri net transitions have a constant effect. Before giving the proof, we provide the reader with a sketch that presents the main arguments in order to make the task of reading the proof easier. Throughout this explanation, we refer to peculiar markings using the same notations as in the proof. The reader is advised to refer to Fig. 2, 3 and 4 to get the intuition of the meaning of these notations.

The proof is constructive. From the fact that the PN accepts the words $B_i w_i^* E_i$ for any $i \geq 1$, we build, by applying Lemma 7, infinite sequences of markings that are ordered (this is the purpose of the two first steps of the proof). Then, at the third step, we exploit these ordering properties, as well as the monotonicity of the PN and the fact that their transitions have constant effect, to show that a sequence of

$$\begin{pmatrix}
\mathbf{m}_1^1 & \mathbf{m}_1^2 & \cdots & \mathbf{m}_1^{N+1} & 0^{|\mathscr{P}|} & \cdots & 0^{|\mathscr{P}|} & 0^{|\mathscr{P}|} & \cdots & 0^{|\mathscr{P}|} & 0^{|\mathscr{P}|} & \cdots\\
\mathbf{m}_2^1 & \mathbf{m}_2^2 & \cdots & \mathbf{m}_2^{N+1} & \mathbf{m}_2^{N+2} & \cdots & \mathbf{m}_2^{2N+1} & 0^{|\mathscr{P}|} & \cdots & 0^{|\mathscr{P}|} & 0^{|\mathscr{P}|} & \cdots\\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & & & &\\
\mathbf{m}_j^1 & \mathbf{m}_j^2 & \cdots & \mathbf{m}_j^{N+1} & \mathbf{m}_j^{N+2} & \cdots & \mathbf{m}_j^{2N+1} & \mathbf{m}_j^{2N+2} & \cdots & \mathbf{m}_j^{jN+1} & 0^{|\mathscr{P}|} & \cdots\\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & &
\end{pmatrix}$$

Fig. 2 An illustration of the construction of $\mathscr{M}_i$.

transitions with the desirable form is firable, and leads to the accepting $\preccurlyeq$-upward-closed set of markings.

Step 1 Let N denote the value $2^{|\mathscr{P}|}+1$. For all $i \geq 1$, we consider the infinite sequence of words $B_i w_i^N E_i$, $B_i w_i^{2N} E_i$, $B_i w_i^{3N} E_i,\ldots$, $B_i w_i^{jN} E_i,\ldots$ For each of these words, we select an accepting sequence of transitions and consider the markings that are reached along this sequence. For instance, when considering the sequence that accepts $B_i w_i^{jN} E_i$, we select $jN+1$ markings $\mathbf{m}_j^k$ $(1 \leq k \leq jN+1)$ s.t.:

$$\mathbf{m}_{init} \xrightarrow{B_i} \mathbf{m}_j^1 \xrightarrow{w_i} \mathbf{m}_j^2 \xrightarrow{w_i} \cdots \xrightarrow{w_i} \mathbf{m}_j^{jN+1} \xrightarrow{E_i}$$

For any $i \geq 1$, we build an infinite matrix $\mathscr{M}_i$. The jth line of $\mathscr{M}_i$ contains all the markings that have been selected along the run accepting $B_i w_i^{jN} E_i$ (in the same order as in the run). Hence, we obtain a matrix with infinitely many lines. In order to obtain infinitely many elements on each line, we pad the matrix with $0^{|\mathscr{P}|} = \langle 0,0,\ldots,0\rangle$ markings. Fig. 2 presents an example of such a matrix.

Then, we apply lemma 7 on $\mathscr{M}_i$ and build N functions $f_{(i,1)}, f_{(i,2)}, \ldots, f_{(i,N)}$. These functions allow us to select elements in the matrix $\mathscr{M}_i$. The selected elements are arranged into a new matrix $\mathscr{M}_i^{\preccurlyeq}$ with N columns and infinitely many lines (see Fig. 3 for an informal illustration of the construction). $\mathscr{M}_i^{\preccurlyeq}$ is built column by column: the j-th column contains the elements selected by $f_{(i,j)}$, i.e., the first element of the j-th columns is the element of $\mathscr{M}_i$ whose coordinate are given by $f_{(i,j)}(1)$, the second element is the element $f_{(i,j)}(2)$ in $\mathscr{M}_i$, and so on.

The new matrix $\mathscr{M}_i^{\preccurlyeq}$ has interesting properties upon with we rely in the rest of the proof. All these properties are direct consequences of Lemma 7. The most important are:

1. Each column of $\mathscr{M}_i^{\preccurlyeq}$ forms an infinitely increasing sequence of markings (according to point 3 of Lemma 7);
2. Each line of $\mathscr{M}_i^{\preccurlyeq}$ is actually a subsequence of one of the lines of $\mathscr{M}_i$ (by point 2 of Lemma 7). Thus, if $\mathbf{m}$ and $\mathbf{m}'$ are two markings taken from the same line of $\mathscr{M}_i^{\preccurlyeq}$ (with $\mathbf{m}$ appearing before $\mathbf{m}'$), we are sure that there exists a sequence of transitions that is firable from $\mathbf{m}$ and produces $\mathbf{m}'$. For that reason, we will sometimes refer to lines of $\mathscr{M}_i^{\preccurlyeq}$ as *runs*.
3. Let us consider two lines ℓ_1 and ℓ_2 of $\mathscr{M}_i^{\preccurlyeq}$ s.t. $\ell_1 < \ell_2$. Let $\mathbf{m}_1$, $\mathbf{m}_2$ be two markings of line ℓ_1 that appear respectively in columns number k_1

$$
\mathscr{M}_i = \left(\begin{array}{ccccccccccccccccc}
\boxed{\mathscr{M}_i(f_{(i,1)}(1))} & \boxed{\boxed{\mathscr{M}_i(f_{(i,2)}(1))}} & \boxed{\boxed{\boxed{\mathscr{M}_i(f_{(i,3)}(1))}}} & 0^{|\mathscr{P}|} & \cdots & & & & & & & & & & & & \\
\times & \times & \times & \times & \times & \times & 0^{|\mathscr{P}|} & \cdots & & & & & & & & & \\
\boxed{\mathscr{M}_i(f_{(i,1)}(2))} & \times & \boxed{\boxed{\mathscr{M}_i(f_{(i,2)}(2))}} & \times & \boxed{\boxed{\boxed{\mathscr{M}_i(f_{(i,3)}(2))}}} & \times & \times & \times & \times & 0^{|\mathscr{P}|} & \cdots & & & & & & \\
\times & \times & \boxed{\mathscr{M}_i(f_{(i,1)}(3))} & \times & \times & \times & \boxed{\boxed{\mathscr{M}_i(f_{(i,2)}(3))}} & \times & \times & \boxed{\boxed{\boxed{\mathscr{M}_i(f_{(i,3)}(3))}}} & \times & \times & 0^{|\mathscr{P}|} & \cdots & & & \\
\times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & 0^{|\mathscr{P}|} & \cdots \\
\vdots & & & \vdots & & & \vdots & & & \vdots & & & \vdots & & & \vdots &
\end{array}\right)
$$

$$
\mathscr{M}_i^{\preccurlyeq} = \left(\begin{array}{ccc}
\boxed{\mathscr{M}_i(f_{(i,1)}(1))} & \boxed{\boxed{\mathscr{M}_i(f_{(i,2)}(1))}} & \boxed{\boxed{\boxed{\mathscr{M}_i(f_{(i,3)}(1))}}} \\
\boxed{\mathscr{M}_i(f_{(i,1)}(2))} & \boxed{\boxed{\mathscr{M}_i(f_{(i,2)}(2))}} & \boxed{\boxed{\boxed{\mathscr{M}_i(f_{(i,3)}(2))}}} \\
\boxed{\mathscr{M}_i(f_{(i,1)}(3))} & \boxed{\boxed{\mathscr{M}_i(f_{(i,2)}(3))}} & \boxed{\boxed{\boxed{\mathscr{M}_i(f_{(i,3)}(3))}}} \\
\vdots & \vdots & \vdots
\end{array}\right)
$$

Fig. 3 An illustration of the construction of $\mathscr{M}_i^{\preccurlyeq}$ for $N = 3$. Each $\times$ represent a marking reached in the PN. Lemma 7 has been applied in order to obtain three functions $f_{(i,1)}$, $f_{(i,2)}$ and $f_{(i,3)}$.

and k_2 with $k_1 < k_2$. Let $\mathbf{m}'_1$ and $\mathbf{m}'_2$ be two markings of line ℓ_2 that appear respectively in columns k_1 and k_2. Let σ and σ' be the sequences s.t. $\mathbf{m}_1 \xrightarrow{\sigma} \mathbf{m}_2$ and $\mathbf{m}'_1 \xrightarrow{\sigma'} \mathbf{m}'_2$. Then, the number of w_i that labels σ' is strictly larger that the number of w_i that labels σ (this stems from point 4 of Lemma 7). This property is important since we want to be able to construct sequences of the form $B_{n_3} w_{n_3}^{i_1} w_{n_1}^{K'} w_{n_2}^{i_2} E_{n_2}$ with K' arbitrarily large.

Step 2 The second step consists to select an infinite subset S of $\{\mathscr{M}_i^{\preccurlyeq} \mid i \geq 1\}$. We do this by building a sequence of runs such that the jth run in the sequence is the first run appearing in $\mathscr{M}_j^{\preccurlyeq}$. Again, we extract the sub-sequence S where markings appearing in different runs are $\preccurlyeq$-ordered by applying successively Lemma 3. In this case, only markings appearing along the $2^{|\mathscr{P}|}+1$ first “columns” are $\preccurlyeq$-ordered.

Step 3 Finally, we show how to split and combine parts of runs appearing in the $\mathscr{M}_i$’s and S to obtain a run that allows the PN to accept a word of the desired form. This is shown in Fig. 4.

In order to build this sequence, we rely on several variables, namely: c_1, c_2, n and x. At the present step of the proof, we introduce some constraints that relate x, and n to c_1, c_2 and K. These constraints are meant to produce a sequence of transitions that accepts a word of the desired form. The main (and most technical) part of step 3 consists to show that these constraints are satisfiable.

The first part of the sequence is the prefix of $\mathscr{M}_{\rho(n)}^{\preccurlyeq}(x)$, up to the “column” c_1 (see Fig. 4). At that point, we are guaranteed that the marking we obtain is larger than $\mathscr{M}_{\rho(1)}^{\preccurlyeq}(K, c_1)$. This allows us to continue the sequence with a part of $\mathscr{M}_{\rho(1)}^{\preccurlyeq}(K)$, starting at “column” c_1 and ending at “column” c_2. Again, by exploiting the properties of the sequences built at steps 1 and 2, as well as the constant effect of PN transitions, we are ensured that the marking we have reached is larger than $\mathscr{M}_{\rho(2)}^{\preccurlyeq}(1, c_2)$. This allow us to finish the sequence with the suffix of $\mathscr{M}_{\rho(2)}^{\preccurlyeq}(1)$. The word accepted by this sequence is of the desired form, since we have correctly chosen the values of x and n (in particular, the central part of the word is longer than K times $|w_{n_1}|$).

We are now ready to present the proof of Lemma 8.

Proof Let $\mathscr{N}$ be a PN with set of places $\mathscr{P}$ and initial marking $\mathbf{m}_{init}$, such that $B_i w_i^* E_i \subseteq L(\mathscr{N}, \mathscr{U})$ for all $i \geq 1$. For technical reason, we assume without loss of generality that $\mathscr{N}$ has a transition $t_\varepsilon = \langle \emptyset, \emptyset, \perp, \perp, 0, \varepsilon \rangle$, i.e. a transition labelled by ε that can be fired from any marking and has no effect. Let N denote the value $2^{|\mathscr{P}|}+1$.

$$\mathscr{M}^{\preccurlyeq}_{\rho(1)}(K) = \mathbf{m}_{init} \xrightarrow{B_{\rho(1)}} \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,1) \cdots\cdots \xrightarrow{w_{\rho(1)}} \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_1) \xRightarrow{w_{\rho(1)}} \vdots\vdots\vdots \xRightarrow{w_{\rho(1)}} \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_2) \xrightarrow{w_{\rho(1)}} \cdots\cdots \xrightarrow{E_{\rho(1)}} \mathbf{n}_{\rho(1),K}$$

$$\mathscr{M}^{\preccurlyeq}_{\rho(n)}(x) = \mathbf{m}_{init} \xRightarrow{B_{\rho(n)}} \mathscr{M}^{\preccurlyeq}_{\rho(n)}(x,1) \xRightarrow{w_{\rho(n)}} \vdots\vdots\vdots \xRightarrow{w_{\rho(n)}} \mathscr{M}^{\preccurlyeq}_{\rho(n)}(x,c_1) \cdots\cdots \xrightarrow{E_{\rho(n)}} \mathbf{n}_{\rho(n),x}$$

$$\mathscr{M}^{\preccurlyeq}_{\rho(2)}(1) = \mathbf{m}_{init} \xrightarrow{B_{\rho(2)}} \mathscr{M}^{\preccurlyeq}_{\rho(2)}(1,1) \cdots\cdots \xrightarrow{w_{\rho(2)}} \mathscr{M}^{\preccurlyeq}_{\rho(2)}(1,c_2) \xRightarrow{w_{\rho(2)}} \vdots\vdots\vdots \xRightarrow{E_{\rho(2)}} \mathbf{n}_{\rho(2),1}$$

Fig. 4 The firable sequence (along the $\Rightarrow$'s) that accepts a word of the form $B_{n_3} w_{n_3}^{i_1} w_{n_1}^{K'} w_{n_2}^{i_2} E_{n_2}$.

Step 1 For any $i \geq 1$, let S_i be the infinite sequence of all the runs accepting the words of the form $B_i w_i^{j \cdot N} E_i$, with $j \geq 1$. That is, S_i is the sequence of runs:

$$
\begin{array}{l}
\mathbf{m}_{init} \xrightarrow{\upsilon_1} \mathbf{m}_1^1 \xrightarrow{\varsigma_1^1} \mathbf{m}_1^2 \xrightarrow{\varsigma_1^2} \cdots \xrightarrow{\varsigma_1^N} \mathbf{m}_1^{N+1} \xrightarrow{\upsilon_1'} \mathbf{n}_{i,1} \\
\mathbf{m}_{init} \xrightarrow{\upsilon_2} \mathbf{m}_2^1 \xrightarrow{\varsigma_2^1} \mathbf{m}_2^2 \xrightarrow{\varsigma_2^2} \cdots \quad \cdots \xrightarrow{\varsigma_2^{2\cdot N}} \mathbf{m}_2^{2\cdot N+1} \xrightarrow{\upsilon_2'} \mathbf{n}_{i,2} \\
\vdots \\
\mathbf{m}_{init} \xrightarrow{\upsilon_j} \mathbf{m}_j^1 \xrightarrow{\varsigma_j^1} \mathbf{m}_j^2 \xrightarrow{\varsigma_j^2} \cdots \quad \cdots \quad \cdots \xrightarrow{\varsigma_j^{j\cdot N}} \mathbf{m}_j^{j\cdot N+1} \xrightarrow{\upsilon_j'} \mathbf{n}_{i,j}
\end{array}
$$

where for any $\ell \geq 1$: $\mathbf{n}_{i,\ell} \in \mathscr{U}$, $\Lambda(\upsilon_\ell) = B_i$ and $\Lambda(\upsilon_\ell') = E_i$. Moreover, $\forall \ell \geq 1$: $\forall 1 \leq k \leq \ell \cdot N : \Lambda(\varsigma_\ell^k) = w_i$. Remark that these executions exist even when $w_i = \varepsilon$, because $\mathscr{N}$ contains the t_ε transition.

