

# Two applications of polylog functions and Euler sums

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## Abstract

Let  $I(n) := \int_0^1 [x^n + (1-x)^n]^{\frac{1}{n}} dx$ . In this paper, we show that  $I(n) = \sum_0^\infty \frac{I_i}{n^i}$ ,  $n \rightarrow \infty$  and we compute  $I_i, i = 0..5$ , obtained by polylog functions and Euler sums. As a corollary, we obtain explicit expressions for some integrals involving functions  $u^i, \exp(-u), (1+\exp(-u))^j, \ln(1+\exp(-u))^k$ . As another asymptotic result, let  $S_0(z) := \frac{Li_m(1)}{Li_m(1)-Li_m(z)}$ , where  $Li_m(z)$  is the polylog function. We provide the asymptotic behaviour of  $S_n, n \rightarrow \infty$  where  $S_n := [z^n]S_0(z)$ . This paper fits within the framework of analytic combinatorics.

**Keywords:** polylog functions, Euler sums, asymptotics, analytic combinatorics

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## 1 Introduction

Some time ago, the following question was circulating among the Mathematical problems aficionados: let

$$I(n) := \int_0^1 [x^n + (1-x)^n]^{\frac{1}{n}} dx,$$

what are  $I_0 := \lim I(n), I_2 := \lim n^2(I(n) - I_0), n \rightarrow \infty$ ? I found it interesting to look at a deeper asymptotic analysis of  $I(n)$  and found actually that, asymptotically,

$$I(n) = \sum_0^\infty \frac{I_i}{n^i}, n \rightarrow \infty,$$

where  $I_i$  are curiously obtained by polylog functions and Euler sums. In this paper we compute  $I_i, i = 0..5$ . As a corollary, we obtain explicit expressions for some integrals involving functions  $u^i, \exp(-u), (1+\exp(-u))^j, \ln(1+\exp(-u))^k$ . About polylog functions, see de Doelder, [2], Apostol, [1], Lewin, [6], and about Euler sums, see Flajolet, Salvy, [3], Xu, [7].

Another problem arose in some work in progress on dynamical systems by Gómez-Aiza and Ward [5]. Ward asked the following question: the polylog function is defined as

$$Li_m(z) := \sum_1^\infty \frac{z^n}{n^m}.$$

Set

$$S_0(z) := \frac{Li_m(1)}{Li_m(1) - Li_m(z)}$$

and

$$S_n := [z^n]S_0(z).$$

What is the asymptotic behaviour of  $S_n, n \rightarrow \infty$ ? In this paper, we provide the asymptotics of  $S_n, m = 3, 4$ , up to the  $1/n^3$  term. Next terms can be mechanically computed.

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## 2 A first analysis of $I_n$

We have

$$I(n) = 2 \int_0^{1/2} [x^n + (1-x)^n]^{\frac{1}{n}} dx = 2 \int_0^{1/2} (1-x) \left[ 1 + \left( \frac{x}{1-x} \right)^n \right]^{\frac{1}{n}} dx,$$

and

$$\begin{aligned} 0 \leq \frac{x}{1-x} \leq 1, \text{ let} \\ F(n) := \left[ 1 + \left( \frac{x}{1-x} \right)^n \right]^{\frac{1}{n}} \sim \exp \left[ \left( \frac{x}{1-x} \right)^n / n \right], \\ \left( \frac{x}{1-x} \right)^n / n \rightarrow 0, n \rightarrow \infty, \text{ exponentially if } x < 1/2, \text{ as } 1/n, \text{ if } x = 1/2. \end{aligned}$$

Hence the asymptotic behaviour of  $I(n)$  is related to the behaviour of  $F(n)$  in the neighbourhood of  $x = 1/2$ . We set  $x = 1/2 - y$  and get

$$I_0 = 2 \int_0^{1/2} (1-x) dx = \frac{3}{4}.$$

We now expand  $I_n$  up to the  $1/n^5$  term.

