Latent Ranking Models
and
Order Polytopes

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Research Themes

- **Geometry and combinatorics of specific polytopes**

  with motivations from mathematical psychology
  (model characterization, data analysis)

  and motivations from optimization
  (finding a best ordering)

- (**Distances between rankings, phylogenetic networks**

  with motivations from biology

- **Knowledge assessment**

- **Approximation algorithms for coloring problems**

- $\ldots)$

The Setting

To be uncovered: some latent ranking of known objects

(alternatives, decisions, etc.)

using one of various types of questions:

- binary choice:
  among two objects, which one is the best?
- multiple choice:
  among some objects, which one is the best?
- best/worst choice:
  among some objects, which one is the best, which one is the worst?
- approval voting:
  which of the objects are acceptable?

A Non-Deterministic Framework

The latent ranking (which governs the choices) is random:

individual variation;
population variation.

Two basic assumptions:

- the choices made are coherent with the actual latent ranking;
- the choice probabilities are computed from the latent ranking probabilities. How?

Take binary choice as a paramount example.

Binary Choice Probabilities

Notation: Z some finite set of cardinality n,

Π the collection of the $n!$ rankings of Z.

For each probability distribution P on Π,

compute the binary choice probabilities as

$$p_{ij} \quad = \quad \sum \{\, P(L) : L \in \Pi \text{ and } i\, L\, j \,\},$$

for $i, j \in Z$ and $i \neq j$.

An example for $Z = \{a, b, c\}$

Can the following data be produced in this way?

$$\begin{aligned} p_{ab} &= 0.12, & p_{ba} &= 0.88, \\ p_{ac} &= 0.56, & p_{ca} &= 0.44, \\ p_{bc} &= 0.75, & p_{cb} &= 0.25. \end{aligned}$$

More precisely: is there some probability distribution P on Π that would give the following?

$$\begin{aligned} 0.12 &= P(abc) + P(acb) + P(cab), \\ 0.88 &= P(bac) + P(bca) + P(cba), \\ 0.56 &= P(abc) + P(acb) + P(bac), \\ 0.44 &= P(bca) + P(cab) + P(cba), \\ 0.75 &= P(abc) + P(bac) + P(bca), \\ 0.25 &= P(acb) + P(cab) + P(cba). \end{aligned}$$

Main Problem: Characterizing Binary Choice Prob.

Given real numbers p_{ij} for all $i, j \in Z$ with $i \neq j$,

can we find some probability distribution P on Π such that the p_{ij}'s are the binary choice probabilities defined by P?

More precisely:

find a necessary and sufficient condition on the p_{ij}'s for the existence of P.

A useful comment:

characterizing binary choice probabilities is ...

... a hopeless problem!

An algorithmically tractable answer would lead to $\mathtt{P} = \mathtt{NP}$.

A Geometric Point of View

Vectors of binary choice probabilities p belong to $\mathbb{R}^{Z \ltimes Z}$

(a space with one real coordinate for each pair (i, j) of distinct objects).

Example

For $Z = \{a, b, c\}$, we have 6-dimensional vectors

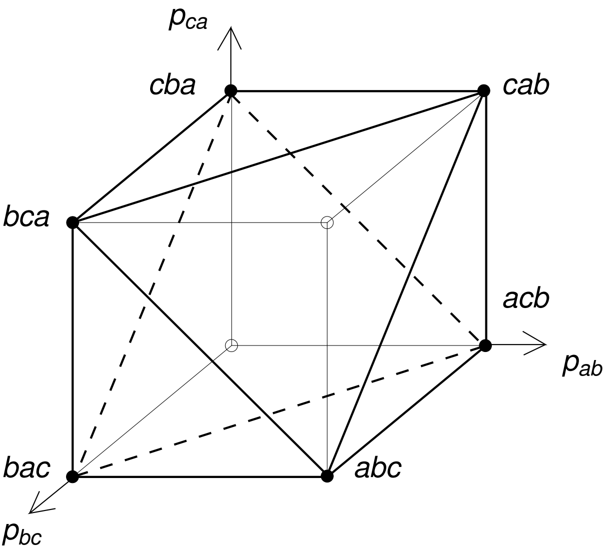
$$\left(p_{ab},\ p_{ba},\ p_{bc},\ p_{cb},\ p_{ac},\ p_{ca} \right).$$