Let $0^{|\mathscr{P}|}$ denote the marking that ranges over $|\mathscr{P}|$ places and assigns 0 token to each place. For any $i \geq 1$, we build, an infinite matrix $\mathscr{M}_i$, whose values are either markings met along the runs of S_i or $0^{|\mathscr{P}|}$. More precisely, for any $j \geq 1, k \geq 1$ we have:

$$
\mathscr{M}_i(j,k) = \begin{cases} \mathbf{m}_j^k & \text{if } 1 \leq k \leq j \cdot N + 1 \\ 0^{|\mathscr{P}|} & \text{otherwise} \end{cases} \tag{5}
$$

For any $i \geq 1$, we can apply Lemma 7 to $\mathscr{M}_i$, and obtain N functions that respect the five points of the lemma. We denote these functions by $f_{(i,1)}, f_{(i,2)}, \ldots f_{(i,N)}$. Thanks to these functions, we build infinitely many sequences of N markings. We represent these sequences under the form of a new matrix $\mathscr{M}_i^{\preccurlyeq}$, with N columns and infinitely many lines (each line corresponds to a sequence). $\mathscr{M}_i^{\preccurlyeq}$ is defined as follows (where $f_{(i,k)}^l(j)$ and $f_{(i,k)}^c(j)$ denote respectively the first and second coordinate of $f_{(i,k)}(j)$):

$$
\forall i \geq 1 : \forall j \geq 1 : \forall 1 \leq k \leq N : \mathscr{M}_i^{\preccurlyeq}(j,k) = \mathscr{M}_i(f_{(i,k)}^l(j), f_{(i,k)}^c(j)) \tag{6}
$$

For any $i, j \geq 1$, let $\mathscr{M}_i^{\preccurlyeq}(j)$ denote $\mathscr{M}_i^{\preccurlyeq}(j,1), \mathscr{M}_i^{\preccurlyeq}(j,2), \ldots \mathscr{M}_i^{\preccurlyeq}(j,N)$, i.e. the sequence of markings that appears on the j-th line of $\mathscr{M}_i^{\preccurlyeq}$. Let us expose several properties of these sequences that will be useful in the sequel of the proof:

1. For any $i \geq 1$, $j \geq 1$, the sequence $\mathscr{M}_i^{\preccurlyeq}(j)$ corresponds to a run of S_i. More precisely, $\mathscr{M}_i^{\preccurlyeq}(j)$ is a subsequence of the markings in the $f_{(i,1)}^l(j)$-th run of S_i. According to the definitions of $\mathscr{M}_i$ and $\mathscr{M}_i^{\preccurlyeq}$ (see (5) and (6)), this can be proved by establishing three points:
   (a) The markings of $\mathscr{M}_i^{\preccurlyeq}(j)$ have all been taken in the same run of S_i: $f_{(i,1)}^l(j) = f_{(i,2)}^l(j) = \cdots = f_{(i,N)}^l(j)$. This is true by point 2 of Lemma 7.
   (b) The ordering of the markings along the run has been preserved. This amounts to show that the sequence $f_{(i,1)}^c(j), f_{(i,2)}^c(j), \ldots, f_{(i,N)}^c(j)$ is strictly increasing. This follows directly from point 4 of Lemma 7.

(c) All the selected markings in $\mathscr{M}_i^{\preccurlyeq}(j)$ exist in the $f^l_{(i,1)}(j)$-th run of S_i, i.e., they are all different from the $0^{|\mathscr{P}|}$ markings we have added when building $\mathscr{M}_i$. Since the ordering of the markings has been preserved, it is sufficient to show that the last marking of $\mathscr{M}_i^{\preccurlyeq}(j)$ corresponds to a marking of the $f^l_{(i,1)}(j)$-th run of S_i, i.e., that $f^c_{(i,N)}(j) \leq N \cdot f^l_{(i,1)}(j) + 1$. By point (a) above, this is equivalent to $f^c_{(i,N)}(j) \leq N \cdot f^l_{(i,N)}(j) + 1$, which is true by point 1 of Lemma 7.

2. Since $\mathscr{M}_i^{\preccurlyeq}(j)$ is a subsequence of markings that appear in a run of S_i, there exists, for any $i \geq 1$, $j \geq 1$ and $1 \leq k_1 < k_2 \leq N$ a sequence of transitions $\sigma_i^j(k_1,k_2)$ s.t.:

$$\mathscr{M}_i^{\preccurlyeq}(j,k_1) \xrightarrow{\sigma_i^j(k_1,k_2)} \mathscr{M}_i^{\preccurlyeq}(j,k_2)$$

Moreover, for any $i \geq 1$, $j \geq 1$ and $1 \leq k \leq N$, there are two sequences of transitions $\sigma_i^j(\cdot,k)$ and $\sigma_i^j(k,\cdot)$ s.t.:

$$\mathbf{m}_{init} \xrightarrow{\sigma_i^j(\cdot,k)} \mathscr{M}_i^{\preccurlyeq}(j,k) \xrightarrow{\sigma_i^j(k,\cdot)} \mathbf{n} \quad (\mathbf{n} \in \mathscr{U})$$

By (5) and (6), these sequences are labelled as follows (for any $i, j \geq 1$):

$$\forall 1 \leq k_1 < k_2 \leq N : \Lambda(\sigma_i^j(k_1,k_2)) = w_i^{(f^c_{(i,k_2)}(j) - f^c_{(i,k_1)}(j))} \tag{7}$$

$$\forall 1 \leq k \leq N : \Lambda(\sigma_i^j(\cdot,k)) = B_i w_i^{(f^c_{(i,k)}(j)-1)} \tag{8}$$

$$\forall 1 \leq k \leq N : \Lambda(\sigma_i^j(k,\cdot)) = w_i^{(j \cdot N+1-f^c_{(i,k)}(j))} E_i \tag{9}$$

Finally, let us introduce the following notation. Let w and $\upsilon \neq \varepsilon$ be two words. Then, we let $\|w\|_\upsilon = i$ iff $w = \upsilon^i$. By (7) and point 4 of Lemma 7, the following holds:

$$\begin{aligned}&\forall i, j \geq 1 : \forall 1 \leq k_1 < k_2 \leq N : \\ &\qquad w_i \neq \varepsilon \text{ implies } \|\Lambda(\sigma_i^j(k_1,k_2))\|_{w_i} < \|\Lambda(\sigma_i^{j+1}(k_1,k_2))\|_{w_i}\end{aligned}$$

That is, when $w_i \neq \varepsilon$, the word that labels the sequence leading from the k_1-th marking of the j-th run of $\mathscr{M}_i^{\preccurlyeq}$ to its k_2-th marking is strictly shorter than the word labelling the corresponding sequence in the $j+1$-th run of $\mathscr{M}_i^{\preccurlyeq}$. In particular, since $w_i \neq \varepsilon$ implies that $\|\Lambda(\sigma_i^1(k_1,k_2))\| \geq 1$, we have:

$$\forall i \geq 1 : \forall 1 \leq k_1 < k_2 \leq N : w_i \neq \varepsilon \text{ implies } \|\Lambda(\sigma_i^j(k_1,k_2))\|_{w_i} \geq j \tag{10}$$

3. Let us first introduce the folowing notation. Let $\mathscr{S}$ be an infinite sequence of runs $\mathscr{S}(1), \mathscr{S}(2), \ldots$, s.t. each run $\mathscr{S}(i)$ is made up of N markings $\mathscr{S}(i,1)$, $\mathscr{S}(i,2)$, $\ldots$, $\mathscr{S}(i,N)$. Then, for any $1 \leq k \leq N$, we denote by $Places(\mathscr{S},k)$ the set of places s.t. $p \in Places(\mathscr{S},k)$ iff, for any $i \geq 1$: $\mathscr{S}(i,k)(p) < \mathscr{S}(i+1,k)(p)$.

By (5) and (6), and by Lemma 7, point 3, for any $1 \leq k \leq N$ and $i \geq 1$, the set $Places(\mathcal{M}_i^{\preccurlyeq}, k) \subseteq \mathcal{P}$ is s.t.:

$$\begin{array}{l} \forall i \geq 1 : \forall 1 \leq k \leq N : \forall p \in \mathcal{P} : \\ \quad p \in Places(\mathcal{M}_i^{\preccurlyeq}, k) \text{ iff } \forall j \geq 1 : \mathcal{M}_i^{\preccurlyeq}(j,k)(p) < \mathcal{M}_i^{\preccurlyeq}(j+1,k)(p) \\ \quad p \notin Places(\mathcal{M}_i^{\preccurlyeq}, k) \text{ iff } \forall j \geq 1 : \mathcal{M}_i^{\preccurlyeq}(j,k)(p) = \mathcal{M}_i^{\preccurlyeq}(j+1,k)(p) \end{array} \tag{11}$$

In particular, this implies that $\mathcal{M}_i^{\preccurlyeq}(1,k), \mathcal{M}_i^{\preccurlyeq}(2,k), \ldots, \mathcal{M}_i^{\preccurlyeq}(j,k), \ldots$ is an increasing sequence (w.r.t. $\preccurlyeq$):

$$\forall i \geq 1 : \forall 1 \leq k \leq N : \forall j \geq 1 : \mathcal{M}_i^{\preccurlyeq}(j,k) \preccurlyeq \mathcal{M}_i^{\preccurlyeq}(j+1,k) \tag{12}$$

$\boxed{\textbf{Step 2}}$ To finish with the construction, we consider the infinite sequence of runs $\mathcal{M}_1^{\preccurlyeq}(1), \mathcal{M}_2^{\preccurlyeq}(1), \ldots$ made up of the first runs (lines) of all $\mathcal{M}_i^{\preccurlyeq}$. From this sequence, we extract the infinite subsequence $S = \mathcal{M}_{\rho(1)}^{\preccurlyeq}(1), \mathcal{M}_{\rho(2)}^{\preccurlyeq}(1), \ldots$ by successively applying Lemma 3. We construct S such that:

1. For any $1 \leq j \leq N$ the sequence $\mathcal{M}_{\rho(1)}^{\preccurlyeq}(1,j), \mathcal{M}_{\rho(2)}^{\preccurlyeq}(1,j), \ldots$ is increasing:
$$\forall k \geq 1 : \mathcal{M}_{\rho(k)}^{\preccurlyeq}(1,j) \preccurlyeq \mathcal{M}_{\rho(k+1)}^{\preccurlyeq}(1,j) \tag{13}$$
2. For any $1 \leq j \leq N$, the places in the set $Places(S,j) \subseteq \mathcal{P}$ strictly increase along the sequence $\mathcal{M}_{\rho(1)}^{\preccurlyeq}(1,j), \mathcal{M}_{\rho(2)}^{\preccurlyeq}(1,j), \ldots$ and all the other places stay constant along the sequence:
$$\forall k \geq 1 : \mathcal{M}_{\rho(k)}^{\preccurlyeq}(1,j)(p) < \mathcal{M}_{\rho(k+1)}^{\preccurlyeq}(1,j)(p) \text{ iff } p \in Places(S,j) \tag{14}$$
Let c_1 and c_2 be such that $1 \leq c_1 < c_2 \leq N$ and $Places(S,c_1) = Places(S,c_2)$. Remark that c_1 and c_2 always exist because there are $2^{|\mathcal{P}|} = N-1$ subsets of $\mathcal{P}$.
3. The sets of strictly increasing places of the selected $\mathcal{M}_i^{\preccurlyeq}$ are equal:
$$\forall 1 \leq j \leq N : \forall k \geq 1 : Places(\mathcal{M}_{\rho(k)}^{\preccurlyeq}, j) = Places(\mathcal{M}_{\rho(k+1)}^{\preccurlyeq}, j) \tag{15}$$
This is possible because there is a finite number of subsets of $\mathcal{P}$.

$\boxed{\textbf{Step 3}}$ The rest of the proof consists in showing that there are $0 < n_1 < n_2 < n_3$ s.t. for any $K \in \mathbb{N}$ there are $i_1 \geq 0$, $i_2 \geq 0$ and $K' \geq K$ and the word $w = B_{n_3} w_{n_3}^{i_1} w_{n_1}^{K'} w_{n_2}^{i_2} E_{n_2}$ is accepted by $\mathcal{N}$.

We first choose the values of n_1 and n_2 as follows: $n_1 = \rho(1)$ and $n_2 = \rho(2)$ (where ρ is the function defined at the begining of step 2). Then, we show how to compute n_3. Actually, we let $n_3 = \rho(n)$ for a well-chosen value of n. We provide a constraint (see equation (16) in the sequel) on n that we prove satisfiable and that we exploit at the end of the proof. Equipped with the values n_1, n_2 and n_3, we show that, for any $K \in \mathbb{N}$, it is possible to compute a value x s.t. the sequence of transitions $\sigma = \sigma_{\rho(n)}^{x}(\cdot, c_1) \cdot \sigma_{\rho(1)}^{K}(c_1, c_2) \cdot \sigma_{\rho(2)}^{1}(c_2, \cdot)$ accepts a word of the desired form.

Choice of n Let $\mathbf{m}_n$ be the marking such that $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1) \xrightarrow{\sigma^1_{\rho(1)}(c_1,c_2)} \mathbf{m}_n$. Remark that, since we are dealing with Petri nets, the sequence $\sigma^1_{\rho(1)}(c_1,c_2)$ has a constant effect (i.e., characterized by a vector of natural constants) equal to $\mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_2) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1)$. As a consequence, we have: $\mathbf{m}_n = \mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1) + \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_2) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1)$. We choose $n > 2$ such that:

$$\mathbf{m}_n = \mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1) + \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_2) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1) \succcurlyeq \mathscr{M}^{\preccurlyeq}_{\rho(2)}(1,c_2) \qquad (16)$$

Let us show that such a n exists. First notice that $\sigma^1_{\rho(1)}(c_1,c_2)$ is firable from $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1)$ for all $n > 2$, because $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1) \succcurlyeq \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1)$ following (13). Then, recall that $Places(S,c_1) = Places(S,c_2)$, i.e. the places that strictly increase along S are the same in columns c_1 and c_2. Let us show that, for any place p, $\mathbf{m}_n(p) \geq \mathscr{M}^{\preccurlyeq}_{\rho(2)}(1,c_2)(p)$. For that purpose, we consider two cases:

1. If $p \in Places(S,c_1)$, then, the sequence $\mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1)(p), \mathscr{M}^{\preccurlyeq}_{\rho(2)}(1,c_1)(p), \ldots$ is strictly growing by (14), and, for any place $p \in Places(S,c_1)$, for any $n \geq 1$, $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1)(p) \geq n-1$. Thus there exists $n \geq 1$ s.t. $\forall p \in Places(S,c_1)$: $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1)(p) \geq \mathscr{M}^{\preccurlyeq}_{\rho(2)}(1,c_2)(p) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_2)(p) + \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1)(p)$. This is equivalent to $\forall p \in Places(S,c_1) : \mathbf{m}_n(p) \geq \mathscr{M}^{\preccurlyeq}_{\rho(2)}(1,c_2)(p)$, by definition of $\mathbf{m}_n$.
2. On the other hand, for any $p \in \mathscr{P} \setminus Places(S,c_1)$, we have: $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1)(p) = \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1)(p)$ and $\mathscr{M}^{\preccurlyeq}_{\rho(2)}(1,c_2)(p) = \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_2)(p)$, by (14) again. Hence, $\forall p \in \mathscr{P} \setminus Places(S,c_1)$: $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1)(p) - \mathscr{M}^{\preccurlyeq}_{\rho(2)}(1,c_2)(p) = \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1)(p) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_2)(p)$, and thus, we have that, for any place $p \in \mathscr{P} \setminus Places(S,c_1)$: $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1)(p) + \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_2)(p) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1)(p) = \mathscr{M}^{\preccurlyeq}_{\rho(2)}(1,c_2)(p)$, hence $\mathbf{m}_n(p) = \mathscr{M}^{\preccurlyeq}_{\rho(2)}(1,c_2)(p)$.

From these two points, we conclude that there exists n s.t. $\mathbf{m}_n \succcurlyeq \mathscr{M}^{\preccurlyeq}_{\rho(2)}(1,c_2)$.

Choice of x We choose $x > K$ such that:

$$\mathscr{M}^{\preccurlyeq}_{\rho(n)}(x,c_1) \succcurlyeq \mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1) + \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_1) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1) \qquad (17)$$

One can prove that such an x always exists by a the same reasoning as in the choice of n, and by the fact that $Places(\mathscr{M}^{\preccurlyeq}_{\rho(n)},c_1) = Places(\mathscr{M}^{\preccurlyeq}_{\rho(1)},c_1)$ (see (15) above). Indeed, $\forall p \in Places(\mathscr{M}^{\preccurlyeq}_{\rho(1)},c_1)$, the sequence $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1)(p), \mathscr{M}^{\preccurlyeq}_{\rho(n)}(2,c_1)(p), \ldots$ is strictly increasing by (11) and (15), and we can thus choose x large enough to have $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(x,c_1)(p) \geq \mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1)(p) + \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_1)(p) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1)(p)$, for any place p in the set $Places(\mathscr{M}^{\preccurlyeq}_{\rho(1)},c_1)$. On the other hand, for any $p \in \mathscr{P} \setminus Places(\mathscr{M}^{\preccurlyeq}_{\rho(1)},c_1)$: $\mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_1)(p) = \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1)(p)$, by (11) and (15). Thus,

$\mathscr{M}^{\preccurlyeq}_{\rho(n)}(x,c_1)(p) \geq \mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1)(p) + \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_1)(p) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1)(p)$ if and only if $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(x,c_1)(p) \geq \mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1)(p)$. This latter point is true by (11). We conclude that $\forall p \in \mathscr{P} : \mathscr{M}^{\preccurlyeq}_{\rho(n)}(x,c_1)(p) \geq \mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1)(p) + \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_1)(p) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1)(p)$.