$$\begin{aligned} I(n) &= 2 \int_0^{1/2} \left( \frac{1}{2} + y \right) \left[ 1 + \left( \frac{1-2y}{1+2y} \right)^n \right]^{\frac{1}{n}} dy \\ &= 2 \int_0^{1/2} \left( \frac{1}{2} + y \right) \left[ 1 + (1-4y+8y^2-16y^3+32y^4+\mathcal{O}(y^5))^n \right]^{\frac{1}{n}} dy, \text{ and with } y = \frac{u}{4n}, \\ &= \frac{2}{4n} \int_0^{2n} \left( \frac{1}{2} + \frac{u}{4n} \right) \left[ 1 + (1-u/n+1/2u^2/n^2-1/4u^3/n^3+1/8u^4/n^4+\mathcal{O}(u^5/n^5))^n \right]^{\frac{1}{n}} du \\ &= \frac{2}{4n} \int_0^{2n} \left( \frac{1}{2} + \frac{u}{4n} \right) \left[ 1 + \exp(-u-1/12u^3/n^2+\mathcal{O}(u^5/n^4)) \right]^{\frac{1}{n}} du \\ &= \frac{2}{4n} \int_0^{2n} \left( \frac{1}{2} + \frac{u}{4n} \right) \left[ 1 + \exp(-u) - 1/12 \exp(-u)u^3/n^2 + \exp(-u)\mathcal{O}(u^5/n^4) \right]^{\frac{1}{n}} du \\ &= \frac{2}{4n} \int_0^{2n} \left( \frac{1}{2} + \frac{u}{4n} \right) \exp \left[ \ln(1+\exp(-u))/n - 1/12 \exp(-u)u^3/((1+\exp(-u))n^3) + \exp(-u)\mathcal{O}(u^5/n^5) \right] du \\ &= \frac{2}{4n} \int_0^{2n} \left( \frac{1}{2} + \frac{u}{4n} \right) \left[ 1 + \ln(1+\exp(-u))/n + 1/2 \ln(1+\exp(-u))^2/n^2 \right. \\ &\quad \left. + 1/12 \left[ -\exp(-u)u^3/(1+\exp(-u)) + 2 \ln(1+\exp(-u))^3 \right] / n^3 \right. \\ &\quad \left. + 1/24 \ln(1+\exp(-u)) \left[ -2 \exp(-u)u^3/(1+\exp(-u)) + \ln(1+\exp(-u))^3 \right] / n^4 \right. \\ &\quad \left. + \exp(-u)\mathcal{O}(u^5/n^5) \right] du \end{aligned}$$

$$\begin{aligned}
&= \frac{3}{4} + \int_0^\infty \left[ 1/4 \ln(1 + \exp(-u))/n^2 + \left[ 1/8u \ln(1 + \exp(-u)) + 1/8 \ln(1 + \exp(-u))^2 \right] /n^3 \right. \\
&+ \left. \left[ 1/16u \ln(1 + \exp(-u))^2 + 1/48 \left[ -\exp(-u)u^3/(1 + \exp(-u)) + 2 \ln(1 + \exp(-u))^3 \right] \right] /n^4 \right. \\
&+ 1/96 \left[ -\exp(-u)u^4/(1 + \exp(-u)) + 2u \ln(1 + \exp(-u))^3 \right. \\
&- \left. \left. 2 \ln(1 + \exp(-u)) \exp(-u)u^3/(1 + \exp(-u)) + \ln(1 + \exp(-u))^4 \right] /n^5 \right. \\
&\left. + \exp(-u)\mathcal{O}(u^6/n^6) \right] du.
\end{aligned}$$

We immediately recover  $I_0$ . The computation of  $I_i, i \geq 1$  is detailed in the next sections.

### 3 Computation of $I_1, I_2, I_3$

We have

$$I_1 = 0,$$

$$I_2 = \int_0^\infty 1/4 \ln(1 + \exp(-u)) du = 1/4 \sum_1^\infty \frac{(-1)^{i+1}}{i^2} = \frac{\pi^2}{48},$$

$$I_3 = \int_0^\infty [1/8u \ln(1 + \exp(-u)) + 1/8 \ln(1 + \exp(-u))^2] du = 11/32\zeta(3) + 1/8I \ln(2)^2\pi + 1/4 \ln(2)Li_2(2) - 1/4Li_3(2),$$

where the polylog function is defined by

$$Li_n(z) = \sum_1^\infty \frac{z^k}{k^n}.$$

But we know that

$$Li_n(z) = -(-1)^n Li_n(1/z) - \frac{(2\pi I)^n}{n!} B_n \left( \frac{1}{2} + \frac{\ln(-z)}{2\pi I} \right), z \notin [0, 1],$$

where  $B_n(x)$  is the  $n$ th Bernoulli polynomial, and

$$Li_2(1/2) = 1/12\pi^2 - 1/2 \ln(2)^2, \text{ hence } Li_2(2) = 1/4\pi^2 - I\pi \ln(2),$$

$$Li_3(1/2) = 7/8\zeta(3) - 1/12\pi^2 \ln(2) + 1/6 \ln(2)^3, \text{ hence } Li_3(2) = 7/8\zeta(3) + 1/4\pi^2 \ln(2) - 1/2I\pi \ln(2)^2.$$

The values of  $Li_k(1/2), k \geq 4$  are not known to be related to classical constants.

This leads to

$$I_3 = \frac{\zeta(3)}{8}.$$

Another, more elegant, way to compute  $I_3$  is to turn to Euler sums. Following Flajolet, Salvy, [3], we have

$$\begin{aligned}
S_{p,q}^{+-} &:= \sum_{k=1}^\infty (-1)^{k-1} \frac{H_k^{(p)}}{k^q}, \quad H_n^{(p)} := \sum_{j=1}^n \frac{1}{j^p}, \\
\bar{\zeta}(s) &:= (1 - 2^{1-s})\zeta(s), \quad \bar{\zeta}(1) := \ln(2),
\end{aligned}$$