As we know $\quad p_{ab} + p_{ba} = 1, \quad p_{ac} + p_{ca} = 1, \quad p_{bc} + p_{cb} = 1,$
we may work with only

$$\left(p_{ab}, \quad p_{bc}, \quad p_{ca} \right).$$

The collection of all (projected) vectors form a polyhedron in $\mathbb{R}^3$:

The Projected Polyhedron for $Z = \{a, b, c\}$



The Linear Ordering Polytope

The binary choice probabilities form a convex polytope, called

the binary choice polytope or linear ordering polytope P^Z_{LO}

of dimension $n \cdot (n-1)/2$,
with one vertex x^L per ranking L of Z.

The Main Problem can be rephrased as:

find an explicit system of linear inequalities for P^Z_{LO}, or better

find the facets of the polytope P^Z_{LO}.

Hopeless!

Origins of the Problem

In mathematical psychology/economics:

Guilbaud (1953), Block and Marschak (1960).

In discrete mathematics:

Megiddo (1977).

In operations research:

Grötschel, Jünger and Reinelt (1985).

In voting theory:

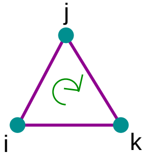
Saari (1999).

Examples of Facet-defining Inequalities for P^n_{LO}

Starting from obvious necessary conditions:

Theorem

The following affine (linear) inequalities on $\mathbb{R}^{Z \ltimes Z}$ *define facets:*



$$
\begin{aligned}
p_{ij} &\ge 0 && \textit{(trivial inequalities)},\\
p_{ij} + p_{jk} + p_{ki} &\le 2 && \textit{(triangular inequalities)}.
\end{aligned}
$$

Many families of nonobvious facets are known today;

here are two examples.

Graphical Inequalities (our first family)

Several research advances led to a marvelous result by Koppen (1995).

Let $G = (V, E)$ be a (simple) graph.

The stability number $\alpha(G)$ of G is the largest number of vertices no two of which are adjacent.

Assume $V, W \subset Z$ with $f : V \to W$ some bijective mapping.

Definition

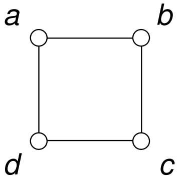
The graphical inequality of G reads

$$\sum_{i \in V} x_{i\,f(i)} \quad - \sum_{\{ij\} \in E} (x_{i\,f(j)} + x_{j\,f(i)}) \quad \leq \quad \alpha(G).$$

An Example of Graphical Inequality

Example

For the graph



with the bijection

$$f : \qquad a \mapsto s, \quad b \mapsto t, \quad c \mapsto u, \quad d \mapsto v,$$

we get the inequality

$$\begin{aligned} & x_{as} + x_{bt} + x_{cu} + x_{dv} \\ & \quad - (x_{at} + x_{bs}) - (x_{bu} + x_{ct}) - (x_{cv} + x_{du}) - (x_{ds} + x_{av}) \\ & \qquad \leq \quad 2. \end{aligned}$$

Koppen's Result

Theorem (Koppen, 1995)

The graphical inequality of G

$$\sum_{i \in V} x_{i\,f(i)} \quad - \sum_{\{ij\} \in E} \left(x_{i\,f(j)} + x_{j\,f(i)} \right) \quad \leq \quad \alpha(G).$$

is valid for the linear ordering polytope.

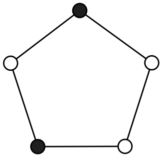
It defines a facet if and only if G is
different from K_2,
connected,
and stability critical.

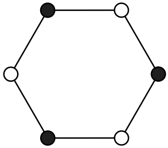
Definition

A graph is stability critical when its stability number increases whenever any of its edges is deleted.