The next step amounts to showing that the sequence σ is firable. From $\mathbf{m}_{init}$, we fire $\sigma^{x}_{\rho(n)}(\cdot,c_1)$ and reach $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(x,c_1)$. From that marking, we can fire the sequence $\sigma^{K}_{\rho(1)}(c_1,c_2)$. This is possible because $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(x,c_1) \succcurlyeq \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_1)$. Indeed, by (17): $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(x,c_1) \succcurlyeq \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_1) + \left(\mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1)\right)$. However, we know that $\left(\mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1)\right) \succcurlyeq 0^{|\mathscr{P}|}$, by (13). This implies that $\mathscr{M}^{\preccurlyeq}_{\rho(n)}(x,c_1) \succcurlyeq \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_1)$ and we have:

$$\mathbf{m}_{init} \xrightarrow{\sigma^{x}_{\rho(n)}(0,c_1)} \mathscr{M}^{\preccurlyeq}_{\rho(n)}(x,c_1) \xrightarrow{\sigma^{K}_{\rho(1)}(c_1,c_2)} \mathbf{m}$$

To finish the sequence, we have to show that $\mathbf{m} \succcurlyeq \mathscr{M}^{\preccurlyeq}_{\rho(2)}(1,c_2)$. Since the effect of $\sigma^{K}_{\rho(1)}(c_1,c_2)$ is constant and equal to $\mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_2) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_1)$, we have:

$$\begin{array}{rll}
\mathbf{m} & = \mathscr{M}^{\preccurlyeq}_{\rho(n)}(x,c_1) + \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_2) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_1) & \\
\Rightarrow \mathbf{m} & \succcurlyeq \mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1) + \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_1) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1) & \\
 & \quad + \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_2) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_1) & \text{by (17)} \\
\Rightarrow \mathbf{m} & \succcurlyeq \mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1) + \mathscr{M}^{\preccurlyeq}_{\rho(1)}(K,c_2) & \\
\Rightarrow \mathbf{m} & \succcurlyeq \mathscr{M}^{\preccurlyeq}_{\rho(n)}(1,c_1) - \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_1) + \mathscr{M}^{\preccurlyeq}_{\rho(1)}(1,c_2) & \text{by (12)} \\
\Rightarrow \mathbf{m} & \succcurlyeq \mathscr{M}^{\preccurlyeq}_{\rho(2)}(1,c_2) & \text{by (16)}
\end{array}$$

We can thus fire $\sigma^{1}_{\rho(2)}(c_2,\cdot)$ from $\mathbf{m}$ and obtain $\mathbf{m}'$ such that $\mathbf{m}' \succcurlyeq \mathbf{n}_{\rho(2),1}$ (by monotonicity), which implies that $\mathbf{m}' \in \mathscr{U}$. Thus, $\mathscr{N}$ accepts $\Lambda(\sigma)$, which is of the form $B_{n_3} w^{i_1}_{n_3} w^{K'}_{n_1} w^{i_2}_{n_2} E_{n_2}$ with $n_1 = \rho(1)$, $n_2 = \rho(2)$ and $n_3 = \rho(n)$, $i_1 \geq 0$, and $i_2 \geq 0$. The former implies that $0 < n_1 < n_2 < n_3$, by definition of ρ. We finish the proof by considering two cases:

1. If $w_{n_1} = \varepsilon$, then clearly, for any $j \geq 0$: $w^{j}_{n_1} = \varepsilon$. In particular $w^{K}_{n_1} = \varepsilon = w^{K'}_{n_1}$. Thus, for any $j \geq 0$, the word $B_{n_3} w^{i_1}_{n_3} w^{j}_{n_1} w^{i_2}_{n_2} E_{n_2}$ satisfies the lemma.
2. If $w_{n_1} \neq \varepsilon$, it remains to show that the central part of the accepted word is long enough, i.e., that $K' \geq K$. This stems from the fact that, by construction, $K' = \|\Lambda(\sigma^{K}_{\rho(1)}(c_1,c_2))\|_{w_i}$ and that $\|\Lambda(\sigma^{K}_{\rho(1)}(c_1,c_2))\|_{w_i} \geq K$ by (10).

In both cases, we conclude that the word we have built, and that is accepted by the PN satisfies the lemma. □

4.3 A pumping lemma for PN+NBA

Let us turn our attention to the third pumping lemma. Its proof relies on the following auxiliary lemma:

Lemma 9 *Let $\mathcal{N} = \langle \mathcal{P}, \mathcal{T}, \Sigma, \mathbf{m}_0 \rangle$ be a* PN+NBA*, and let σ be a finite sequence of transitions of $\mathcal{N}$ that contains n occurrences of transitions in $\mathcal{T}_e$. Let $\mathbf{m}_1$, $\mathbf{m}'_1$, $\mathbf{m}_2$ and $\mathbf{m}'_2$ be four makings such that (i) $\mathbf{m}_1 \xrightarrow{\sigma} \mathbf{m}'_1$, (ii) $\mathbf{m}_2 \xrightarrow{\sigma} \mathbf{m}'_2$ and (iii) $\mathbf{m}_2 \succcurlyeq \mathbf{m}_1$. Then, for every place $p \in \mathcal{P}$: $\mathbf{m}'_2(p) - \mathbf{m}'_1(p) \geq \mathbf{m}_2(p) - \mathbf{m}_1(p) - n$.*

Proof Let us consider a place $p \in \mathcal{P}$. First, we remark that when we fire σ from $\mathbf{m}_2$ instead of $\mathbf{m}_1$, its Petri net arcs will have the same effect on p. On the other hand, since we want to find a lower bound on $\mathbf{m}'_2(p) - \mathbf{m}'_1(p)$, we consider the situation where no non-blocking arcs affect p when σ is fired from $\mathbf{m}_1$, but they all remove one token from p when σ is fired from $\mathbf{m}_2$. In the latter case, the effect of σ on p is $\mathbf{m}'_1(p) - \mathbf{m}_1(p) - n$. We obtain thus: $\mathbf{m}'_2(p) \geq \max\{\mathbf{m}_2(p) + \mathbf{m}'_1(p) - \mathbf{m}_1(p) - n, 0\}$. Hence $\mathbf{m}'_2(p) \geq \mathbf{m}'_1(p) + \mathbf{m}_2(p) - \mathbf{m}_1(p) - n$, and thus: $\mathbf{m}'_2(p) - \mathbf{m}'_1(p) \geq \mathbf{m}_2(p) - \mathbf{m}_1(p) - n$. □

We can now state our pumping lemma for PN+NBA:

Lemma 10 *Let $\mathcal{N}$ be a* PN+NBA *and $\mathcal{U}$ be an $\preccurlyeq$-upward-closed set of markings of $\mathcal{N}$.* **If** *there exists an infinite sequence of words $w_1, w_2, \ldots$ such that for any $i \geq 1$, there exist two words B_i, E_i with $B_i w_i^* E_i \subseteq L(\mathcal{N}, \mathcal{U})$,* **then** *there exist $i_1 \geq 0$, $i_2 > 0$, $i_3 \geq 0$ and $0 < n_1 < n_2 < n_3$ such that the word $B_{n_3} w_{n_3}^{i_1} w_{n_1}^{i_2} w_{n_2}^{i_3} E_{n_2}$ is in $L(\mathcal{N}, \mathcal{U})$.*

Once again, since the proof of Lemma 10 is rather technical, we first sketch it informally. The proof may be decomposed into two steps:

Step 1 We build an infinite sequence of runs whose $i-th$ element is a run that accepts the word $B_i w_i^{2^{|\mathcal{P}|}} E_i$ (where $\mathcal{P}$ is the set of places of the PN+NBA considered). Then, we build a sub-sequence of these runs by applying successively Lemma 3. Those sub-sequences have the property that markings appearing in different runs are $\preccurlyeq$-ordered. The increasing sequences appear along the $2^{|\mathcal{P}|} + 1$ first "columns".

Step 2 Finally, we show how to split and combine parts of runs appearing in the runs in order to obtain a new run that allows the PN+NBA to accept a word of the desired form.
In order to build this sequence, we rely on several variables, namely: c_1, c_2 and n. At the present step of the proof, we present several constraints on c_1, c_2 and n. These constraints are meant to produce a sequence of transitions that accepts a word of the desired form. The main (and most technical) part of step 2 consists to show that these constraints are satisfiable.

Proof Let $\mathcal{N}$ be a PN+NBA with set of places $\mathcal{P}$ and initial marking $\mathbf{m}_{init}$ such that $B_i w_i^* E_i \subseteq L(\mathcal{N}, \mathcal{U})$. For technical reason, we assume without loss of generality that $\mathcal{N}$ has a transition $t_\varepsilon = \langle \emptyset, \emptyset, \perp, \perp, 0, \varepsilon \rangle$, i.e. a transition labelled by ε that can be fired from any marking and has no effect.

$\boxed{\textbf{Step 1}}$ Since $B_i w_i^* E_i \subseteq L(\mathcal{N}, \mathcal{U})$ for all $i \geq 1$, the word $B_i w_i^{2^{|\mathcal{P}|}} E_i$ is accepted by $\mathcal{N}$. Let us consider the infinite sequence of runs that accept the words: $B_1 w_1^{2^{|\mathcal{P}|}} E_1$,

$B_2w_2^{2^{|\mathscr{P}|}}E_2,\ldots, B_jw_j^{2^{|\mathscr{P}|}}E_j,\ldots$, i.e.,

$$\begin{array}{l}
\mathbf{m}_{init} \xrightarrow{\upsilon_1} \mathbf{m}_1^1 \xrightarrow{\varsigma_1^1} \mathbf{m}_1^2 \xrightarrow{\varsigma_1^2} \cdots \xrightarrow{\varsigma_1^{2^{|\mathscr{P}|}}} \mathbf{m}_1^{2^{|\mathscr{P}|}+1} \xrightarrow{\upsilon_1'} \mathbf{n}_1\\
\mathbf{m}_{init} \xrightarrow{\upsilon_2} \mathbf{m}_2^1 \xrightarrow{\varsigma_2^1} \mathbf{m}_2^2 \xrightarrow{\varsigma_2^2} \cdots \xrightarrow{\varsigma_2^{2^{|\mathscr{P}|}}} \mathbf{m}_2^{2^{|\mathscr{P}|}+1} \xrightarrow{\upsilon_2'} \mathbf{n}_2\\
\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots\\
\mathbf{m}_{init} \xrightarrow{\upsilon_i} \mathbf{m}_i^1 \xrightarrow{\varsigma_i^1} \mathbf{m}_i^2 \xrightarrow{\varsigma_i^2} \cdots \xrightarrow{\varsigma_i^{2^{|\mathscr{P}|}}} \mathbf{m}_i^{2^{|\mathscr{P}|}+1} \xrightarrow{\upsilon_i'} \mathbf{n}_i\\
\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots
\end{array}$$

where for any $i \geq 1$: $\Lambda(\upsilon_i) = B_i$, $\Lambda(\upsilon_i') = E_i$, $\mathbf{n}_i \in \mathscr{U}$ and for any $1 \leq j \leq 2^{|\mathscr{P}|}$: $\Lambda(\varsigma_i^j) = w_i$. Remark that these executions exist even when $w_i = \varepsilon$, because $\mathscr{N}$ contains the t_ε transition.

By applying Lemma 3 successively, we can construct an infinite subsequence of that sequence:

$$\begin{array}{l}
\mathbf{m}_{init} \xrightarrow{\upsilon_{\rho(1)}} \mathbf{m}_{\rho(1)}^1 \xrightarrow{\varsigma_{\rho(1)}^1} \mathbf{m}_{\rho(1)}^2 \xrightarrow{\varsigma_{\rho(1)}^2} \cdots \xrightarrow{\varsigma_{\rho(1)}^{2^{|\mathscr{P}|}}} \mathbf{m}_{\rho(1)}^{2^{|\mathscr{P}|}+1} \xrightarrow{\upsilon_{\rho(1)}'} \mathbf{n}_{\rho(1)}\\
\mathbf{m}_{init} \xrightarrow{\upsilon_{\rho(2)}} \mathbf{m}_{\rho(2)}^1 \xrightarrow{\varsigma_{\rho(2)}^1} \mathbf{m}_{\rho(2)}^2 \xrightarrow{\varsigma_{\rho(2)}^2} \cdots \xrightarrow{\varsigma_{\rho(2)}^{2^{|\mathscr{P}|}}} \mathbf{m}_{\rho(2)}^{2^{|\mathscr{P}|}+1} \xrightarrow{\upsilon_{\rho(2)}'} \mathbf{n}_{\rho(2)}\\
\ldots
\end{array}$$

such that, for any $1 \leq j \leq 2^{|\mathscr{P}|}+1$, the sequence $\mathbf{m}_{\rho(1)}^j\mathbf{m}_{\rho(2)}^j\ldots$ is increasing:

$$\forall 1 \leq j \leq 2^{|\mathscr{P}|}+1 : \forall k \geq 1 : \mathbf{m}_{\rho(k)}^j \preccurlyeq \mathbf{m}_{\rho(k+1)}^j \tag{18}$$

and, for any $1 \leq j \leq 2^{|\mathscr{P}|}+1$ there exists a set of places, noted $Places(j)$ that strictly increase along the sequence $\mathbf{m}_{\rho(1)}^j\mathbf{m}_{\rho(2)}^j\ldots$ while the other places stay constant:

$$\forall 1 \leq j \leq 2^{|\mathscr{P}|}+1 : \forall k \geq 1 : \mathbf{m}_{\rho(k)}^j(p) < \mathbf{m}_{\rho(k+1)}^j(p) \text{ iff } p \in Places(j) \tag{19}$$

Since, there are $2^{|\mathscr{P}|}$ subsets of $\mathscr{P}$, there exist $1 \leq c_1 < c_2 \leq 2^{|\mathscr{P}|}+1$ such that $Places(c_1) = Places(c_2)$.

In the following, we denote by $\sigma_{\rho(j)}(k_1,k_2)$ with $k_1 < k_2$ the sequence $\varsigma_{\rho(j)}^{k_1} \cdot \ldots \cdot \varsigma_{\rho(j)}^{k_2-1}$. We also denote by $\sigma_{\rho(j)}(\cdot,k)$, the sequence $\upsilon_{\rho(j)} \cdot \varsigma_{\rho(j)}^1 \cdot \ldots \cdot \varsigma_{\rho(j)}^{k-1}$; and by $\sigma_{\rho(j)}(k,\cdot)$ the sequence $\varsigma_{\rho(j)}^k \cdot \ldots \cdot \varsigma_{\rho(j)}^{\rho(j)} \cdot \upsilon_{\rho(j)}'$

Step 2 The rest of the proof consists in devising a word of $L(\mathscr{N},\mathscr{U})$ that is of the form $B_{n_3}w_{n_3}^{i_1}w_{n_1}^{i_2}w_{n_2}^{i_3}E_{n_2}$, with $i_1 \geq 0$, $i_2 > 0$, $i_3 \geq 0$ and $0 < n_1 < n_2 < n_3$. The sequence of transitions that accepts this word (called σ) is built as follows:

$$\sigma = \sigma_{\rho(n)}(\cdot,c_1) \cdot \sigma_{\rho(1)}(c_1,c_2) \cdot \sigma_{\rho(2)}(c_2,\cdot)$$

for a well-chosen value of n. We next explain how to compute this value.