$$\begin{aligned}
2S_{1,q}^{+-} &= (q+1)\bar{\zeta}(q+1) - \zeta(q+1) - 2 \sum_{k=1}^{q/2-1} \bar{\zeta}(k)\zeta(q+1-2k), 1+q \text{ odd}, \\
2S_{1,2}^{+-} &= \frac{5}{4}\zeta(3), \\
2S_{1,4}^{+-} &= 59/16\zeta(5) - 1/6\pi^2\zeta(3).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^\infty u \ln(1 + \exp(-u)) du &= \sum_1^\infty \frac{(-1)^{i+1}}{i^3} = \frac{3}{4}\zeta(3), \\
\int_0^\infty \ln(1 + \exp(-u))^2 du &= \sum_1^\infty \sum_1^\infty \frac{(-1)^{i+j}}{ij(i+j)} = \sum_{k=2}^\infty \sum_{i=1}^{k-1} \frac{(-1)^k}{ki(k-i)} = \sum_{k=2}^\infty \frac{(-1)^k}{k^2} 2H_{k-1} \\
&= 2 \sum_{k=1}^\infty \frac{(-1)^k}{k^2} H_k - 2 \sum_{k=1}^\infty \frac{(-1)^k}{k^3} = -2S_{1,2}^{+-} + \frac{3}{2}\zeta(3) = \frac{\zeta(3)}{4}.
\end{aligned}$$

We immediately recover  $I_3$ .

Another similar sum will be used in the next section: for  $p+q$  odd,

$$\begin{aligned}
S_{p,q}^{+-} &:= \left[ (1 - (-1)^p)\zeta(p)\bar{\zeta}(q) + \bar{\zeta}(p+q) + 2 \sum_{k=0}^{\lfloor p/2 \rfloor} \binom{q+p-2k-1}{q-1} (-1)^{p-2k+1} \bar{\zeta}(q+p-2k)\bar{\zeta}(2k) \right. \\
&\quad \left. + 2(-1)^p \sum_0^{\lfloor q/2 \rfloor} \binom{p+q-2k-1}{p-1} \zeta(p+q-2k)\bar{\zeta}(2k) \right] / 2, \\
S_{2,3}^{+-} &= -11/32\zeta(5) + 5/48\zeta(3)\pi^2.
\end{aligned}$$

## 4 Computation of $I_4$

Now we have

$$\begin{aligned}
S_1 &:= \int_0^\infty \frac{u^3 e^{-u}}{1 + e^{-u}} du = \frac{7\pi^4}{120}, \\
S_2 &:= \int_0^\infty \ln(1 + \exp(-u))^3 du = \ln(2)^3 \pi I + 3 \ln(2)^2 Li_2(2) - 6 \ln(2) Li_3(2) + 6 Li_4(2) - 1/15\pi^4 \\
&= 1/4\pi^2 \ln(2)^2 - 21/4 \ln(2)\zeta(3) - 6 Li_4(1/2) + 1/15\pi^4 - 1/4 \ln(2)^4, \\
S_3 &:= \int_0^\infty u \ln(1 + \exp(-u))^2 du = \sum_{k=2}^\infty \frac{(-1)^k}{k^3} 2H_{k-1} = 2 \sum_{k=1}^\infty \frac{(-1)^k}{k^3} H_k - 2 \sum_{k=1}^\infty \frac{(-1)^k}{k^4} = -2S_{1,3}^{+-} - 2 \left( -\frac{7}{720}\pi^4 \right), \\
S_{1,3}^{+-} &= -2 Li_4(1/2) + 11/4\zeta(4) + 1/2\zeta(2) \ln(2)^2 - 1/12 \ln(2)^4 - 7/4\zeta(3) \ln(2), \text{ this is } \mu_1 \text{ in [3]}, \\
S_3 &= 4 Li_4(1/2) - 1/24\pi^4 - 1/6\pi^2 \ln(2)^2 + 1/6 \ln(2)^4 + 7/2 \ln(2)\zeta(3).
\end{aligned}$$

Hence

$$I_4 := \frac{1}{48} [-S_1 + 2S_2 + 3S_3] = -\frac{\pi^4}{960}.$$

## 5 Computation of $I_5$

We compute

$$S_4 := \int_0^\infty \frac{u^4 e^{-u}}{1 + e^{-u}} du = 45/2\zeta(5).$$

Now we turn to  $S_5$ , which is the most intricate case of our integral expressions:

$$\begin{aligned} S_5 &:= \int_0^\infty u \ln(1 + \exp(-u))^3 du = \sum_{i=1}^\infty \sum_{j=1}^\infty \sum_{\ell=1}^\infty \frac{(-1)^{i+j+\ell+1}}{ij\ell(i+j+\ell)^2} = \sum_{k=3}^\infty \frac{(-1)^{k+1}}{k^2} \sum_{v=2}^{k-1} \frac{1}{k-v} \sum_{i=1}^{v-1} \frac{1}{i(v-i)} \\ &= \sum_{v=2}^\infty 2 \frac{H_{v-1}}{v} \sum_{k=v+1}^\infty \frac{(-1)^{k+1}}{k^2(k-v)} \\ &= 2 \sum_{v=2}^\infty \frac{(-1)^{k-1}}{k^2} \sum_{j=1}^{k-1} \frac{H_{j-1}}{j(k-j)} \\ &= 2 \sum_{v=2}^\infty \frac{(-1)^{k-1}}{k^3} \sum_{j=1}^{k-1} H_{j-1} \left[ \frac{1}{j} + \frac{1}{k-j} \right] \\ &= \sum_{v=2}^\infty \frac{(-1)^{k-1}}{k^3} \left[ H_{k-1}^2 - H_{k-1}^{(2)} + 2H_{k-1}^2 - 2H_{k-1}^{(2)} \right] \\ &= 3 \sum_{v=2}^\infty \frac{(-1)^{k-1}}{k^3} \left[ H_{k-1}^2 - H_{k-1}^{(2)} \right]. \end{aligned}$$

