Graphes et Optimisation Mathématique

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Présentation de l'équipe

Research activities of GOM

The research activities of the unit "Graphs and Mathematical Optimization" cover many aspects of Mathematical Programming with a special emphasis on combinatorial and Network Optimization. The main fields of applications considered are location theory, communication network planning, traffic and transportation models, bilevel programming and production planning.

Martine Labbé

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Polyhedral approaches to integer programming problems

- Most optimization problems we will encounter can be formulated as (mixed) integer programs.

$$(IP) \quad \min \quad c^T x \\ \text{s.t.} \quad Ax \geq b \\ x \geq 0, \ \text{integer}$$

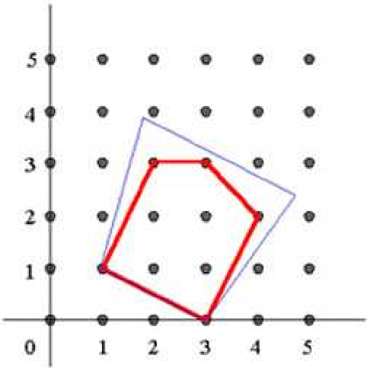
  where $A \in \mathbb{R}^{m \times n}$, $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.
- Let $X = \{x \in \mathbb{Z}^n : \ Ax \geq b, \ x \geq 0\}$ be the set of feasible solutions and z_{IP} the value of the optimal solution.

The convex hull of feasible solutions

- (IP) is equivalent to

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & x \in \mathrm{conv}(X), \end{array}$$

  where $\mathrm{conv}(X)$ is the convex hull of points in X.
- $\mathrm{conv}(X)$ is a polyhedron $\{x \in \mathbb{R}^n : \tilde{A}x \geq \tilde{b}\}$ but ...
  - for hard problems, we do not know $\tilde{A}$ and $\tilde{b}$;
  - the number of inequalities is exponential in the size of the problem.



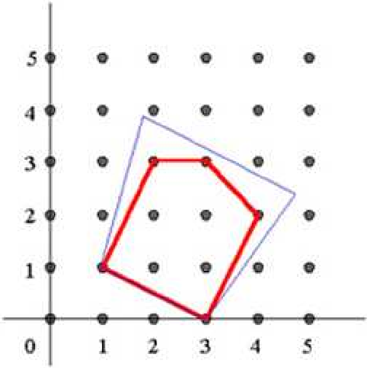
Relaxations

- Use relaxations of $\mathrm{conv}(X)$:

$$(REL) \quad \min \; c^T x \text{ s.t. } x \in P,$$

  where P is a polyhedron such that $\mathrm{conv}(X) \subseteq P$.
- $z_{REL} \leq z_{IP}$ and inequalities defining P are valid inequalities for IP.
- Simplest relaxation: linear relaxation.

$$\begin{array}{lll} (LP) & \min & c^T x \\ & \text{s.t.} & Ax \geq b \\ & & x \geq 0. \end{array}$$



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Cutting planes

- An inequality $\alpha^t x \geq \beta$ is a valid inequality for $\mathrm{conv}(X)$ if it is satisfied by all $x \in X$.
- Try to strenghten the LP relaxation by adding valid inequalities.
- The best inequalities are those which form a minimal set describing $\mathrm{conv}(X)$: facet defining inequalities.
- Two approaches:
    - Generic valid inequalities (Gomory cuts, covers, MIR, ...)
    - Problem specific inequalities: classes of strong valid inequalities coming from the structure of the problem. Usually, one tries to prove these inequalities are facet defining.

Topological network design

Data:

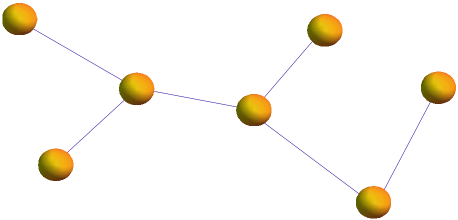
- an undirected graph $G = (V, E)$;
- set of vertices V: offices, routers, ADMs, ... that must be connected;
- set of edges E: possible links;
- for each edge $e \in E$: installation cost $c_e \geq 0$.

Objective:

Minimize the sum of the costs of installed edges.

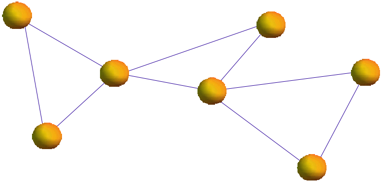
Simple connectivity

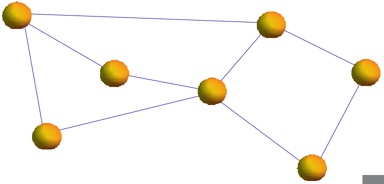
- Without additional constraints: Minimum cost connected network.
- Minimum spanning tree.



Survivability

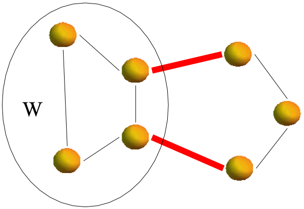
- A single link failure disconnects a tree.
- Connectivity requirements: Prespecified number of disjoint paths required between any pair of nodes.
- Usually, 2 paths are sufficient (probability of two simultaneous failures is really small).
- Paths may be edge-disjoint or node-disjoint.