We choose $n > 2$ such that, when firing $\sigma_{\rho(1)}(c_1, c_2)$ from $\mathbf{m}^{c_1}_{\rho(n)}$, we reach a marking $\mathbf{m} \succcurlyeq \mathbf{m}^{c_2}_{\rho(2)}$. Let us show that such a n always exists. First, remark that for any $n > 2$: $\sigma_{\rho(1)}(c_1, c_2)$ is firable from $\mathbf{m}^{c_1}_{\rho(n)}$ since, by (18), $\mathbf{m}^{c_1}_{\rho(n)} \succcurlyeq \mathbf{m}^{c_1}_{\rho(1)}$. Let k be the number of non-blocking arcs in $\sigma_{\rho(1)}(c_1, c_2)$. By Lemma 9, we have that

$$\forall p \in \mathscr{P} : \mathbf{m}(p) \geq \mathbf{m}^{c_1}_{\rho(n)}(p) + \mathbf{m}^{c_2}_{\rho(1)}(p) - \mathbf{m}^{c_1}_{\rho(1)}(p) - k \tag{20}$$

But, since $Places(c_1) = Places(c_2)$, we can state the following. For any place $p \in Places(c_1)$ and for any $n \geq 1$: $\mathbf{m}^{c_1}_{\rho(n)}(p) \geq n - 1$, since by (19) the sequence $\mathbf{m}^{c_1}_{\rho(1)}(p), \mathbf{m}^{c_1}_{\rho(2)}(p), \ldots$ is strictly increasing. In particular, if we choose n such that

$$n > \max_{p \in Places(c_1)} \left(\mathbf{m}^{c_2}_{\rho(2)}(p) - \mathbf{m}^{c_2}_{\rho(1)}(p) + \mathbf{m}^{c_1}_{\rho(1)}(p) \right) + k$$

we have $\forall p \in Places(c_1) : \mathbf{m}^{c_1}_{\rho(n)}(p) \geq \mathbf{m}^{c_2}_{\rho(2)}(p) - \mathbf{m}^{c_2}_{\rho(1)}(p) + \mathbf{m}^{c_1}_{\rho(1)}(p) + k$ and thus:

$$\forall p \in Places(c_1) : \mathbf{m}^{c_1}_{\rho(n)}(p) + \mathbf{m}^{c_2}_{\rho(1)}(p) - \mathbf{m}^{c_1}_{\rho(1)}(p) - k \geq \mathbf{m}^{c_2}_{\rho(2)}(p) \tag{21}$$

By (20) and (21), we obtain:

$$\forall p \in Places(c_1) : \mathbf{m}(p) \geq \mathbf{m}^{c_2}_{\rho(2)}(p) \tag{22}$$

On the other hand, for any place p, the monotonicity property of $\mathsf{PN{+}NBA}$ implies that $\mathbf{m}(p) \geq \mathbf{m}^{c_2}_{\rho(1)}(p)$. And since, by (19): $\forall p \in \mathscr{P} \setminus Places(c_1) : \mathbf{m}^{c_2}_{\rho(1)}(p) = \mathbf{m}^{c_2}_{\rho(2)}(p)$, we obtain:

$$\forall p \in \mathscr{P} \setminus Places(c_1) : \mathbf{m}(p) \geq \mathbf{m}^{c_2}_{\rho(2)}(p) \tag{23}$$

By (22) and (23), we conclude that $\mathbf{m} \succcurlyeq \mathbf{m}^{c_2}_{\rho(2)}$.

Thus, the sequence of transitions $\sigma = \sigma_{\rho(n)}(\cdot, c_1) \cdot \sigma_{\rho(1)}(c_1, c_2) \cdot \sigma_{\rho(2)}(c_2, \cdot)$ is firable from $\mathbf{m}_{init}$ (with n computed as explained above) and leads to a marking $\mathbf{m}'$, i.e $\mathbf{m}_{init} \xrightarrow{\sigma} \mathbf{m}'$. Since $\mathbf{m} \succcurlyeq \mathbf{m}^{c_2}_{\rho(2)}$, we also have that $\mathbf{m}' \succcurlyeq \mathbf{n}_{\rho(2)}$, by monotonicity. Hence $\mathbf{m}' \in \mathscr{U}$, and the word $\Lambda(\sigma) \in L(\mathscr{N}, \mathscr{U})$. It is not difficult to see that by the previous construction this word is of the form: $B_{n_3} w^{i_1}_{n_3} w^{i_2}_{n_1} w^{i_3}_{n_2} E_{n_2}$ with *(i)* $n_3 = \rho(n)$, $n_1 = \rho(1)$ and $n_2 = \rho(2)$, hence $0 < n_1 < n_2 < n_3$, and *(ii)* $i_1 \geq 0$, $i_2 = c_2 - c_1 > 0$, $i_3 \geq 0$. $\square$

5 Properties of WSL

In this section, we apply the pumping lemmata of the previous section to obtain several results about WSL and languages of EPN. Section 5.1 presents properties of WSL that can be proved thanks to Lemma 6. Then, the pumping lemmata on PN and PN+NBA are exploited in sections 5.2 and 5.3 to prove a strict hierarchy among the languages of PN, PN+NBA and PN+T; as well as in section 5.4, to obtain closure properties of languages of EPN. Finally, section 5.5 shows that some of the results that have been obtained thanks to the pumping lemma on WSL can also be obtained thanks to the pumping lemmata on PN and PN+NBA.

5.1 Consequences of Lemma 6

We first study several classical languages and show that they are not well-structured. These languages are: the set of all words of the form $\mathtt{a}^n\mathtt{b}^n$, the set of all words of the form $\mathtt{a}^n\mathtt{b}^m$ with $m \geq n$, and the set of all palindromes.

- $\mathscr{L} = \{\mathtt{a}^n\mathtt{b}^n | n \geq 1\} \notin L^G(\mathsf{WSTS})$. Suppose that $\mathscr{L} \in L^G(\mathsf{WSTS})$. Since, $\forall k \geq 1 : \mathtt{a}^k\mathtt{b}^k \in \mathscr{L}$, we can apply Lemma 6 (letting $B_k = \mathtt{a}^k$ and $E_k = \mathtt{b}^k$, for any $k \geq 1$). We conclude that there is $i < j$ s.t. $\mathtt{a}^j\mathtt{b}^i \in \mathscr{L}$, which is a contradiction. Notice that this results is also a consequence of Theorem 2 and Theorem 1, following the reasoning given in [16, pages 175–176].
- $\mathscr{L}^{\geq} = \{\mathtt{a}^n\mathtt{b}^m | m \geq n\} \notin L^G(\mathsf{WSTS})$. The proof is similar to the previous one.
- $\mathscr{L}^R = \{w \cdot w^R\} \notin L^G(\mathsf{WSTS})$. Let Σ be an alphabet and $w = \mathtt{a}_1 \cdot \ldots \cdot \mathtt{a}_n \in \Sigma^*$, we define the mirror of w, as the word $w^R = \mathtt{a}_n \cdot \ldots \cdot \mathtt{a}_1$. Let us suppose $\mathscr{L}^R \in L^G(\mathsf{WSTS})$. Since $\{\mathtt{a}^n\mathtt{bba}^n \mid n \geq 0\} \subseteq \mathscr{L}^R$, we can apply Lemma 6 (letting $B_k = \mathtt{a}^k\mathtt{b}$ and $E_k = \mathtt{ba}^k$, for any $k \geq 1$). We conclude that there exist $i < j$ such that $\mathtt{a}^j\mathtt{bba}^i \in \mathscr{L}^R$, which is a contradiction. Hence $\mathscr{L}^R \notin L^G(\mathsf{WSTS})$.

These results allow us to show that neither the class of WSL, nor $L^G(\mathsf{PN})$, nor $L^G(\mathsf{PN{+}NBA})$, nor $L^G(\mathsf{PN{+}T})$ are closed under complement.

Proposition 4 *$L^G(\mathsf{WSTS}), L^G(\mathsf{PN}), L^G(\mathsf{PN{+}NBA})$ and $L^G(\mathsf{PN{+}T})$ are not closed under complement.*

Proof It is not difficult to devise a PN $\mathscr{N}$ and an $\preccurlyeq$-upward-closed set $\mathscr{U}$ such that $L(\mathscr{N}, \mathscr{U}) = \{\mathtt{a}^n\mathtt{b}^m \mid m < n\}$. It is well-known [16] that $L^G(\mathsf{PN})$ is closed under union and that the regular languages are all in $L^G(\mathsf{PN})$. Hence, $\{\mathtt{a}^n\mathtt{b}^m \mid m < n\} \cup \big((\mathtt{a}+\mathtt{b})^* \setminus \mathtt{a}^*\mathtt{b}^*\big)$ is in $L^G(\mathsf{PN})$, but also in $\mathsf{PN{+}NBA}$ and in $\mathsf{PN{+}T}$, since PN is a syntactic subclass of theirs. However, its complement is $\mathscr{L}^{\geq} = \{\mathtt{a}^n\mathtt{b}^m \mid m \geq n\}$, which is not a WSL (see above). □

Finally, we can also exploit the previous results to show that the class of WSL is incomparable to the class of Context Free Languages (C.F.L., for short).

Proposition 5 *The class $L^G(\mathsf{WSTS})$ is incomparable to the class of context-free languages.*

Proof C.F.L. $\not\subseteq L^G(\mathsf{WSTS})$ stems from the fact that $\mathscr{L}$, which is well-known to be a C.F.L., is not in $L^G(\mathsf{WSTS})$. We prove that $L^G(\mathsf{WSTS}) \not\subseteq$ C.F.L. thanks to $\mathscr{L}_1 = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i \geq j \geq k \geq 0\}$. It is not difficult to devise a PN that accepts $\mathscr{L}_1$ for some $\preccurlyeq$-upward-closed set. On the other hand, we prove that $\mathscr{L}_1$ is not a C.F.L. thanks to the classical pumping lemma for C.F.L.

For that purpose, we have to devise, for any constant $n \in \mathbb{N}$, a word $\omega_n \in \mathscr{L}_1$ such that $|\omega_n| \geq n$ and, for any words u, v, w, x and y respecting (i) $\omega = u \cdot v \cdot w \cdot x \cdot y$, (ii) $|v \cdot w \cdot x| \leq n$ and (iii) $|v \cdot x| > 0$, we can find $i \geq 0$ s.t. $u \cdot v^i \cdot w \cdot x^i \cdot y \notin \mathscr{L}_1$.

For any $n \geq 0$, we let $\omega_n = \mathtt{a}^n\mathtt{b}^n\mathtt{c}^n$. Clearly $\omega_n \in \mathscr{L}_1$ and $|\omega_n| \geq n$, for any n. Let us consider all the possible values of u, v,..., y that respect the three conditions above, and let us show that, for all these values, there exists an $i \geq 0$ such that $u \cdot v^i \cdot w \cdot x^i \cdot y \notin \mathscr{L}_1$.

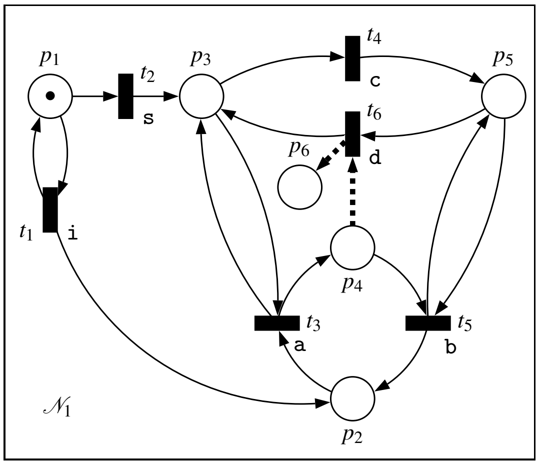

Fig. 5 The PN+NBA used in the proof of Theorem 4.

- *If either v or x contain at least two different characters*, the word $u\cdot v^2\cdot w\cdot x^2\cdot y$ is clearly not a word of $\mathscr{L}_1$.
- *If $v\in\mathtt{a}^*$*, then, since $|v\cdot w\cdot x|\leq n$, there are two possibilities. Either $x\in\mathtt{a}^*$. In that case, we choose $i=0$ and the word $u\cdot v^0\cdot w\cdot x^0\cdot y$ is of the form $\mathtt{a}^{n-|v\cdot x|}\mathtt{b}^n\mathtt{c}^n$, and is clearly not in $\mathscr{L}_1$, since $|v\cdot x|>0$. Otherwise, $x\in\mathtt{b}^*$. In that case, we choose $i=0$ again and we obtain a word of the form $\mathtt{a}^{n-|v|}\mathtt{b}^{n-|x|}\mathtt{c}^n$, which is not in $\mathscr{L}_1$ because $|v\cdot x|>0$.
- *Otherwise, i.e., $v\in\mathtt{b}^*$ or $v\in\mathtt{c}^*$*, we choose $i=2$, and the word $u\cdot v^2\cdot w\cdot x^2\cdot y$ contains either more $\mathtt{b}$'s or more $\mathtt{c}$'s than $\mathtt{a}$'s. Hence, it does not belong to $\mathscr{L}_1$. □

5.2 PN+NBA are more expressive than PN

In this section we prove that the class of languages accepted by PN+NBA strictly contains the class of languages accepted by PN (when the acceptance condition is an $\preccurlyeq$-upward-closed set). Since the class PN forms a syntactic subclass of PN+NBA, we obtain this result by showing that there is a language accepted by a PN+NBA that cannot be accepted by any PN.

Separation of PN+NBA *and* PN The strategy adopted in the proof is as follows. We look into the PN+NBA $\mathcal{N}_1$ of Fig. 5 with initial marking $\mathbf{m}_0$ such that $\mathbf{m}_0(p_1)=1$ and $\mathbf{m}_0(p)=0$ for $p\in\{p_2,p_3,p_4,p_5,p_6\}$, and prove it accepts every word of the form $\mathtt{i}^k\mathtt{s}\left(\mathtt{a}^k\mathtt{c}\mathtt{b}^k\mathtt{d}\right)^j$, for $k\geq 0$ and $j\geq 0$ (Lemma 11), but not those of the form $\mathtt{i}^{n_3}\mathtt{s}\mathtt{a}^{n_3}\mathtt{c}(\mathtt{b}^{n_3}\mathtt{d}\mathtt{a}^{n_3}\mathtt{c})^{i_1}\left(\mathtt{b}^{n_1}\mathtt{d}\mathtt{a}^{n_1}\mathtt{c}\right)^k\left(\mathtt{b}^{n_2}\mathtt{d}\mathtt{a}^{n_2}\mathtt{c}\right)^{i_2}\mathtt{b}^{n_2}\mathtt{d}$, for k big enough, and $0<n_1<n_2<n_3$ (Lemma 12). Then we invoke Lemma 8 (pumping lemma on PN) to prove that every PN accepting the words of the first form also accepts words of the latter, which implies that no PN accepts $L(\mathcal{N}_1,\mathbb{N}^6)$.

Lemma 11 *For any $k \geq 0$, for any $j \geq 0$, the word* $\mathtt{i}^k\mathtt{s}\left(\mathtt{a}^k\mathtt{c}\mathtt{b}^k\mathtt{d}\right)^j$ *is in* $L(\mathcal{N}_1, \mathbb{N}^6)$.

Proof Remark that, since the $\preccurlyeq$-upward-closed set considered here is $\mathbb{N}^6$, we just need to show that a sequence of transitions labelled by $\mathtt{i}^k\mathtt{s}\left(\mathtt{a}^k\mathtt{c}\mathtt{b}^k\mathtt{d}\right)^j$ is firable in $\mathcal{N}_1$ to get the Lemma.

The following holds for any $k \geq 0$. After firing the transitions $t_1^k t_2$ from the initial marking of $\mathcal{N}_1$, we reach the marking $\mathbf{m}_1$ such that $\mathbf{m}_1(p_2) = k$, $\mathbf{m}_1(p_3) = 1$, and $\mathbf{m}_1(p_j) = 0$ for $j \in \{1,4,5,6\}$. Then, we can fire $t_3^k t_4$ from $\mathbf{m}_1$. This leads to the marking $\mathbf{m}_2$ such that $\mathbf{m}_2(p_4) = k$, $\mathbf{m}_2(p_5) = 1$, and $\mathbf{m}_2(p_j) = 0$ for $j \in \{1,2,3,6\}$. From $\mathbf{m}_2$, t_5^k can be fired. This sequence of transitions moves the k tokens from p_4 to p_2. Then, from the resulting marking, t_6 can be fired. Since, p_4 is now empty, the effect of t_6 only consists in moving the token from p_5 to p_3 (its non-blocking arc has no effect) and we reach $\mathbf{m}_1$ again. Thus, the sequence of transitions $t_3^k t_4 t_5^k t_6$, labelled by $\mathtt{a}^k\mathtt{c}\mathtt{b}^k\mathtt{d}$, can be fired arbitrarily often from $\mathbf{m}_1$, and reaches the same marking. Hence the word $\mathtt{i}^k\mathtt{s}\left(\mathtt{a}^k\mathtt{c}\mathtt{b}^k\mathtt{d}\right)^j$ is in $L(\mathcal{N}_1, \mathbb{N}^6)$, for any $k \geq 0$, any $j \geq 0$. □

Lemma 12 *Let n_1, n_2 and n_3 be three natural numbers such that $0 < n_1 < n_2 < n_3$. The words*

$$\mathtt{i}^{n_3}\mathtt{s}\mathtt{a}^{n_3}\mathtt{c}\left(\mathtt{b}^{n_3}\mathtt{d}\mathtt{a}^{n_3}\mathtt{c}\right)^{i_1}\left(\mathtt{b}^{n_1}\mathtt{d}\mathtt{a}^{n_1}\mathtt{c}\right)^k\left(\mathtt{b}^{n_2}\mathtt{d}\mathtt{a}^{n_2}\mathtt{c}\right)^{i_2}\mathtt{b}^{n_2}\mathtt{d}$$

are not in $L(\mathcal{N}_1, \mathbb{N}^6)$, for all $i_1 \geq 0$, $k \geq n_3 - n_1$ and $i_2 \geq 0$.