But

$$\begin{aligned} H_{n-1}^2/n^3 &= [H_n^2 - 2H_n/n + 1/n^2]/n^3, \\ H_{n-1}^{(2)}/n^3 &= [H_n^{(2)} - 1/n^2]/n^3. \end{aligned}$$

Hence

$$S_5 = -3(T_1 + T_2), \text{ with}$$

$$T_2 = - \left[ -S_{2,3}^{+-} + 15/16\zeta(5) \right] = 5/48\zeta(3)\pi^2 - \frac{41}{32}\zeta(5),$$

$$T_1 = T_3 + 2S_{1,4}^{+-} - 15/16\zeta(5),$$

$$T_3 = \sum_{k=1}^\infty (-1)^k \frac{H_k^2}{k^3}$$

$$= -(4Li_5(1/2) + 4\ln(2)Li_4(1/2) + 2/15\ln(2)^5 + 7/4\zeta(3)\ln(2)^2$$

$$- 19/32\zeta(5) - 2/3\zeta(2)\ln(2)^3 - 11/8\zeta(2)\zeta(3)), \text{ see [7], where many recent references can be found, so}$$

$$T_1 = -4Li_5(1/2) - 4\ln(2)Li_4(1/2) - 2/15\ln(2)^5 - 7/4\zeta(3)\ln(2)^2 + \frac{107}{32}\zeta(5) + 1/9\pi^2\ln(2)^3 + 1/16\zeta(3)\pi^2,$$

and finally

$$S_5 = 12Li_5(1/2) + 12\ln(2)Li_4(1/2) + 2/5\ln(2)^5 + 21/4\zeta(3)\ln(2)^2 - \frac{99}{16}\zeta(5) - 1/3\pi^2\ln(2)^3 - 1/2\zeta(3)\pi^2,$$

$$\begin{aligned} S_6 &:= \int_0^\infty \frac{u^3 e^{-u} \ln(1 + \exp(-u))}{1 + e^{-u}} du = \int_0^\infty u^3 \sum_{k=2}^\infty e^{-uk} (-1)^k \sum_{i=1}^{k-1} \frac{1}{i} du = 3! \sum_{k=2}^\infty \frac{1}{k^4} (-1)^k H_{k-1} \\ &= 3! \left[ \sum_{k=1}^\infty \frac{1}{k^4} (-1)^k H_k - \sum_{k=1}^\infty \frac{1}{k^5} (-1)^k \right] = 3! \left[ -S_{1,4}^{+-} + \frac{15}{16}\zeta(5) \right] = -87/16\zeta(5) + 1/2\pi^2\zeta(3), \end{aligned}$$

$$\begin{aligned}
S_7 &:= \int_0^\infty \ln(1 + \exp(-u))^4 du \\
&= 2/3\pi^2 \ln(2)^3 - 21/2 \ln(2)^2 \zeta(3) - 24 \ln(2) Li_4(1/2) - 4/5 \ln(2)^5 - 24 Li_5(1/2) + 24\zeta(5).
\end{aligned}$$

Hence

$$I_5 = \frac{1}{96} [-S_4 + 2S_5 - 2S_6 + S_7] = -1/48\zeta(3)\pi^2.$$

Let us summarize our results in the following theorem:

**Theorem 5.1** *Let*

$$I(n) := \int_0^1 [x^n + (1-x)^n]^{\frac{1}{n}} dx.$$

*We have*

$$I(n) = \sum_0^\infty \frac{I_i}{n^i}, n \rightarrow \infty, I_0 = \frac{3}{4}, I_1 = 0, I_2 = \frac{\pi^2}{48}, I_3 = \frac{\zeta(3)}{8}, I_4 = -\frac{\pi^4}{960}, I_5 = -1/48\zeta(3)\pi^2.$$

We also have the following corollary (many other similar results can be found in Xu [7])