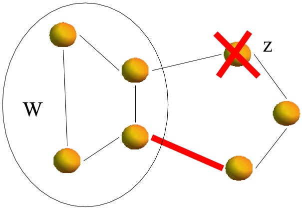


Two-connected network problem

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e \\
\text{s.t.} \quad & x(\delta(W)) \geq 2 && W \subset V,\ \emptyset \neq W \neq V, \\
& x(\delta_{G-z}(W)) \geq 1 && z \in V,\ W \subset V \setminus \{z\},\ \emptyset \neq W \neq V \setminus \{z\}, \\
& x_e \in \{0, 1\} && e \in E.
\end{aligned}
$$

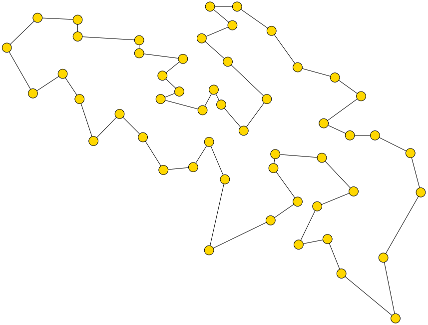


Cut constraint



Node-cut constraint

Application to the Belgian network



- Minimum-cost two-connected network is a Hamiltonian cycle.
- Need for additional survivability requirements.

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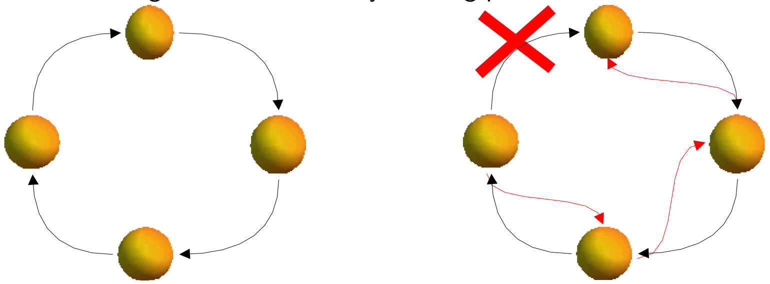
Bounded rings

Additional requirement:

Each edge belongs to a cycle of bounded length.

Bounded rings: condition added to limit the increase of the distance in case of rerouting. Also motivated by the ring protection of the network.



The problem

Minimum cost two-connected network with bounded rings

Given :

- V, a set of n vertices and $E \subseteq V \times V$, a set of possible edges;
- $c_e \geq 0$, a fixed cost associated to edge $e \in E$;
- $d_e \geq 0$, a fixed length associated to edge $e \in E$.

Find the minimum total cost two-connected network spanning V such that each edge belongs to a cycle of length at most K.

- Formulation.
- Valid inequalities and separation procedures.
- Branch-and-cut algorithm and heuristics.

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Traffic engineering in IP networks

- Basic model: Find shortest paths according to a given metric.
- Arc metrics are statically fixed by the operator.
- Multiple same-cost paths allowed: for simplicity we assume here that the flow is equally split on all outgoing links that belong to at least one shortest path. (Some operators forbid this behavior.)

Traffic engineering in IP networks

- Collaboration with AT&T Labs-Research
- Aim: adapt routing (taking the demand into account) to avoid congestion.
- Common belief among operators: OSPF not well suited for TE (motivation for using MPLS).
- CISCO recommendation: use weights inversely proportionnal to the arc capacities
  - $\rightarrow$ demand is not taken into account.
- What happens if we try to optimize the weights knowing the demand?

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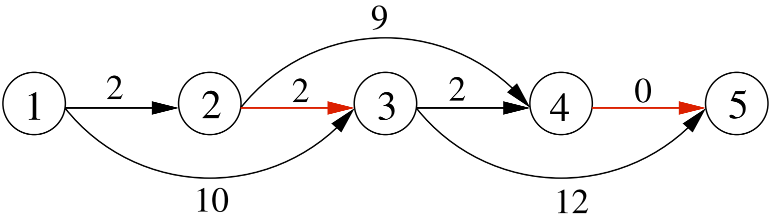
Static case: fixed demands, topology

- Local search heuristic.
- Efficient algorithms for shortest paths and routing update.
- Main results:
  - Our heuristic allows to deal with an increase of demands of 55 % compared to simple OSPF strategies.
  - Optimal routing can cope with a 3 % increase in demands only, so it would be preferable to increase the hardware capacity.
  - For realistic network topologies, OSPF works well (with our heuristic).

Network pricing



- The user never pay more than 22.
- The optimal path with zero tolls has a cost of 6.

$\Rightarrow$ Maximal revenue $= 22 - 6 = 16$.

Bilevel model (Labbé, Marcotte, Savard, 1998):

$$(\text{TSP}) \qquad \max_{t} \sum_{k \in K} \sum_{(i,j) \in A} \eta^k t_{ij} x_{ij}^k$$

$$\text{s.t. } t_{ij} \geq 0 \qquad \forall (i,j) \in A$$

$$(x,y) \in \text{argmin} \sum_{k \in K} \left(\sum_{(i,j) \in A} (c_{ij} + t_{ij}) x_{ij}^k + \sum_{(i,j) \in B} d_{ij} y_{ij}^k \right)$$

$$\text{s.t. } \sum_{i:(i,j) \in A} x_{ij}^k + \sum_{i:(i,j) \in B} y_{ij}^k - \sum_{l:(j,l) \in A} x_{jl}^k - \sum_{l:(j,l) \in B} y_{jl}^k = \left\{ \begin{array}{ll} -1 & \text{if } j = o_k \\ 1 & \text{if } j = d_k \\ 0 & \text{otherwise} \end{array} \right\}$$

$$\forall k \in K, j \in N$$

$$x_{ij}^k, y_{ij}^k \in \{0,1\} \qquad \forall k \in K, (i,j) \in A \cup B$$

Applications

- Yield management;
- Transportation companies (airline, railway);
- Pricing of products;
- . . .

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