Proof In this proof, we will identify a sequence of transitions with the word it accepts (all the transitions have different labels). Clearly (see the proof of Lemma 11), for any $n_3 \geq 0$, $m \geq 0$, the firing of $\mathtt{i}^{n_3}\mathtt{s}\left(\mathtt{a}^{n_3}\mathtt{c}\mathtt{b}^{n_3}\mathtt{d}\right)^m$ from $\mathbf{m}_0$ leads to a marking $\mathbf{m}_1$ such that $\mathbf{m}_1(p_2) = n_3$, $\mathbf{m}_1(p_3) = 1$, and $\forall i \in \{1,4,5,6\} : \mathbf{m}_1(p_i) = 0$ (the non-blocking arc of t_6 hasn't consumed any token in p_4). By firing $\mathtt{a}^{n_3}\mathtt{c}\mathtt{b}^{n_1}\mathtt{d}$ from $\mathbf{m}_1$, we now have n_1 tokens in p_2, $n_3 - n_1 - 1$ tokens in p_4 and one token in p_6 (this time the non-blocking arc has moved one token since $n_1 < n_3$). Clearly, at each subsequent firing of $\mathtt{a}^{n_1}\mathtt{c}\mathtt{b}^{n_1}\mathtt{d}$, the non-blocking arc of t_6 will remove one token from p_4 and the marking of this place will strictly decrease until p_4 becomes empty. Let $\ell = n_3 - n_1 - 1$. It is easy to see that that firing $\mathtt{a}^{n_3}\mathtt{c}\mathtt{b}^{n_1}\mathtt{d}\left(\mathtt{a}^{n_1}\mathtt{c}\mathtt{b}^{n_1}\mathtt{d}\right)^\ell$ from $\mathbf{m}_1$ leads to a marking $\mathbf{m}_2$ with $\mathbf{m}_2(p_2) = n_1$, $\mathbf{m}_2(p_3) = 1$, $\mathbf{m}_2(p_6) = n_3 - n_1$ and $\forall j \in \{1,4,5\} : \mathbf{m}_2(p_j) = 0$. This characterization also implies that we can fire $\mathtt{a}^{n_1}\mathtt{c}\mathtt{b}^{n_1}\mathtt{d}$ an arbitrary number of times from $\mathbf{m}_2$ because $\mathbf{m}_2 \xrightarrow{\mathtt{a}^{n_1}\mathtt{c}\mathtt{b}^{n_1}\mathtt{d}} \mathbf{m}_2$. On the other hand, it is not possible to fire $\mathtt{a}^{n_1}\mathtt{c}\mathtt{b}^{n_2}\mathtt{d}$, with $n_2 > n_1$, from $\mathbf{m}_2$. Indeed $\mathbf{m}_2 \xrightarrow{\mathtt{a}^{n_1}\mathtt{c}\mathtt{b}^{n_1}} \mathbf{m}_3$, with $\mathbf{m}_3(p_5) = 1$, $\mathbf{m}_3(p_2) = n_1$, $\mathbf{m}_3(p_6) = n_3 - n_1$ and $\forall j \in \{1,3,4\} : \mathbf{m}_3(p_j) = 0$, which does not allow to fire the $\mathtt{b}$-labelled transition t_5 anymore. We conclude that, $\forall k \geq n_3 - n_1$, a sequence labelled by $\mathtt{i}^{n_3}\mathtt{s}\left(\mathtt{a}^{n_3}\mathtt{c}\mathtt{b}^{n_3}\mathtt{d}\right)^m\mathtt{a}^{n_3}\mathtt{c}\left(\mathtt{b}^{n_1}\mathtt{d}\mathtt{a}^{n_1}\mathtt{c}\right)^k\mathtt{b}^{n_2}\mathtt{d}\mathtt{a}^{n_2}\mathtt{c}$, is not firable in $\mathcal{N}_1$. Thus, we will not find in $L(\mathcal{N}_1, \mathbb{N}^6)$ any word with this prefix, hence the Lemma. □

Thanks to these lemmata, we can prove Proposition 6.

Proposition 6 *There is no* PN $\mathcal{N}$ *with an $\preccurlyeq$-upward-closed set $\mathcal{U}$ such that $L(\mathcal{N}, \mathcal{U}) = L(\mathcal{N}_1, \mathbb{N}^6)$.*

Proof By Lemma 11, any PN $\mathcal{N}$ s.t. $L(\mathcal{N},\mathcal{U}) = L(\mathcal{N}_1,\mathbb{N}^6)$ for some $\preccurlyeq$-upward-closed set of accepting markings $\mathcal{U}$, must accept $\mathtt{i}^k\mathtt{s}\left(\mathtt{a}^k\mathtt{c}\mathtt{b}^k\mathtt{d}\right)^j$, for any $k \geq 1$ and $j \geq 0$. Hence, we can apply Lemma 8, by letting $B_k = \mathtt{i}^k\mathtt{s}\mathtt{a}^k\mathtt{c}$, $E_k = \mathtt{b}^k\mathtt{d}$ and $w_k = \mathtt{b}^k\mathtt{d}\mathtt{a}^k\mathtt{c}$, for any $k \geq 1$. We conclude that $\mathcal{N}$ also accepts a word of the form: $\mathtt{i}^{n_3}\mathtt{s}\mathtt{a}^{n_3}\mathtt{c}\left(\mathtt{b}^{n_3}\mathtt{d}\mathtt{a}^{n_3}\mathtt{c}\right)^{i_1}\left(\mathtt{b}^{n_1}\mathtt{d}\mathtt{a}^{n_1}\mathtt{c}\right)^{L'}\left(\mathtt{b}^{n_2}\mathtt{d}\mathtt{a}^{n_2}\mathtt{c}\right)^{i_2}\mathtt{b}^{n_2}\mathtt{d}$ such that $0 < n_1 < n_2 < n_3$ and $L' \geq n_3 - n_1$. Since it is not in $L(\mathcal{N}_1,\mathcal{U})$, by Lemma 12, there is no PN $\mathcal{N}$ and no $\preccurlyeq$-upward-closed set $\mathcal{U}$ s.t. $L(\mathcal{N},\mathcal{U}) = L(\mathcal{N}_1,\mathbb{N}^6)$. □

Thus, we conclude that:

Theorem 4 $L^G(\mathsf{PN}) \subset L^G(\mathsf{PN{+}NBA})$.

Proof $L^G(\mathsf{PN}) \subseteq L^G(\mathsf{PN{+}NBA})$ is trivial since PN is a syntactic subclass of PN+NBA. The strictness of the inclusion is given by Proposition 6. □

5.3 PN+T are more expressive than PN+NBA

Let us now prove a similar result about the classes PN+NBA and PN+T: the class of languages that can be accepted by some PN+T strictly contains the class of languages accepted by any given PN+NBA. For this purpose, we first show that a PN+T can always *simulate* a PN+NBA, hence $L^G(\mathsf{PN{+}NBA}) \subseteq L^G(\mathsf{PN{+}T})$. Then, we prove, thanks to Lemma 10, that there is a language that can be recognized by a PN+T, but not by a PN+NBA, which implies the strictness of the inclusion.

Simulation of a PN+NBA *by a* PN+T Lemma 13 below states that any PN+NBA can be simulated by a PN+T. The proof of this lemma is based on the following construction. Let us consider a PN+NBA $\mathcal{N} = \langle \mathcal{P}, \mathcal{T}, \Sigma, \mathbf{m}_0 \rangle$, and an $\preccurlyeq$-upward-closed set $\mathcal{U}$ of markings, and let us show how to transform them into a PN+T $\mathcal{N}'$ and an $\preccurlyeq$-upward-closed set $\mathcal{U}'$ such that $L(\mathcal{N},\mathcal{U}) = L(\mathcal{N}',\mathcal{U}')$.

Let us consider the partition of $\mathcal{T}$ into $\mathcal{T}_e$ and $\mathcal{T}_r$ as defined in Section 2, and a new place p_{Tr} (the trash place). We show now how to build $\mathcal{N}' = \langle \mathcal{P}', \mathcal{T}', \Sigma, \mathbf{m}'_0 \rangle$ and $\mathcal{U}'$. First, $\mathcal{P}' = \mathcal{P} \cup \{p_{Tr}\}$. For each transition $t = \langle I, O, s, d, 1, \lambda \rangle$ in $\mathcal{T}_e$, we put in $\mathcal{T}'$: $t_l = \langle I, O, s, p_{Tr}, +\infty, \lambda \rangle$ and $t_e = \langle I_e, O_e, \perp, \perp, 0, \lambda \rangle$, two new transitions, such that: $\forall p \in \mathcal{P} : \left(p \neq s \Rightarrow I_e(p) = I(p) \wedge p \neq d \Rightarrow O_e(p) = O(p)\right)$, $I_e(s) = I(s) + 1$ and $O_e(d) = O(d) + 1$. We also add into $\mathcal{T}'$ all the transitions of $\mathcal{T}_r$ (extended to p_{Tr} such that they have no guard and no effect on p_{Tr}). Finally, $\forall p \in \mathcal{P} = \mathbf{m}'_0(p) = \mathbf{m}_0(p)$, $\mathbf{m}'_0(p_{Tr}) = 0$ and $\mathcal{U}' = \{\mathbf{m} \mid \exists \mathbf{m}' \in \mathcal{U} : \forall p \in \mathcal{P} : \mathbf{m}(p) = \mathbf{m}'(p)\}$.

Example 2 Fig. 6 illustrates the above construction. ◇

Lemma 13 *For any* PN+NBA $\mathcal{N}$ *with an* $\preccurlyeq$*-upward-closed set* $\mathcal{U}$*, it is possible to construct a* PN+T $\mathcal{N}'$ *and an* $\preccurlyeq$*-upward closed set* $\mathcal{U}'$ *s.t.:* $L(\mathcal{N},\mathcal{U}) = L(\mathcal{N}',\mathcal{U}')$.

Proof Let us consider the previous construction and let us prove that $L(\mathscr{N},\mathscr{U}) = L(\mathscr{N}',\mathscr{U}')$.

$\boxed{L(\mathscr{N},\mathscr{U}) \subseteq L(\mathscr{N}',\mathscr{U}')}$ We show that, for every sequence of transitions σ of $\mathscr{N}$ that leads into a marking $\mathbf{m} \in \mathscr{U}$, we can find a sequence of transitions σ' of $\mathscr{N}'$ that leads into a marking $\mathbf{m}' \in \mathscr{U}'$ such that $\Lambda(\sigma) = \Lambda(\sigma')$.

Let us define the function $f : \mathscr{T} \times \mathbb{N}^{|\mathscr{P}|} \to \mathscr{T}'$ such that $\forall t \in \mathscr{T}_r : f(t,\mathbf{m}) = t$ and $\forall t = \langle O, I, s, d, 1, \lambda\rangle \in \mathscr{T}_e : f(t,\mathbf{m}) = t_e$, if $\mathbf{m}(s) > I(s)$ (the non-blocking arc still has an effect after the firing of the Petri part of the transition); and $f(t,\mathbf{m}) = t_l$, otherwise.

Let $\sigma = \mathbf{m}_0 \xrightarrow{t_1} \mathbf{m}_1 \xrightarrow{t_2} \ldots \xrightarrow{t_n} \mathbf{m}_n \xrightarrow{t_{n+1}} \mathbf{m}_{n+1}$ be a sequence of $\mathscr{N}$ such that $\mathbf{m}_{n+1} \in \mathscr{U}$. Then we may see that $\sigma' = \mathbf{m}'_0 \xrightarrow{f(t_1,\mathbf{m}'_0)} \mathbf{m}'_1 \xrightarrow{f(t_2,\mathbf{m}'_1)} \ldots \xrightarrow{f(t_n,\mathbf{m}'_{n-1})} \mathbf{m}'_n \xrightarrow{f(t_{n+1},\mathbf{m}'_n)} \mathbf{m}'_{n+1}$ is a sequence of $\mathscr{N}'$, where $\forall 1 \le i \le n+1 : \mathbf{m}'_i$ is such that $\mathbf{m}'_i(p) = \mathbf{m}_i(p)$ for all $p \in \mathscr{P}$ and $\mathbf{m}'_i(p_{Tr}) = 0$. Hence, $\mathbf{m}'_{n+1} \in \mathscr{U}'$ and $\Lambda(\sigma')$ is accepted. Since we have $\forall 1 \le i \le n+1 : \Lambda(t_i) = \Lambda(f(t_i,\mathbf{m}_{i-1}))$, we conclude that $\Lambda(\sigma) = \Lambda(\sigma')$, hence $L(\mathscr{N},\mathscr{U}) \subseteq L(\mathscr{N}',\mathscr{U}')$.

$\boxed{L(\mathscr{N}',\mathscr{U}') \subseteq L(\mathscr{N},\mathscr{U})}$ We show that, for every sequence of transitions σ' of $\mathscr{N}'$ that leads into a marking $\mathbf{m}' \in \mathscr{U}'$, we can find a sequence of transitions σ of $\mathscr{N}$ that leads into a marking $\mathbf{m} \in \mathscr{U}$ such that $\Lambda(\sigma') = \Lambda(\sigma)$.

We define the function $g : \mathscr{T}' \to \mathscr{T}$ such that for all $t \in \mathscr{T}_r$: $g(t) = t$ and for all $t \in \mathscr{T}_e : g(t_e) = g(t_l) = t$. Moreover, we define the relation $\preccurlyeq_{\mathscr{P}}$ that compares two markings only on the places that are in $\mathscr{P}$. Thus, if $\mathbf{m}$ is defined on $\mathscr{P}$ and $\mathbf{m}'$ on $\mathscr{P}'$ (remember that $\mathscr{P} \subseteq \mathscr{P}'$), $\mathbf{m}' \preccurlyeq_{\mathscr{P}} \mathbf{m}$ iff $\forall p \in \mathscr{P} : \mathbf{m}'(p) \le \mathbf{m}(p)$.

Let $\sigma' = \mathbf{m}'_0 \xrightarrow{t_1} \mathbf{m}'_1 \xrightarrow{t_2} \ldots \xrightarrow{t_n} \mathbf{m}'_n \xrightarrow{t_{n+1}} \mathbf{m}'_{n+1}$ be a sequence of $\mathscr{N}'$ such that $\mathbf{m}'_{n+1} \in \mathscr{U}'$. Then, there exist $\mathbf{m}_1, \mathbf{m}_2, \ldots \mathbf{m}_{n+1}$ in $\mathscr{N}$ such that we have $\mathbf{m}_0 \xrightarrow{g(t_1)} \mathbf{m}_1 \xrightarrow{g(t_2)} \ldots \xrightarrow{g(t_n)} \mathbf{m}_n \xrightarrow{g(t_{n+1})} \mathbf{m}_{n+1}$ and $\mathbf{m}_{n+1} \in \mathscr{U}$. To prove that the sequence of markings exists, we show by induction on the indexes, that $\mathbf{m}'_i \preccurlyeq_{\mathscr{P}} \mathbf{m}_i$ for all i such that $0 \le i \le n+1$. That implies that $\forall 1 \le i \le n+1 : g(t_i)$ is firable from $\mathbf{m}_{i-1}$ because $g(t_i)$ consumes no more tokens in any place p than t_i does.

Base case: $j = 0$. The base case is trivially verified.

Induction step: $j = k$. By induction hypothesis, we have: $\forall 0 \le j \le k-1 : \mathbf{m}'_j \preccurlyeq_{\mathscr{P}} \mathbf{m}_j$. In the case where $t_k = \langle I, O, s, d, b, \lambda\rangle$ (from $\mathbf{m}'_{k-1}$) has the same effect on $\mathscr{P}$ than $g(t_k)$ (from $\mathbf{m}_{k-1}$), we directly have that $\mathbf{m}'_k \preccurlyeq_{\mathscr{P}} \mathbf{m}_k$. This happens if t_k is a regular Petri transition or if $\mathbf{m}_{k-1}(s) = \mathbf{m}'_{k-1}(s) = I(s)$.