**Corollary 5.2**

$$\begin{aligned}
\int_0^\infty 1/4 \ln(1 + \exp(-u)) du &= \frac{\pi^2}{48}, \\
\int_0^\infty u \ln(1 + \exp(-u)) du &= \frac{3}{4}\zeta(3), \\
\int_0^\infty \ln(1 + \exp(-u))^2 du &= \frac{\zeta(3)}{4}, \\
\int_0^\infty \frac{u^3 e^{-u}}{1 + e^{-u}} du &= \frac{7\pi^4}{120}, \\
\int_0^\infty \ln(1 + \exp(-u))^3 du &= 1/4\pi^2 \ln(2)^2 - 21/4 \ln(2)\zeta(3) - 6Li_4(1/2) + 1/15\pi^4 - 1/4 \ln(2)^4, \\
\int_0^\infty u \ln(1 + \exp(-u))^2 du &= 4Li_4(1/2) - 1/24\pi^4 - 1/6\pi^2 \ln(2)^2 + 1/6 \ln(2)^4 + 7/2 \ln(2)\zeta(3), \\
\int_0^\infty \frac{u^4 e^{-u}}{1 + e^{-u}} du &= 45/2\zeta(5), \\
\int_0^\infty u \ln(1 + \exp(-u))^3 du \\
&= 12Li_5(1/2) + 12 \ln(2) Li_4(1/2) + 2/5 \ln(2)^5 + 21/4\zeta(3) \ln(2)^2 - \frac{99}{16}\zeta(5) - 1/3\pi^2 \ln(2)^3 - 1/2\zeta(3)\pi^2, \\
\int_0^\infty \frac{u^3 e^{-u} \ln(1 + \exp(-u))}{1 + e^{-u}} du &= -87/16\zeta(5) + 1/2\pi^2\zeta(3), \\
\int_0^\infty \ln(1 + \exp(-u))^4 du \\
&= 2/3\pi^2 \ln(2)^3 - 21/2 \ln(2)^2 \zeta(3) - 24 \ln(2) Li_4(1/2) - 4/5 \ln(2)^5 - 24 Li_5(1/2) + 24\zeta(5).
\end{aligned}$$

We have two Open problem:

Open problem 1: how to explain the relatively simple  $I_i$  expressions?

Open problem 2: can we find ‘easily’ similar computations for  $I_i, i \geq 6$ ?

Let

$$\begin{aligned}
I_{n,2} &:= I_0 + I_2/n^2, \\
I_{n,3} &:= I_0 + I_2/n^2 + I_3/n^3, \\
I_{n,4} &:= I_0 + I_2/n^2 + I_3/n^3 + I_4/n^4.
\end{aligned}$$

To check the quality of our asymptotics, we display, in Figure 1,  $I_0, I(n), I_{n,2}, I_{n,3}, I_{n,4}$ .

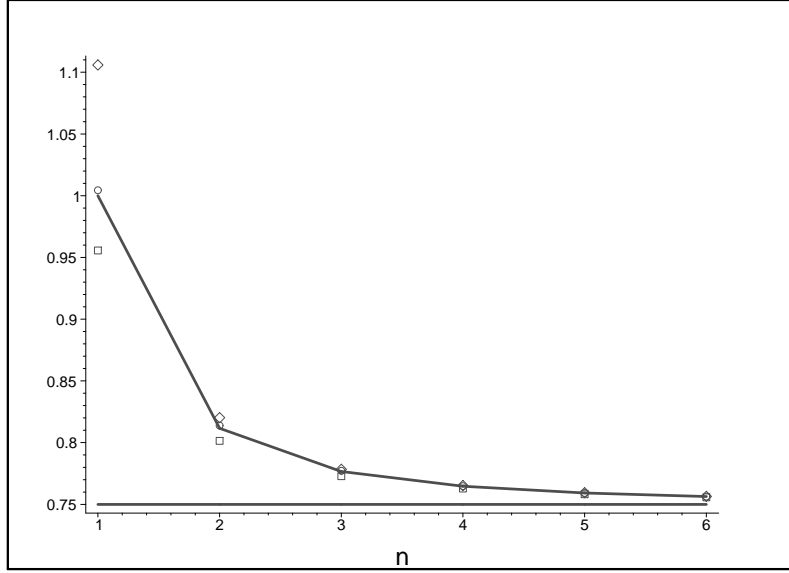


Figure 1:  $I(n)$ (line),  $I_0$ (line),  $I_{n,2}$ (box),  $I_{n,3}$ (diamond),  $I_{n,4}$ (circle)

## 6 A first analysis of $S_n$

We know that

$$Li_m(z) = \frac{(-1)^m}{(m-1)!} w^{m-1} (\ln(w) - H_{m-1}) + \sum_{j \neq m-1, j \geq 0} \frac{(-1)^j}{j!} \zeta(m-j) w^j, \text{ see [4], VI.20, with } \quad (1)$$

$$w := -\ln(z).$$

The singularity  $z = 1$  in  $S_0(z)$  leads to the desired expansion of  $S_n$ . Set

$$L := \ln \left( \frac{1}{1-z} \right) = Li_1(z),$$

$$L_{k,n} := [z^n] L^k, \text{ with}$$

$$L_{1,n} = \frac{1}{n}.$$

Let

$$\varepsilon := 1 - z,$$

$$S(\varepsilon, L) := \text{expansion of } S_0(1 - \varepsilon) \text{ w.r.t. } \varepsilon,$$

we compute successively

$$D_{i,j} := [\varepsilon^i L^j] S(\varepsilon, L), \text{ which depends on } m,$$

$$G_{i,j,n} := [z^n] \varepsilon^i L^j, \text{ which is independent of } m,$$

$$T_{i,j,n} = D_{i,j} G_{i,j,n} = [z^n \varepsilon^i L^j] S(\varepsilon, L),$$

$$S_n := [z^n] S_0(z),$$

$$T_n := \text{asymptotics of } S_n = \sum_i \sum_j T_{i,j,n}, n \rightarrow \infty.$$

We also define

$$C_{n,k} := \text{asymptotics of } S_n \text{ up to the } 1/n^k \text{ term.}$$

In this paper, we will compute  $C_{n,k}, k = 0..3, m = 3, 4$ , but as we will see, more terms can be mechanically obtained. We will also show some graphs of  $C_{n,k}$ .