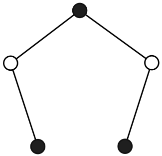
An Example of Stability-Critical Graph

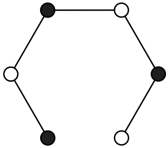
Examples





Delete any edge:





Thus: the 5-cycle is stability critical but the 6-cycle is not.

A Weighted Generalization

Doignon, Fiorini and Joret (2005) generalize Koppen's result:

weights are assigned to the vertices of the graph
(and used in the generalized graphical inequality).

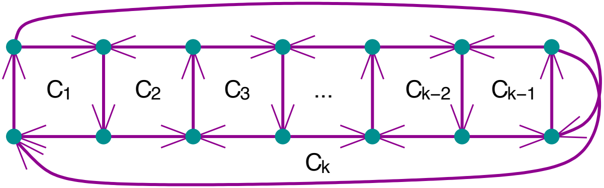
There results an interesting class of weighted graphs, which extend stability-critical graphs ...

... and lead to many new problems.

Möbius-Type Inequalities (our second family)

Other facet defining inequalities for the binary choice polytope include Möbius Ladders Inequalities.

For k **odd**, $k \geq 3$: $$\sum_{ij \in M} x_{ij} \geq \frac{k+1}{2}$$ with M the arc set



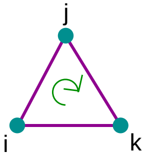
Theorem (Grötschel, Jünger and Reinelt, 1985)

If M is a Möbius ladder generated by k directed cycles then it defines a facet of P^n_{LO}.

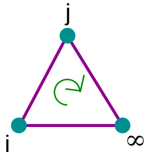
An Alternative Interpretation of the Inequality

The inequality is "obtained by summing up"

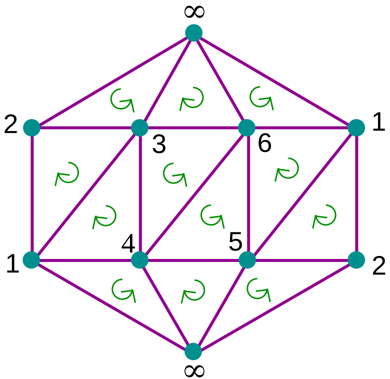
$x_{ij} + x_{jk} + x_{ki} \geq 1 \rightarrow$



$x_{ij} \geq 0 \rightarrow$



and thus come from (notice point-identification)

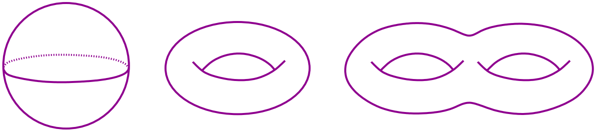


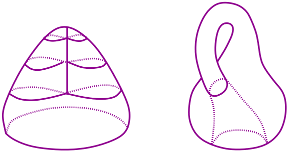
An oriented triangulation of the projective plane!

Non-Orientable Surfaces

Theorem (Fiorini, 2007)

Any nonorientable surface has some facet-defining directed triangulation.

Orientable:  . . .

Nonorientable:  . . .

Open question: how to characterize the “good” triangulations?

What About the Other Choice Types?

In each case, a polytope whose vertices are known and facets are to be uncovered.

- Binary Choice:
  as said, hopeless, but many interesting, related problems are there around;
- Multiple Choice:
  completely solved by Falmagne (1978),
  a new proof by Fiorini (2004);
- Best/Worst Choice:
  presently under investigation;
- Approval Voting:
  Doignon and Fiorini (2003, 2004) present a solution
  …which leads to new problems on some specific classes of graphs.

Conclusion

The choice polytopes are special cases of $0/1$-polytopes obtained as the convex hull of characteristic vectors

(other examples: the traveling salesperson problem, network polytopes, etc.).

They are investigated from geometric, combinatorial, etc. points of view,

using or not computers
(softwares are available for chasing faces in small dimensions, also handmade programs are needed).

Thank you!