Otherwise t_k has a transfer arc and we must consider two cases:

- The transfer of t_k has no effect and the non-blocking arc of $g(t_k)$ moves one token from the source s to the target d, hence $I(s) = \mathbf{m}'_{k-1}(s) < \mathbf{m}_{k-1}(s)$. Since t_k and $g(t_k)$ have the same effect except that $g(t_k)$ removes one more token from s and adds one more token in d, and since $\mathbf{m}'_{k-1} \preccurlyeq_{\mathscr{P}} \mathbf{m}_{k-1}$ with $\mathbf{m}'_{k-1}(s) < \mathbf{m}_{k-1}(s)$, we conclude that $\mathbf{m}'_k \preccurlyeq_{\mathscr{P}} \mathbf{m}_k$.
- The transfer of t_k moves at least one token from the source s to p_{Tr} and the non-blocking arc of $g(t_k)$ moves one token from s to d. Since t_k and $g(t_k)$

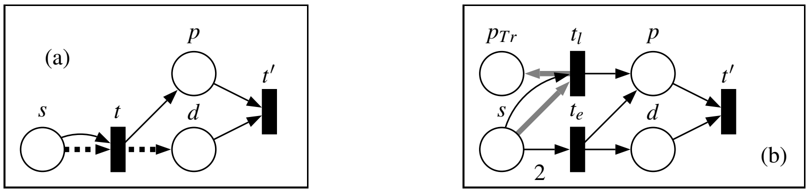

Fig. 6 A PN+NBA $\mathscr{N}$ (a) and the corresponding PN+T $\mathscr{N}'$ (b)

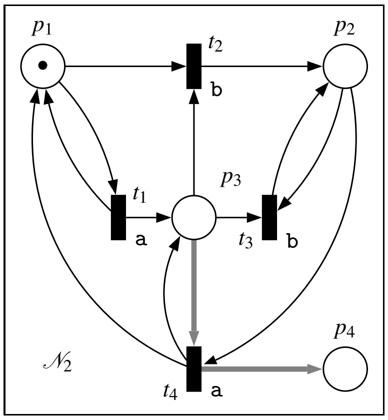

Fig. 7 The PN+T used in the proof of Theorem 5.

have the same effect on the places in $\mathscr{P}$ except that $g(t_k)$ adds one more token in d and t_k may remove more tokens from s, and since $\mathbf{m}'_{k-1} \preccurlyeq_{\mathscr{P}} \mathbf{m}_{k-1}$, we conclude that $\mathbf{m}'_k \preccurlyeq_{\mathscr{P}} \mathbf{m}_k$.

Thus, there are $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_{n+1}$ s.t. $\mathbf{m}_0 \xrightarrow{g(t_1)} \mathbf{m}_1 \xrightarrow{g(t_2)} \ldots \xrightarrow{g(t_n)} \mathbf{m}_n \xrightarrow{g(t_{n+1})} \mathbf{m}_{n+1}$ in $\mathscr{N}$ and $\forall 1 \le i \le n+1 : \mathbf{m}'_i \preccurlyeq_{\mathscr{P}} \mathbf{m}_i$. Thus, $\mathbf{m}_{n+1} \in \mathscr{U}$. Since $\Lambda(t_i) = \Lambda(g(t_i))$ for all $1 \le i \le n+1$, we conclude that $\Lambda(\sigma') = \Lambda(\sigma)$, hence $L(\mathscr{N}', \mathscr{U}') \subseteq L(\mathscr{N}, \mathscr{U})$. $\square$

Separation of PN+T *and* PN+NBA Let us now prove that $L^G(\mathsf{PN{+}NBA})$ is *strictly* included in $L^G(\mathsf{PN{+}T})$. We consider the PN+T $\mathscr{N}_2$ presented in Fig.7 with the initial marking $\mathbf{m}_0(p_1) = 1$ and $\mathbf{m}_0(p) = 0$ for $p \in \{p_2, p_3, p_4\}$. The two following Lemmata allow us to better understand the behaviour of $\mathscr{N}_2$.

Lemma 14 *For any $k \ge 1$, for any $j \ge 0$, the word $\left(\mathtt{a}^k\mathtt{b}^k\right)^j$ is in $L(\mathscr{N}_2, \mathbb{N}^4)$.*

Proof Remark that, since the $\preccurlyeq$-upward-closed set considered here is $\mathbb{N}^4$, we just need to show that a sequence of transitions labelled by $\left(\mathtt{a}^k\mathtt{b}^k\right)^j$ $(j \ge 0)$ is firable in $\mathscr{N}_2$ to get the lemma.

The following holds for any $k \ge 1$. From the initial marking $\mathbf{m}_0$ of $\mathscr{N}_2$, we can fire $t_1^k t_2 t_3^{k-1}$ (which is labelled by $\mathtt{a}^k\mathtt{b}^k$), and obtain the marking $\mathbf{m}_1$ such that

$\mathbf{m}_1(p_2) = 1$ and $\forall p \in \{p_1, p_3, p_4\} : \mathbf{m}_1(p) = 0$. Thus, t_4 is firable from $\mathbf{m}_1$ and does not transfer any token, but produces a token in p_3 and moves the token from p_2 to p_1. It is thus not difficult to see that $t_4 t_1^{k-1} t_2 t_3^{k-1}$, labelled by $\mathtt{a}^k\mathtt{b}^k$, can be fired from $\mathbf{m}_1$. The marking one obtains is $\mathbf{m}_1$ again. Hence, we can fire a sequence labelled by $\mathtt{a}^k\mathtt{b}^k$ arbitrarily often from $\mathbf{m}_1$. Thus, any word of the form $\left(\mathtt{a}^k\mathtt{b}^k\right)^j$ is in $L(\mathcal{N}_2, \mathbb{N}^4)$. □

Lemma 15 *Let n_1, n_2, n_3 be three natural numbers such that $0 < n_1 < n_2 < n_3$. For any $i_1 \geq 0$, $i_2 > 0$ and $i_3 \geq 0$, the words of the form:*

$$\mathtt{a}^{n_3}(\mathtt{b}^{n_3}\mathtt{a}^{n_3})^{i_1}(\mathtt{b}^{n_1}\mathtt{a}^{n_1})^{i_2}(\mathtt{b}^{n_2}\mathtt{a}^{n_2})^{i_3}\mathtt{b}^{n_2}$$

are not in $L(\mathcal{N}_2, \mathbb{N}^4)$.

Proof The following holds for any n_1, n_2, n_3 with $0 < n_1 < n_2 < n_3$. From the initial marking of $\mathcal{N}_2$, the only sequence of transitions labelled by $\mathtt{a}^{n_3}$ is $t_1^{n_3}$. Firing this sequence leads to the marking $\mathbf{m}_1$ such that $\mathbf{m}_1(p_1) = 1, \mathbf{m}_1(p_3) = n_3$ and $\mathbf{m}_1(p) = 0$ if $p \in \{p_2, p_4\}$. From $\mathbf{m}_1$ the only firable sequence of transitions labelled by $\mathtt{b}^{n_3}$ is $t_2 t_3^{n_3-1}$. This leads to the marking $\mathbf{m}_2$ such that $\mathbf{m}_2(p_2) = 1$ and $\mathbf{m}_2(p) = 0$ if $p \neq p_2$. The only sequence of transitions firable from $\mathbf{m}_2$ and labelled by $\mathtt{a}^{n_3}$ is $t_4 t_1^{n_3-1}$. Since $\mathbf{m}_2(p_3) = 0$, the transfer of t_4 has no effect when fired from $\mathbf{m}_2$. Hence, we reach $\mathbf{m}_1$ again after firing $t_4 t_1^{n_3-1}$. By repeating the reasoning, we conclude that the only sequence of transitions firable from the initial marking and labelled by $(\mathtt{a}^{n_3}\mathtt{b}^{n_3})^{i_1}\mathtt{a}^{n_3}$ (when $i_1 > 0$) is $t_1^{n_3} t_2 t_3^{n_3-1}(t_4 t_1^{n_3-1} t_2 t_3^{n_3-1})^{i_1-1} t_4 t_1^{n_3-1}$ and leads to $\mathbf{m}_1$. In the case where $i_1 = 0$, the sequence $t_1^{n_3}$ is firable and leads to $\mathbf{m}_1$ too. From $\mathbf{m}_1$, the only firable sequence of transitions labelled by $\mathtt{b}^{n_1}$ is $t_2 t_3^{n_1-1}$. This leads to a marking similar to $\mathbf{m}_2$, noted $\mathbf{m}_2'$, except that p_3 contains $n_3 - n_1$ tokens. Then, the only firable sequence of transitions labelled by $\mathtt{a}^{n_1}$ is $t_4 t_1^{n_1-1}$. In this case, the transfer of t_4 moves the $n_3 - n_1$ tokens from p_3 to p_4 and we reach a marking similar to $\mathbf{m}_1$, noted $\mathbf{m}_1'$, except that p_4 contains $n_3 - n_1$ tokens and p_3 contains n_1 tokens. From $\mathbf{m}_1'$, the only firable sequence of transitions labelled by $\mathtt{b}^{n_1}\mathtt{a}^{n_1}$ is $t_2 t_3^{n_1-1} t_4 t_1^{n_1-1}$ and leads to $\mathbf{m}_1'$. Hence, the sequence $(t_2 t_3^{n_1-1} t_4 t_1^{n_1-1})^{i_2}$ is firable from $\mathbf{m}_1'$.

However, after firing $t_2 t_3^{n_1-1}$ from $\mathbf{m}_1'$, we reach a marking $\mathbf{m}_2''$ similar to $\mathbf{m}_2$ except that p_4 contains $n_3 - n_1$ tokens and from which no transition labelled by $\mathtt{b}$ is firable. Since $n_2 > n_1$, we conclude that there is no sequence of transitions labelled by $\mathtt{b}^{n_2}$ that is firable from $\mathbf{m}_1'$, hence $\mathtt{a}^{n_3}(\mathtt{b}^{n_3}\mathtt{a}^{n_3})^{i_1}(\mathtt{b}^{n_1}\mathtt{a}^{n_1})^{i_2}(\mathtt{b}^{n_2}\mathtt{a}^{n_2})^{i_3}\mathtt{a}^{n_2}$ with $i_1 \geq 0, i_2 > 0, i_3 \geq 0$ is not in $L(\mathcal{N}_2, \mathbb{N}^4)$. □

Thanks to these two lemmata, and thanks to Lemma 10, we can now prove Proposition 7, that states that no PN+NBA can accept the language of $\mathcal{N}_2$.

Proposition 7 *There is no* PN+NBA *with an $\preccurlyeq$-upward-closed set $\mathcal{U}$ such that $L(\mathcal{N}, \mathcal{U}) = L(\mathcal{N}_2, \mathbb{N}^4)$.*

Proof By Lemma 14, any PN+NBA $\mathcal{N}$ s.t. $L(\mathcal{N}, \mathcal{U}) = L(\mathcal{N}_2, \mathbb{N}^4)$ for some $\preccurlyeq$-upward-closed set $\mathcal{U}$, accepts $\left(\mathtt{a}^j\mathtt{b}^j\right)^k$, for any $j \geq 1, k \geq 1$. Thus, we can apply Lemma 10, by letting $B_i = \mathtt{a}^i$, $E_i = \mathtt{b}^i$ and $w_i = \mathtt{b}^i\mathtt{a}^i$, for all $i \geq 1$, and obtain that

$\mathcal{N}$ accepts a word of the form: $\mathtt{a}^{n_3}(\mathtt{b}^{n_3}\mathtt{a}^{n_3})^{i_1}(\mathtt{b}^{n_1}\mathtt{a}^{n_1})^{i_2}(\mathtt{b}^{n_2}\mathtt{a}^{n_2})^{i_3}\mathtt{b}^{n_2}$ with $0 < n_1 < n_2 < n_3$ and $i_2 > 0$. Since, by Lemma 15, this word is not in $L(\mathcal{N}_2, \mathbb{N}^4)$, there can be no PN+NBA $\mathcal{N}$ and no $\preccurlyeq$-upward-closed-set $\mathcal{U}$ s.t.: $L(\mathcal{N}, \mathcal{U}) = L(\mathcal{N}_2, \mathbb{N}^4)$. □

The two last propositions allow us to conclude that:

Theorem 5 $L^G(\mathsf{PN{+}NBA}) \subset L^G(\mathsf{PN{+}T})$

Proof $L^G(\mathsf{PN{+}NBA}) \subseteq L^G(\mathsf{PN{+}T})$ is given by Lemma 13. The strictness of the inclusion is given by Proposition 7. □

5.4 Closure Properties of EPN

The pumping lemmata on PN and PN+NBA can also be used to show that neither $L^G(\mathsf{PN})$ nor $L^G(\mathsf{PN{+}NBA})$ are closed under iteration.

Theorem 6 $L^G(\mathsf{PN})$ *and* $L^G(\mathsf{PN{+}NBA})$ *are not closed under iteration.*

Proof It is easy to show that $L = \{\mathtt{a}^n\mathtt{b}^m | n \geq m\} \in L^G(\mathsf{PN})$ (hence, L is also in $L^G(\mathsf{PN{+}NBA})$). Let us show, by contradiction, that $L^+ \notin L^G(\mathsf{PN})$. Suppose that there is a PN $\mathcal{N}$ and an upward-closed set $\mathcal{U}$ s.t. $L(\mathcal{N}, \mathcal{U}) = L^+$. Let $B_i = \mathtt{a}^i$, $w_i = \mathtt{b}^i\mathtt{a}^i$ and $E_i = \mathtt{b}_i$ for all $i \geq 1$. Thanks to Lemma 8, we obtain that $L(\mathcal{N}, \mathcal{U})$ contains a word of the form:

$$\mathtt{a}^{n_3}(\mathtt{b}^{n_3}\mathtt{a}^{n_3})^{i_1}(\mathtt{b}^{n_1}\mathtt{a}^{n_1})^{K}(\mathtt{b}^{n_2}\mathtt{a}^{n_2})^{i_2}\mathtt{b}^{n_2}$$

with $n_1 < n_2 < n_3$, $K \geq 1$, which is not in L^+. Hence the contradiction. A similar proof for PN+NBA invokes Lemma 10. □

In [16], Peterson proves that $L^L(\mathsf{PN})$ is not closed under iteration, but does not treat the case of $L^G(\mathsf{PN})$ (which we have solved here) and mentions the case of $L^P(\mathsf{PN})$ as an open problem (see page 186 of [16]). It is possible to adapt the proof of Theorem 6 to show that $L^P(\mathsf{PN})$ is not closed under iteration. Indeed, L is also in $L^P(\mathsf{PN})$. Then, suppose that there exists a PN $\mathcal{N}$ with set of places $\mathcal{P}$ s.t. $L(\mathcal{N}, \mathbb{N}^{|\mathcal{P}|}) = L^+$. Since $L(\mathcal{N}, \mathbb{N}^{|\mathcal{P}|}) \in L^P(\mathsf{PN}) \subseteq L^G(\mathsf{PN})$, we can apply Lemma 8, and conclude that $L(\mathcal{N}, \mathbb{N}^{|\mathcal{P}|})$ too contains a word that is not in L^+. Hence $L^P(\mathsf{PN})$ is not closed under iteration.

Following Definition 1, Theorem 6 allows us to deduce that:

Corollary 2 $L^G(\mathsf{PN})$ *and* $L^G(\mathsf{PN{+}NBA})$ *are not full* AFL.