## 7 Some asymptotics for $L_{k,n}$

Set

$$H := \Gamma(n + \alpha)/(\Gamma(\alpha)\Gamma(n + 1)), \text{ see [1], VI.7,}$$

$$\frac{\partial^2 H(\alpha)}{\partial \alpha^2} = [\psi(1, n + \alpha) + \psi(n + \alpha)^2 - 2\psi(n + \alpha)\psi(\alpha) + \psi(\alpha)^2 - \psi(1, \alpha)]\Gamma(n + \alpha)/(\Gamma(\alpha)\Gamma(n + 1)),$$

where  $\psi(n, x)$  is the  $n$ th polygamma function, which is the  $n$ th derivative of the digamma function, we have

$$L_{2,n} = \lim_{\alpha \rightarrow 0} \frac{\partial^2 H(\alpha)}{\partial \alpha^2} = (2\psi(n) + 2\gamma)/n,$$

$$L_{2,n} = (2\ln(n) + 2\gamma)/n - 1/n^2 - 1/(6n^3) + 1/(60n^5) + \mathcal{O}(1/n^6), \text{ see [1], Figure VI.5 for the first terms,}$$

$$\begin{aligned} \frac{\partial^3 H(\alpha)}{\partial \alpha^3} &= [\psi(2, n + \alpha) + 3\psi(1, n + \alpha)\psi(n + \alpha) - 3\psi(1, n + \alpha)\psi(\alpha) \\ &\quad + \psi(n + \alpha)^3 - 3\psi(n + \alpha)^2\psi(\alpha) + 3\psi(n + \alpha)\psi(\alpha)^2 \\ &\quad - 3\psi(n + \alpha)\psi(1, \alpha) - \psi(\alpha)^3 + 3\psi(\alpha)\psi(1, \alpha) - \psi(2, \alpha)]\Gamma(n + \alpha)/(\Gamma(\alpha)\Gamma(n + 1)), \end{aligned}$$

$$L_{3,n} = \lim_{\alpha \rightarrow 0} \frac{\partial^3 H(\alpha)}{\partial \alpha^3} = 1/2(6\psi(1, n) + 12\psi(n)\gamma - \pi^2 + 6\gamma^2 + 6\psi(n)^2)/n,$$

$$L_{3,n} = (3\ln(n)^2 + 6\ln(n)\gamma - 1/2\pi^2 + 3\gamma^2)/n + (3 - 3\ln(n) - 3\gamma)/n^2 + (-1/2\gamma + 9/4 - 1/2\ln(n))/n^3 + 3/(4n^4) + (1/20\gamma + 1/48 + 1/20\ln(n))/n^5 + \mathcal{O}(1/n^6).$$

## 8 The case $m = 3, 4$

We have, for  $m = 3$ , by (1),

$$Li_3(z) = -1/2w^2(\ln(w) - 3/2) + \zeta(3) - 1/6\pi^2w + 1/12w^3 - 1/288w^4 + 1/86400w^6 - 1/10160640w^8 + \mathcal{O}(w^9),$$

$$S_0(z) = \zeta(3)/(1/2w^2(\ln(w) - 3/2) + 1/6\pi^2w - 1/12w^3 + 1/288w^4 - 1/86400w^6 + 1/10160640w^8 + \mathcal{O}(w^9)),$$

we have the expansions

$$w = \varepsilon + 1/2\varepsilon^2 + 1/3\varepsilon^3 + 1/4\varepsilon^4 + \mathcal{O}(\varepsilon^5),$$

$$\ln(w) = -L + 1/2\varepsilon + 5/24\varepsilon^2 + 1/8\varepsilon^3 + \mathcal{O}(\varepsilon^4).$$

Hence

$$\begin{aligned} S(\varepsilon, L) &= 6\zeta(3)/(\pi^2\varepsilon) + 1/8\zeta(3)(-24\pi^6 + 144\pi^4L + 216\pi^4)/\pi^8 \\ &\quad + 1/8\zeta(3)(-4\pi^6 - 48\pi^4 + 432\pi^2L^2 + 1296\pi^2L + 972\pi^2)/\pi^8\varepsilon \\ &\quad + 1/8\zeta(3)(-2\pi^6 - 19\pi^4 + 360\pi^2L + 216\pi^2L^2 + 54\pi^2 + 1296L^3 + 5832L^2 + 8748L + 4374)/\pi^8\varepsilon^2 + \mathcal{O}(\varepsilon^3). \end{aligned}$$

This leads successively to

$$D_{-1,0} = 6\zeta(3)/\pi^2,$$

$$D_{0,1} = 18\zeta(3)/\pi^4,$$



$$\begin{aligned}
D_{1,1} &= 162\zeta(3)/\pi^6, \\
D_{1,2} &= 54\zeta(3)/\pi^6, \\
T_{-1,0,n} &= 6\zeta(3)/\pi^2, T_{-1,k,n} = 0, k > 0, \\
T_{0,1,n} &= 18\zeta(3)/(\pi^4 n), T_{0,k,n} = 0, k > 1,
\end{aligned}$$