On the other hand, it is easy to show that:

Theorem 7 $L^G(\mathsf{PN{+}T})$ *is a full* AFL*, closed under intersection.*

Proof We consider two PN+T $\mathcal{N}_1 = \langle P_1, T_1, \Sigma_1, \mathbf{m}_0^1\rangle$ and $\mathcal{N}_2 = \langle P_2, T_2, \Sigma_2, \mathbf{m}_0^2\rangle$ and two upward-closed sets $\mathcal{U}_1$ and $\mathcal{U}_2$, and we assume that the set of places and transitions of this two nets are disjoint. For each property to prove we show how to build an upward-closed set $\mathcal{U}$ and a PN+T $\mathcal{N} = \langle P, T, \Sigma, \mathbf{m}_0\rangle$ s.t. $L(\mathcal{N}, \mathcal{U})$ is the desired language. Since the proofs that $\mathcal{N}$ accepts the right language are quite

immediate, we do not provide them here. We rather report the main ideas of the construction which should be clear enough to convince the reader.
Union: $L(\mathscr{N}_1,\mathscr{U}_1)\cup L(\mathscr{N}_2,\mathscr{U}_2)\in L^G(\mathsf{PN+T})$. We build $\mathscr{N}$ as follows. $P=P_1\uplus P_2\uplus\{p_{init},p_1,p_2\}$. For each transition $t=\langle I,O,s,d,b,\lambda\rangle\in T_1$, we put in T a transition $t'=\langle I\cup\{p_1\},O\cup\{p_1\},s,d,b,\lambda\rangle$. Symmetrically, for each transition $t=\langle I,O,s,d,b,\lambda\rangle\in T_2$, we put in T a transition $t'=\langle I\cup\{p_2\},O\cup\{p_2\},s,d,b,\lambda\rangle$. We also add to T the two following transitions: $t_1=\langle\{p_{init}\},O_1,\perp,\perp,0,\varepsilon\rangle$ and $t_2=\langle\{p_{init}\},O_2,\perp,\perp,0,\varepsilon\rangle$ where $O_1(p)=\mathbf{m}_0^1(p)$ for all $p\in P_1$, $O_1(p_1)=1$ and $O_1(p)=0$ for all $p\in P_2\cup\{p_2\}$; and $O_2(p)=\mathbf{m}_0^2(p)$ for all $p\in P_2$, $O_2(p_2)=1$ and $O_2(p)=0$ for all $p\in P_1\cup\{p_1\}$. We let $\Sigma=\Sigma_1\cup\Sigma_2$. The accepting upward-closed set is:

$$\mathscr{U}=\{\mathbf{m}\mid\mathbf{m}\in_{P_1}\mathscr{U}_1\}\cup\{\mathbf{m}\mid\mathbf{m}\in_{P_2}\mathscr{U}_2\}$$

where $\mathbf{m}\in_P\mathscr{U}$ means that the projection of the marking $\mathbf{m}$ on the set of places P is in $\mathscr{U}$. More precisely, let $\mathbf{m}':P\mapsto\mathbb{N}$ be the marking s.t. for any $p\in P$: $\mathbf{m}'(p)=\mathbf{m}(p)$. Then, $\mathbf{m}\in_P\mathscr{U}$ iff $\mathbf{m}'$ is in $\mathscr{U}$. Remark that $\{\mathbf{m}\mid\mathbf{m}\in_{P_1}\mathscr{U}_1\}$ is upward-closed because $\mathscr{U}_1$ is upward-closed. Similarly, $\{\mathbf{m}\mid\mathbf{m}\in_{P_2}\mathscr{U}_2\}$ is upward-closed too. We conclude that $\mathscr{U}$ is upward-closed because the union of two upward-closed sets is an upward-closed set. Finally, we let $\mathbf{m}_0$ be s.t. $\mathbf{m}_0(p_{init})=1$ and $\mathbf{m}_0(p)=0$ for any $p\neq p_{init}$.

It is not difficult to see that $\mathscr{N}$ accepts exactly $L(\mathscr{N}_1,\mathscr{U}_1)\cup L(\mathscr{N}_2,\mathscr{U}_2)$. Indeed, any transition of $\mathscr{N}$ that corresponds to a transition of $\mathscr{N}_1$ (resp. $\mathscr{N}_2$) can be fired only if there is a token in p_1 (p_2). In the initial marking, only t_1 and t_2 are enabled. Firing t_1 puts a token in p_1 which enables the sub-net that corresponds to $\mathscr{N}_1$ (and accepts words from $L(\mathscr{N}_1,\mathscr{U}_1)$ only). Symmetrically, t_2 enables the subnet that corresponds to $\mathscr{N}_2$.
Concatenation: $L(\mathscr{N}_1,\mathscr{U}_1)\cdot L(\mathscr{N}_2,\mathscr{U}_2)\in L^G(\mathsf{PN+T})$. We build $\mathscr{N}$ as follows. $P=P_1\uplus P_2\uplus\{p_1,p_2\}$. For any transition $t=\langle I,O,s,d,b,\lambda\rangle$ in T_1, we put in T a transition $t'=\langle I\cup\{p_1\},O\cup\{p_1\},s,d,b,\lambda\rangle$. For each transition $t=\langle I,O,s,d,b,\lambda\rangle$ in T_2, we put in T a transition $t'=\langle I\cup\{p_2\},O\cup\{p_2\},s,d,b,\lambda\rangle$. We also add to T a transition $t_{\mathbf{m}}$ for any $\mathbf{m}\in\min(\mathscr{U}_1)$, where $t_{\mathbf{m}}=\langle I,O,\perp,\perp,0,\varepsilon\rangle$ s.t.:

$$\forall p\in P: I(p)=\begin{cases}1 & \text{if } p=p_1\\ \mathbf{m}(p) & \text{if } p\in P_1\\ 0 & \text{otherwise}\end{cases}\qquad O(p)=\begin{cases}1 & \text{if } p=p_2\\ \mathbf{m}_0^2(p) & \text{if } p\in P_2\\ 0 & \text{otherwise}\end{cases}$$

Notice that since $\preccurlyeq$ is a wqo, $\min(\mathscr{U}_1)$ is finite. Hence, we only add a finite number of transitions $t_{\mathbf{m}}$.

We also let $\Sigma=\Sigma_1\cup\Sigma_2$. The initial marking $\mathbf{m}_0$ is s.t.

$$\forall p\in P:\mathbf{m}_0(p)=\begin{cases}1 & \text{if } p=p_1\\ \mathbf{m}_0^1(p) & \text{if } p\in P_1\\ 0 & \text{otherwise}\end{cases}$$

Finally, the accepting upward-closed set $\mathscr{U}$ is: $\mathscr{U}=\{\mathbf{m}\mid\mathbf{m}\in_{P_2}\mathscr{U}_2\}$

It is rather straightforward to see that $L(\mathscr{N},\mathscr{U})=L(\mathscr{N}_1,\mathscr{U}_1)\cdot L(\mathscr{N}_2,\mathscr{U}_2)$. Indeed, in the initial marking, a token is present in p_1, which enables the transitions that corresponds to those of $\mathscr{N}_1$ but no token is present in p_2, which inhibits all the transitions that correspond to transitions of $\mathscr{N}_2$. Moreover, $\mathbf{m}_0$ corresponds to

$\mathbf{m}_0^1$ as far as the places of $\mathcal{N}_1$ are concerned. Hence, a sequence of transitions that accepts a word from $L(\mathcal{N}_1, \mathcal{U}_1)$ can be fired from $\mathbf{m}_0$. When a marking that corresponds to an accepting marking of $\mathcal{N}_1$ is reached, one of the $t_{\mathbf{m}}$ transitions can fire (and they can fire in this case only). Indeed, since $\preccurlyeq$ is a wqo, all the accepting markings of $\mathcal{N}_1$ are greater to at least one $\mathbf{m} \in \mathsf{min}(\mathcal{U}_1)$ (and only those markings are). This firing moves the token from p_1 to p_2 and creates a marking that corresponds to $\mathbf{m}_0^2$ on the places of $\mathcal{N}_2$. This inhibits the subnet that corresponds to $\mathcal{N}_1$ and enables the subnet that corresponds to $\mathcal{N}_2$. That subnet is then ready to accept a word from $L(\mathcal{N}_2, \mathcal{U}_2)$.

Intersection: $L(\mathcal{N}_1, \mathcal{U}_1) \cap L(\mathcal{N}_2, \mathcal{U}_2) \in L^G(\mathsf{PN{+}T})$. We build $\mathcal{N}$ as follows. For any transition t, let λ_t be the label of t. We let

$$P = P_1 \uplus P_2 \uplus \{p_{lock}\} \uplus \{p_{t_1,t_2} \mid t_1 \in T_1 \wedge t_2 \in T_2 \wedge \lambda_{t_1} = \lambda_{t_2} \neq \varepsilon\}$$

That is, P contains all the places of $\mathcal{N}_1$ and $\mathcal{N}_2$, a special place p_{lock} that we will use to inhibit transitions of $\mathcal{N}$, and a place p_{t_1,t_2} per pair of transitions from $\mathcal{N}_1$ and $\mathcal{N}_2$ that have the same label (different from ε).

For each ε-labelled transition $t = \langle I, O, s, d, b, \varepsilon\rangle$ of $T_1 \cup T_2$, we add to T the transition $t' = \langle I \cup \{p_{lock}\}, O \cup \{p_{lock}\}, s, d, b, \varepsilon\rangle$. Thus, t' can fire if and only if a token is present in place p_{lock}. Beside this, its effect is the same as in $\mathcal{N}_1$ or $\mathcal{N}_2$.

For every transition $t_1 = \langle I_1, O_1, s_1, d_1, b_1, \lambda_1\rangle$ of T_1 and every transition $t_2 = \langle I_2, O_2, s_2, d_2, b_2, \lambda_2\rangle$ of T_2 s.t. $\lambda_1 = \lambda_2 \neq \varepsilon$, we add to T two transitions $t = \langle I_1 \cup \{p_{lock}\}, O_1 \cup \{p_{t_1,t_2}\}, s_1, d_1, b_1, \lambda_1\rangle$ and $t' = \langle I_2 \cup \{p_{t_1,t_2}\}, O_2 \cup \{p_{lock}\}, s_2, d_2, b_2, \varepsilon\rangle$. Remark that these two transitions t and t' are meant to fire sequentially, and that, once t has fired, no other transition can fire before the corresponding t' fires (because t consumes the token in p_{lock}).

The initial markings is $\mathbf{m}_0$ defined as follows:

$$\forall p \in P : \mathbf{m}_0(p) = \begin{cases} \mathbf{m}_0^1(p) & \text{if } p \in P_1 \\ \mathbf{m}_0^2(p) & \text{if } p \in P_2 \\ 1 & \text{if } p = p_{lock} \\ 0 & \text{otherwise} \end{cases}$$

The accepting upward-closed set is defined as:

$$\mathcal{U} = \{\mathbf{m} \mid \mathbf{m} \in_{P_1} \mathcal{U}_1 \text{ and } \mathbf{m} \in_{P_2} \mathcal{U}_2 \text{ and } \mathbf{m}(p_{lock}) \geq 1\}$$

$\mathcal{U}$ is indeed upward-closed. Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be two markings s.t. $\mathbf{m}_1 \in \mathcal{U}$ and $\mathbf{m}_1 \preccurlyeq \mathbf{m}_2$, and let us show that $\mathbf{m}_2 \in \mathcal{U}$. Since $\mathbf{m}_1 \preccurlyeq \mathbf{m}_2$, we have (i) for any $p \in P_1$: $\mathbf{m}_1(p) \leq \mathbf{m}_2(p)$; (ii) for any $p \in P_2$: $\mathbf{m}_1(p) \leq \mathbf{m}_2(p)$; and (iii) $\mathbf{m}_1(p_{lock}) \leq \mathbf{m}_2(p_{lock})$. Since $\mathcal{U}_1$ is upward-closed and since $\mathbf{m}_1 \in_{P_1} \mathcal{U}_1$, point (i) implies that $\mathbf{m}_2 \in_{P_1} \mathcal{U}_1$. Similarly, we deduce that $\mathbf{m}_2 \in_{P_2} \mathcal{U}_2$ from point (ii). Finally, since $1 \leq \mathbf{m}_1(p_{lock})$, we have $1 \leq \mathbf{m}_2(p_{lock})$. Hence $\mathbf{m}_2 \in \mathcal{U}$.

It is not difficult to see that $L(\mathcal{N}, \mathcal{U}) = L(\mathcal{N}_1, \mathcal{U}_1) \cap L(\mathcal{N}_2, \mathcal{U}_2)$. Indeed, for any pair of transitions t_1 and t_2 respectively from $\mathcal{N}_1$ and $\mathcal{N}_2$ that have the same label, there are two transitions in $\mathcal{N}$ that, when fired sequentially, have the same effect than t_1 and t_2 on their respective input and output places. The place p_{lock} ensures that the two transitions of $\mathcal{N}$ that correspond to t_1 and t_2 will fire sequentially. The transitions of $\mathcal{N}_1$ and $\mathcal{N}_2$ that are labelled by ε do not require any synchronisation and can thus fire independently. Hence, any pair of executions of

$\mathcal{N}_1$ and $\mathcal{N}_2$ that have the same label can be simulated by an execution of $\mathcal{N}$, and any execution of $\mathcal{N}$ (ending in a marking $\mathbf{m}$ s.t. $\mathbf{m}(p_{lock}) = 1$) corresponds to a pair of executions of $\mathcal{N}_1$ and $\mathcal{N}_2$ with the same label.

Iteration: $L^+(\mathcal{N}_1, \mathcal{U}_1) \in L^G(\mathsf{PN{+}T})$. The idea is similar to the construction for the concatenation. Let us assume that $P_1 = \{p_1, p_2, \ldots p_n\}$. We build $\mathcal{N}$ as follows. The set of places $P = P_1 \uplus \{p_{lock}, p_{Tr}, p'_1, p'_2, \ldots p'_n\}$. The set of transitions is:

$$\begin{aligned} T = \quad & \{\langle I \cup \{p_{lock}\}, O \cup \{p_{lock}\}, s, d, b, \lambda\rangle \mid \langle I, O, s, d, b, \lambda\rangle \in T_1\} \\ & \uplus \{t_{\mathbf{m}} \mid \mathbf{m} \in \min(\mathcal{U}_1)\} \\ & \uplus \{t'_1, t'_2, \ldots, t'_n\} \end{aligned}$$

where the transitions t'_i and $t_{\mathbf{m}}$ are defined as follows. For any $1 \le i < n$, $t'_i = \langle \{p'_i\}, O_i, p_i, p_{Tr}, +\infty, \varepsilon\rangle$ with:

$$\forall p \in P : O_i(p) = \begin{cases} \mathbf{m}^1_0(p) & \text{if } p = p_i \\ 1 & \text{if } p = p'_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

The transition t'_n is $\langle \{p'_n\}, O_n, p_n, p_{Tr}, +\infty, \varepsilon\rangle$ with:

$$\forall p \in P : O_n(p) = \begin{cases} \mathbf{m}^1_0(p) & \text{if } p = p_n \\ 1 & \text{if } p = p_{lock} \\ 0 & \text{otherwise} \end{cases}$$

Finally, for every $\mathbf{m} \in \min(\mathcal{U})$, $t_{\mathbf{m}} = \langle I_{\mathbf{m}}, \{p'_1\}, \perp, \perp, 0, \varepsilon\rangle$, with:

$$\forall p \in P : I_{\mathbf{m}}(p) = \begin{cases} \mathbf{m}(p) & \text{if } p \in P_1 \\ 1 & \text{if } p = p_{lock} \\ 0 & \text{otherwise} \end{cases}$$

The initial marking $\mathbf{m}_0$ is s.t. $\mathbf{m}_0(p_{lock}) = 1$, for every $p \in P_1$, $\mathbf{m}_0(p) = \mathbf{m}^1_0(p)$ and for every $p \in P \setminus (P_1 \cup \{p_{lock}\})$, $\mathbf{m}_0(p) = 0$. Finally, the accepting upward-closed set is $\mathcal{U} = \{\mathbf{m} \mid \exists \mathbf{m}' \in \mathcal{U}_1 : \forall p \in P_1 : \mathbf{m}'(p) \le \mathbf{m}(p)\}$.

Let us show why the construction is correct. $\mathcal{N}$ contains all the transitions of $\mathcal{N}_1$ (with the same labels), that have been adapted in order to fire only if there is at least one token in p_{lock}, which is true initially. Hence, $\mathcal{N}$ can start its execution by firing a sequence of transitions that is labelled by a word in $L(\mathcal{N}_1, \mathcal{U}_1)$ and put into the places of P_1 a marking that is in $\mathcal{U}_1$. At that point, the global marking of $\mathcal{N}$ is thus in $\mathcal{U}$. Thus, the word read so far (which is indeed in $L(\mathcal{N}_1, \mathcal{U}_1)^*$) is accepted. Nevertheless, the net can continue its execution, because, once a marking of $\mathcal{U}$ has been reached, one of the $t_{\mathbf{m}}$ transitions can fire, by monotonicity. This removes the token from p_{lock}, which inhibits all the (adapted) transitions from $\mathcal{N}_1$. At that point, the only firable sequence of transitions is $t'_1 t'_2 t'_3 \ldots t'_n$ (labelled by ε). Each t'_i transition has the effect to restore the initial marking of p_i, by first transferring all the tokens from p_i to p_{Tr} (a trash can place), and then, produce into p_i exactly $\mathbf{m}^1_0(p_i)$ tokens. The last transition t'_n of the sequence also produces a token into p_{lock}, which allows the (adapted) transitions from $\mathcal{N}_1$ to fire anew. Since the initial marking has been restored, a new word from $L(\mathcal{N}_1, \mathcal{U}_1)$ can be read. This allows to reach again a marking in $\mathcal{U}$, and so on. Thus, every word in $L(\mathcal{N}_1, \mathcal{U}_1)^+$ is in $L(\mathcal{N}, \mathcal{U})$.