The general form of  $G_{i,j}$  is given in [4], Equ. (27). The detailed computation goes as follows,

$$\begin{aligned}
G_{1,1,n} &= L_{1,n} - L_{1,n-1} = -1/n^2 - 1/n^3 - 1/n^4 - 1/n^5 + \mathcal{O}(1/n^6), \\
T_{1,1,n} &= D_{1,1}G_{1,1,n} = -162\zeta(3)/(\pi^6 n^2) - 162\zeta(3)/(\pi^6 n^3) + \mathcal{O}(1/n^4), \\
G_{1,2,n} &= L_{2,n} - L_{2,n-1} = (2 - 2\ln(n) - 2\gamma)/n^2 + (5 - 2\ln(n) - 2\gamma)/n^3 \\
&\quad + (43/6 - 2\ln(n) - 2\gamma)/n^4 + (55/6 - 2\ln(n) - 2\gamma)/n^5 + \mathcal{O}(1/n^6), \\
T_{1,2,n} &= D_{1,2}G_{1,2,n} = 54(2 - 2\ln(n) - 2\gamma)\zeta(3)/(\pi^6 n^2) + 54(5 - 2\ln(n) - 2\gamma)\zeta(3)/(\pi^6 n^3) + \mathcal{O}(1/n^4), \\
D_{2,3} &= 162\zeta(3)/\pi^8, \\
D_{2,2} &= 27\zeta(3)(27 + \pi^2)/\pi^8, \\
D_{2,1} &= 9/2\zeta(3)(10\pi^2 + 243)/\pi^8, \\
G_{2,3,n} &= L_{3,n} - 2L_{3,n-1} + L_{3,n-2} = (6 - 18\gamma - \pi^2 - 18\ln(n) + 6\gamma^2 + 6\ln(n)^2 + 12\ln(n)\gamma)/n^3 + \mathcal{O}(1/n^4), \\
G_{2,2,n} &= L_{2,n} - 2L_{2,n-1} + L_{2,n-2} = (-6 + 4\ln(n) + 4\gamma)/n^3 + \mathcal{O}(1/n^4), \\
G_{2,1,n} &= L_{1,n} - 2L_{1,n-1} + L_{1,n-2} = 2/n^3 + \mathcal{O}(1/n^4), \\
T_{2,3,n} &= D_{2,3}G_{2,3,n} = 162(6 - 18\gamma - \pi^2 - 18\ln(n) + 6\gamma^2 + 6\ln(n)^2 + 12\ln(n)\gamma)\zeta(3)/(n^3\pi^8) + \mathcal{O}(1/n^4), \\
T_{2,2,n} &= D_{2,2}G_{2,2,n} = 27(-6 + 4\ln(n) + 4\gamma)\zeta(3)(27 + \pi^2)/(n^3\pi^8) + \mathcal{O}(1/n^4), \\
T_{2,1,n} &= D_{2,1}G_{2,1,n} = 9\zeta(3)(10\pi^2 + 243)/(n^3\pi^8) + \mathcal{O}(1/n^4).
\end{aligned}$$

Finally

$$T_n = T_{-1,0,n} + T_{0,1,n} + \sum_i \sum_j T_{i,j,n}.$$

This leads to the following theorem:

### Theorem 8.1

Let

$$S_0(z) := \frac{Li_3(1)}{Li_3(1) - Li_3(z)}, \text{ then}$$

$$\begin{aligned}
S_n &:= [z^n]S_0(z) = 6\zeta(3)/\pi^2 + 18\zeta(3)/(\pi^4 n) + 3\zeta(3)(-18\pi^2 - 36\pi^2 \ln(n) - 36\pi^2 \gamma)/(\pi^8 n^2) \\
&\quad + 3\zeta(3)(-42\pi^2 - 405 + 324\gamma^2 + 324\ln(n)^2 + 648\ln(n)\gamma)/(\pi^8 n^3) + \mathcal{O}(1/n^4).
\end{aligned}$$

This gives

$$\begin{aligned}
C_{n,0} &= 6\zeta(3)/\pi^2, \\
C_{n,1} &= 6\zeta(3)/\pi^2 + 18\zeta(3)/(\pi^4 n), \\
C_{n,2} &= 6\zeta(3)/\pi^2 + 18\zeta(3)/(\pi^4 n) + 3\zeta(3)(-18\pi^2 - 36\pi^2 \ln(n) - 36\pi^2 \gamma)/(\pi^8 n^2), \\
C_{n,3} &= 6\zeta(3)/\pi^2 + 18\zeta(3)/(\pi^4 n) + 3\zeta(3)(-18\pi^2 - 36\pi^2 \ln(n) - 36\pi^2 \gamma)/(\pi^8 n^2) \\
&\quad + 3\zeta(3)(-42\pi^2 - 405 + 324\gamma^2 + 324\ln(n)^2 + 648\ln(n)\gamma)/(\pi^8 n^3).
\end{aligned}$$

To check the quality of our asymptotics, we display, in Figure 2,  $S_n, C_{n,0}, C_{n,1}, C_{n,2}, C_{n,3}$ .