On the other hand, in the case where the sequence of (adapted) transitions from $\mathcal{N}_1$ does not produce a marking $\mathbf{m}$ that corresponds to a marking of $\mathcal{U}_1$, then, (i) the marking $\mathbf{m}$ is not in $\mathcal{U}$ and is thus not accepting, and (ii) no transition of the form $t_{\mathbf{m}}$ can fire. Hence, the net is blocked until a marking corresponding to an accepting marking of $\mathcal{N}_1$ is reached. We conclude that $L(\mathcal{N},\mathcal{U}) \subseteq L(\mathcal{N}_1,\mathcal{U}_1)^+$. Hence, $L(\mathcal{N},\mathcal{U}) = L(\mathcal{N}_1,\mathcal{U}_1)^+$.

Arbitrary homomorphism: $h(L(\mathcal{N}_1,\mathcal{U}_1)) \in L^G(\mathsf{PN{+}T})$. Let h be a homomorphism that maps each character a of Σ_1 to a sequence of characters $h(a)$ of an alphabet Σ' (and ε to itself). Again, we denote the label of any transition t by λ_t. We build $\mathcal{N}$ as follows. We let $\Sigma = \Sigma'$. We define the set of places P as:

$$P = P_1 \uplus \{p_{lock}\} \uplus \bigcup_{t \in T_1} \{p_{t,i} | 1 \leq i < |h(\lambda_t)|\}$$

As usual, the place p_{lock} is meant to lock the net, i.e., prevent undesired transitions to fire, when necessary. The places $p_{t,i}$ act as intermediary states when reading the word $h(\lambda_t)$ for any $t \in T_1$ with $|h(\lambda_t)| \geq 1$. More precisely, a token in $p_{t,i}$ means that the net has accepted the prefix of length i of $h(\lambda_t)$ so far.

T is built according to these ideas. For any transition $t = \langle I,O,s,d,b,\lambda\rangle$ of T_1, we consider two cases. If $h(\lambda) = \varepsilon$ or $h(\lambda) \in \Sigma_1$, we add to T a single transition $t' = \langle I \cup \{p_{lock}\}, O \cup \{p_{lock}\}, s,d,b,h(\lambda)\rangle$. Otherwise $|h(\lambda)| > 1$, and we assume that $h(\lambda) = w_1 w_2 \cdots w_n$. We add to T the n transitions $t_1,t_2,\ldots t_n$ defined as follows. $t_1 = \{I \cup \{p_{lock}\}, O \cup \{p_{t,1}\}, s,d,b,w_1\}$. For any $1 < i < n$, $t_i = \langle \{p_{t,i-1}\},\{p_{t,i}\},\perp,\perp,0,w_i\rangle$. Finally, $t_n = \langle \{p_{t,n-1}\},\{p_{lock}\},\perp,\perp,0,w_n\}\rangle$.

The initial marking $\mathbf{m}_0$ is s.t.:

$$\forall p \in P : \mathbf{m}_0(p) = \begin{cases} \mathbf{m}_0^1(p) & \text{if } p \in P_1 \\ 1 & \text{if } p = p_{lock} \\ 0 & \text{otherwise} \end{cases}$$

The accepting upward-closed set $\mathcal{U}$ is $\{\mathbf{m} | \mathbf{m} \in_{P_1} \mathcal{U}_1 \text{ and } \mathbf{m}(p_{lock}) \geq 1\}$. For the justification that $\mathcal{U}$ is upward-closed, we refer the reader to the arguments used in the case of the intersection.

Clearly, $L(\mathcal{N},\mathcal{U}) = h\big(L(\mathcal{N}_1,\mathcal{U}_1)\big)$. Indeed, each transition t of $\mathcal{N}_1$ with label λ and s.t. $h(\lambda) \leq 1$, is replaced by a transition t' with label $h(\lambda)$, that has the same effect on the places of P_1 but which can be fired only if the token is present in p_{lock}. Moreover, each transition t of $\mathcal{N}_1$ with label λ and s.t. $h(\lambda) > 1$ is replaced by a set of transitions that, when fired sequentially, accept $h(\lambda)$ and have the same effect as t on the places of P_1. Thanks to the places of the form $p_{t,i}$ and thanks to p_{lock}, we ensure that these transitions are indeed fired sequentially.

Inverse homomorphism: $h^{-1}(L(\mathcal{N}_1,\mathcal{U}_1)) \in L^G(\mathsf{PN{+}T})$. Let Σ' be an alphabet and let h be a homomorphism that maps any word on Σ' to a word on Σ_1. The $\mathsf{PN{+}T}$ $\mathcal{N}$ is built as follows. First of all, we build a PN $\mathcal{N}_o = \langle P_o, T_o, \Sigma_1 \uplus \{\alpha_a \mid a \in \Sigma'\}, \mathbf{m}_0^o\rangle$ that will act as an observer and repeatedly accepts all the words of the form $h(a)$ for any $a \in \Sigma'$.

More precisely, $\mathcal{N}_o$ is defined as follows. Its set of places is:

$$P_o = \{p_{init}\} \uplus \{p_{a,i} \mid a \in \Sigma' \wedge 1 \leq i \leq |h(a)|\}$$

The set of transitions is:

$$T_o = \{t_{a,i} \mid a \in \Sigma' \wedge 1 \leq i \leq |h(a)|\} \cup \{t_a^h \mid a \in \Sigma'\}$$

where, for any $a \in \Sigma'$ s.t. $h(a) = w_1 w_2 \cdots w_n$: (i) $t_a^h = \langle \{p_{a,n}\}, \{p_{init}\}, \perp, \perp, 0, \alpha_a \rangle$; (ii) $t_{a,1} = \langle \{p_{init}\}, \{p_{a,1}\}, \perp, \perp, 0, w_1 \rangle$; and (iii) for any $1 < i \leq n$, we let: $t_{a,i} = \langle \{p_{a,i-1}\}, \{p_{a,i}\}, \perp, \perp, 0, w_i \rangle$. Moreover, for any $a \in \Sigma'$ s.t. $h(a) = \varepsilon$, we have $t_a^h = \langle \{p_{init}\}, \{p_{init}\}, \perp, \perp, 0, \alpha_a \rangle$. The initial marking $\mathbf{m}_0^o$ puts a token in p_{init} only. The accepting set is $\mathscr{U}_o = \{\mathbf{m} \mid \mathbf{m}(p_{init}) \geq 1\}$. Thus, any accepting sequence of transitions of $\mathscr{N}_o$ is labelled by a word of the form $h(a_1) \cdot \alpha_{a_1} \cdot h(a_2) \cdot \alpha_{a_2} \cdots h(a_n) \cdot \alpha_{a_n}$, where all the a_i's belong to Σ' (remark that it holds when $h(a) = \varepsilon$ too).

The next step amounts to computing a new PN+T $\mathscr{N}'$ and a new upward-closed set $\mathscr{U}'$ from $\mathscr{N}_1$, $\mathscr{N}_o$, $\mathscr{U}_1$ and $\mathscr{U}_o$ by applying the same procedure as in the case of the intersection, *except that* we treat all the transitions labelled by α_a for some $a \in \Sigma'$ as if they were labelled by ε (in other words, we replace all the α_a labels in $\mathscr{N}_o$ by ε, compute the intersection, then restore the labels. Remember that the ε-labelled transitions are unaffected by the construction we have presented for the intersection. Thus, all the transitions of the form t_a^h appear *as is* in the resulting net). What we obtain is a net that accepts all the words of the form $h(a_1) \cdot \alpha_{a_1} \cdot h(a_2) \cdot \alpha_{a_2} \cdots h(a_n) \cdot \alpha_{a_n}$ such that $h(a_1) \cdot h(a_2) \cdots h(a_n) = h(a_1 \cdot a_2 \cdots a_n)$ is in $L(\mathscr{N}_1, \mathscr{U}_1)$. We obtain $\mathscr{N}$ by replacing the label λ_t of any transition t in $\mathscr{N}'$ as follows: if $\lambda_t = \alpha_a$ for $a \in \Sigma'$, we let $\lambda_t = a$, otherwise, we let $\lambda_t = \varepsilon$. We also let $\mathscr{U} = \mathscr{U}'$. Hence, $L(\mathscr{N}, \mathscr{U})$ is the set of all the words of the form $a_1 \cdot a_2 \cdots a_n$ s.t. $h(a_1 \cdot a_2 \cdots a_n) \in L(\mathscr{N}_1, \mathscr{U}_1)$. This is exactly $h^{-1}\big(L(\mathscr{N}_1, \mathscr{U}_1)\big)$. □

Remark 5 $L^P(\mathsf{PN{+}T})$ is not a full AFL. The justification is the same as in the case of $L^P(\mathsf{WSTS})$. That is, let us consider the language $L = \{\varepsilon, \mathtt{a}\}$ on the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, and the homomorphism h s.t. $h(\mathtt{a}) = \mathtt{bb}$. Then, $L \in L^P(\mathsf{PN{+}T})$, but $h(L) = \{\varepsilon, \mathtt{bb}\} \notin L^P(\mathsf{PN{+}T})$ because it is not prefix-closed (it does not contain the prefix b of bb).

5.5 Some remarks about the pumping lemmata

It is interesting to compare, on the one hand, Lemma 6, and, on the other hand, Lemma 8 and Lemma 10. Indeed, Lemma 6 provides us with a property that holds on any WSL, where Lemma 8 and Lemma 10 deal with restricted subclasses of WSTS (namely, PN and PN+NBA). Because they focus on these two peculiar classes, these two lemmata allow us to state more precise properties than the one that is given by Lemma 6.

Nevertheless, when we restrict ourselves to the class PN, Lemma 8 is more general than Lemma 6. By letting $w_i = \varepsilon$ for any $i \geq 1$ in Lemma 8, we re-obtain Lemma 6. In particular, we can obtain thanks to Lemma 8 several results[3] that we had previously proved with Lemma 6 in section 5.1. From our point of view, this is another argument in favor of the interest of Lemma 8. A similar conclusion can be drawn when comparing Lemma 6 to Lemma 10 for the class PN+NBA.

[3] The results we allude to are: $\mathscr{L} \notin L^G(\mathsf{PN})$, $\mathscr{L}^{\geq} \notin L^G(\mathsf{PN})$, $\mathscr{L}^R \notin L^G(\mathsf{PN})$ and $L^G(\mathsf{PN})$ is not closed under complement

6 Conclusion

The (labelled) well-structured transition systems are a well-known class of infinite-state transition systems, that enjoy monotonicity properties and whose set of states is well-quasi ordered. In the present work, we have studied several properties of the classes of languages that can be recognized by WSTS, and some of their subclasses, such as the EPN. We have proved three pumping lemmata by exploiting specific properties of the WSTS (which is, to the best of our knowledge, original in this context). These lemmata have allowed us mainly to strictly separate the expressiveness of three important classes of EPN: the PN, the PN+NBA, and the PN+T.

These different models have been used in different works to modelize behaviours of concurrent systems [6,5,20]. Roughly speaking, in these modelizations, each process is represented by a token and the place in which each token is present encodes the state of the corresponding process. The peculiar features of PN+T or PN+NBA have been regarded as natural ways to express the communication procedures between the processes of the system. For instance, a transfer arc is perfectly suited to represent a *broadcast*, i.e., a message that is sent to all the processes in a given state, and that modifies at once the state of all these processes. Such broadcast are intrinsic features of some programming languages, such as JAVA (through the `notifyAll` keyword). From our point of view, it is thus important to have a precise knowledge of the expressivity of these models, and to be able to compare these expressivity. By strictly separating the expressive powers of PN, PN+NBA and PN+T, our results demonstrate the meaningfulness of these different communication procedures.

Acknowledgements The authors would like to express their gratitude to the anonymous referee, whose comments have definitely contributed to improve the present paper.

References

1. P. A. Abdulla, K. Cerans, B. Jonsson, and Y.-K. Tsay. General Decidability Theorems for Infinite-state Systems. In *Proceedings of the 11th Annual Symposium on Logic in Comuter Science (LICS'96)*, pages 313–321. IEEE Computer Society Press, 1996.
2. P.A. Abdulla, A Bouajjani, and B Jonsson. On-the-Fly Analysis of Systems with Unbounded, Lossy FIFO Channels. In *Proceedings of the 10th International Conference on Computer Aided Verification (CAV'98)*, volume 1427 of *LNCS*, pages 305–318. Springer, 1998.
3. Parosh Abdulla, Aurore Annichini, and Ahmed Bouajjani. Symbolic verification of lossy channel systems: Application to the bounded retransmission protocol. In *Proc. 5th Intern. Conf. on Tools and Algorithms for the Construction and Analysis of Systems (TACAS'99)*, number 1579 in LNCS, pages 208–222. Springer-Verlag, 1999.
4. G. Ciardo. Petri nets with marking-dependent arc multiplicity: properties and analysis. In *Proceedings of the 15th International Conference on Applications and Theory of Petri Nets (ICATPN 94)*, volume 815 of *LNCS*, pages 179–198. Springer, 1994.
5. G. Delzanno. Automatic Verification of Parameterized Cache Coherence Protocols. In *Proceedings of the 12th International Conference on Computer Aided Verification (CAV 2000)*, volume 1855 of *LNCS*, pages 53–68. Springer, 2000.

6. G. Delzanno, J.-F. Raskin, and L. Van Begin. Towards the Automated Verification of Multithreaded Java Programs. In *Proceedings of the International Conference on Tools and Algorithms for Construction and Analysis of Systems (TACAS 2002)*, volume 2280 of *LNCS*, pages 173–187. Springer, 2002.
7. L.E. Dickson. Finiteness Of The Odd Perfect And Primitive Abundant Numbers With r Distinct Prime Factors. *American Journal of Mathematics*, 35:413–422, 1913.
8. E. A. Emerson and K. S. Namjoshi. On Model Checking for Non-deterministic Infinite-state Systems. In *Proceedings of the 13th Annual Symposium on Logic in Computer Science (LICS '98)*, pages 70–80. IEEE Computer Society Press, 1998.
9. A. Finkel, G. Geeraerts, J.-F. Raskin, and L. Van Begin. On the omega-language expressive power of extended Petri nets. In *Proceedings of EXPRESS'04, 11th International Workshop on Expressiveness in Concurrency, London, Great Britain*, volume 128(2) of *Electronic Notes in Theoretical Computer Science*, pages 87–101. Elsevier Publishing, 2004.
10. A. Finkel and P. Schnoebelen. Well-structured transition systems everywhere! *Theoretical Computer Science*, 256(1-2):63–92, 2001.
11. S. M. German and A. P. Sistla. Reasoning about Systems with Many Processes. *Journal of ACM*, 39(3):675–735, 1992.
12. Seymour Ginsburg. *Algebraic and Automata-Theoretic Properties of Formal Languages*. Elsevier Science Inc., 1975.
13. John Hopcroft, Rajeev Motwani, and Jeffrey Ullman. *Introduction to Automata Theory, Languages, and Computation, second edition*. Addison-Wesley, 2001.
14. Matthias Jantzen. Language theory of petri nets. In Wilfried Brauer, Wolfgang Reisig, and Grzegorz Rozenberg, editors, *Proc. of Advances in Petri Nets 1986*, volume 254 of *Lecture Notes in Computer Science*, pages 397–412. Springer, 1986.
15. N.M. Minsky. *Finite and Infinite Machines*. Englewood Cliffs, N.J., Prentice-Hall, 1967.
16. J. L. Peterson. *Petri Net Theory and the Modeling of Systems*. Prentice Hall, 1981.
17. J.-F. Raskin and L. Van Begin. Petri Nets with Non-blocking Arcs are Difficult to Analyse. In *Proceedings of the 5th International Workshop on Verification of Infinite-state Systems (INFINITY 2003)*, volume 96(1) of *ENTCS*. Elsevier, 2003.
18. Arto Salomaa. *Formal Languages*. Academic Press, 1973.
19. Arto Salomaa. *Computation and Automata*, volume 25 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1985.
20. L. Van Begin. *Efficient Verification of Counting Abstractions for Parametric systems*. PhD thesis, Université Libre de Bruxelles, Belgium, 2003.