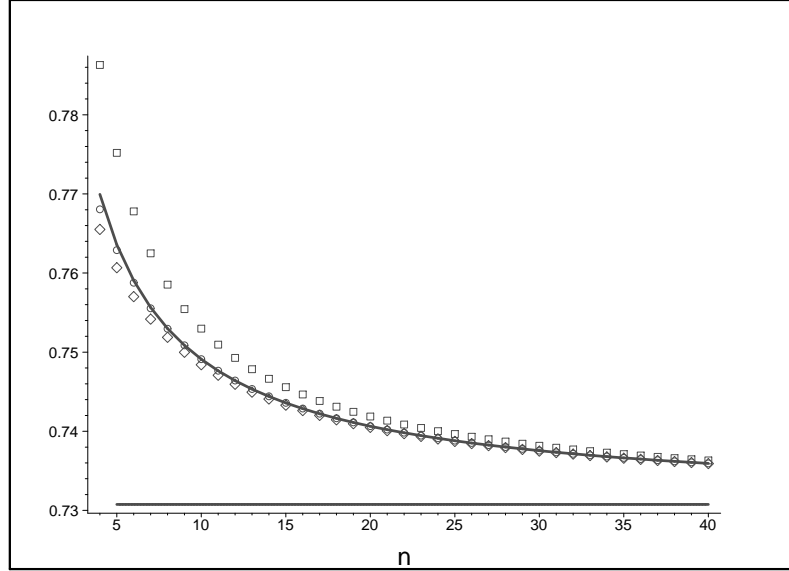


Figure 2:  $m = 3$ ,  $S_n$ (line),  $C_{n,0}$ (line),  $C_{n,1}$ (box),  $C_{n,2}$ (diamond),  $C_{n,3}$ (circle)

The convergence of  $S_n$  to  $C_{n,0}$  is rather slow: we have  $C_{n,0} = 0.7307629692\dots$  and  $S_{100} = 0.7329\dots$

The case  $m = 4$  is mechanically treated like the case  $m = 3$ . We obtain

$$\begin{aligned} C_{n,0} &= 1/90\pi^4/\zeta(3), \\ T_{0,1,n} &= 0, \\ C_{n,2} &= 1/90\pi^4/\zeta(3) + 1/540\pi^4/(\zeta(3)^2n^2), \\ C_{n,3} &= 1/90\pi^4/\zeta(3) + 1/540\pi^4(\zeta(3)^2n^2) - 1/1620\pi^6/(\zeta(3)^3n^3), \end{aligned}$$

we display, in Figure 3,  $S_n, C_{n,0}, C_{n,2}, C_{n,3}$ . The convergence of  $S_n$  to  $C_{n,0}$  is faster.

Remarks

1. in order to expand  $S_0(1 - \varepsilon)$ , we first expand w.r.t  $\varepsilon$  as  $\varepsilon = o(1/L^k), k > 0$
2.  $G_{k,j}$  starts with a  $1/n^{k+1}$  term, which allows an easy expansion
3. more and more terms in the expansion of  $S(\varepsilon, L)$  are needed when  $m$  increases: the first terms don't contain any  $L$  terms. For instance, for  $m = 6$ , only the  $\varepsilon^3$  contains a linear  $L$  contribution, and the asymptotics of  $S_n$  starts as  $1/945\pi^6/\zeta(5) + 1/18900\pi^6/(\zeta(5)^2n^4)$ . More terms can be mechanically computed.

## 9 Acknowledgements

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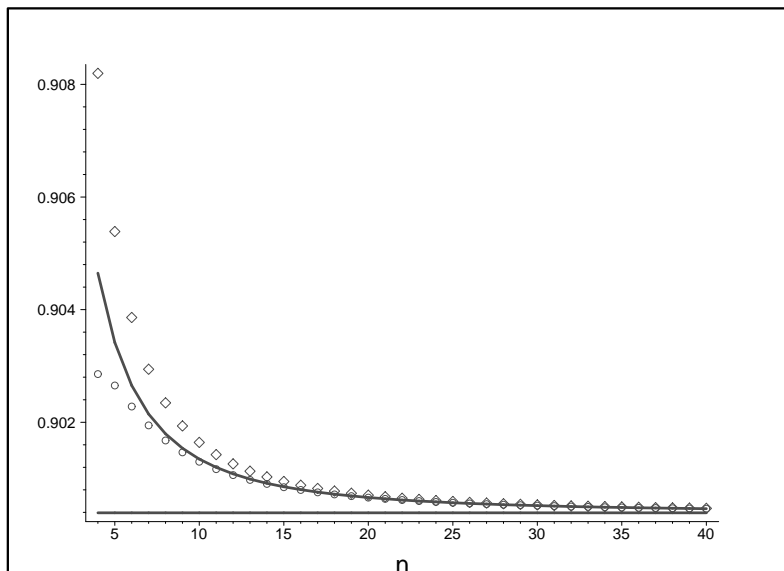


Figure 3:  $m = 4$ ,  $S_n$ (line),  $C_{n,0}$ (line),  $C_{n,2}$ (diamond),  $C_{n,3}$ (circle)

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