

An Asymptotic Series for an Integral

Michael E. Hoffman, Markus Kuba, Moti Levy, and Guy Louchard

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Abstract

We obtain an asymptotic series $\sum_{j=0}^{\infty} \frac{I_j}{n^j}$ for the integral $\int_0^1 [x^n + (1-x)^n]^{\frac{1}{n}} dx$ as $n \rightarrow \infty$, and compute I_j in terms of alternating (or “colored”) multiple zeta value. We also show that I_j is a rational polynomial the ordinary zeta values, and give explicit formulas for $j \leq 12$. As a byproduct, we obtain precise results about the convergence of norms of random variables and their moments. We study $\|(U, 1-U)\|_n$ as n tends to infinity and we also discuss $\|(U_1, U_2, \dots, U_r)\|_n$ for standard uniformly distributed random variables.

1 Introduction

Let

$$I(n) = \int_0^1 [x^n + (1-x)^n]^{\frac{1}{n}} dx. \quad (1)$$

We shall obtain an asymptotic series

$$I(n) = I_0 + \frac{I_1}{n} + \frac{I_2}{n^2} + \frac{I_3}{n^3} + \dots$$

This integral has been discussed in [9] (together with a different problem proposed by M.D. Ward). Therein, it is treated by a different approach using Euler sums and polylogarithms, leading to the first few terms I_0 up I_7 in terms of multiple zeta values.

Here, we give a complete expansion of $I(n)$. The coefficients I_k can be written in terms of alternating or “colored” multiple zeta values. The multiple zeta values are defined by

$$\zeta(i_1, \dots, i_k) = \sum_{n_1 > \dots > n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}$$

for positive integers i_1, \dots, i_k with $i_1 > 1$. This notation can be extended to alternating or “colored” multiple zeta values by putting a bar over those exponents with an associated sign in the numerator, as in

$$\zeta(\bar{3}, \bar{1}, 1) = \sum_{n_1 > n_2 > n_3 \geq 1} \frac{(-1)^{n_1+n_2}}{n_1^3 n_2 n_3}.$$

Note that $\zeta(a_1, a_2, \dots, a_k)$ converges unless a_1 is an unbarred 1. We have $\zeta(\bar{1}) = -\log 2$ and

$$\zeta(\bar{n}) = (2^{1-n} - 1)\zeta(n)$$

for $n \geq 2$. Alternating multiple zeta values have been extensively studied, and some identities for them are established in [2]. Our formula for I_k , $k \geq 2$, can be stated as

$$I_k = \frac{(-1)^k}{2} \sum_{j=2}^k E_{2\lfloor \frac{j-1}{2} \rfloor + 1}(0) \zeta(\bar{j}, \underbrace{1, \dots, 1}_{k-j}), \quad (2)$$

where E_n is the n th Euler polynomial. But in fact the right-hand side of Eq. (2) can always be rewritten as a rational polynomial in the ordinary zeta values $\zeta(i)$, $i \geq 2$. This follows from an identity of Kölbig [8] that relates alternating multiple zeta values $\zeta(\bar{n}, 1, \dots, 1)$ and multiple zeta values $\zeta(n, 1, \dots, 1)$.

After our main result, we interpret the integral $I(n)$ as the expected value of a certain random variable Z_n , defined in terms of the n th norm of the random vector $(U, 1 - U)$. Here, U denotes a standard uniformly distributed random variable. We complement our analysis of $I(n) = \mathbb{E}(Z_n)$ by studying the positive real moments $\mathbb{E}(Z_n^s)$ in terms of (alternating) multiple zeta values, as n tends to infinity. Moreover, we also discuss as a counterpart the n th norm of the random vector (U_1, U_2, \dots, U_r) for $r \geq 2$ and derive its moments in terms of multiple zeta values and related sums.

2 Main result: a complete expansion of $I(n)$

Because of the symmetry around $x = \frac{1}{2}$ in (1), one can write

$$I(n) = 2 \int_0^{\frac{1}{2}} [x^n + (1-x)^n]^{\frac{1}{n}} dx = 2 \int_0^{\frac{1}{2}} (1-x) \left[1 + \left(\frac{x}{1-x} \right)^n \right]^{\frac{1}{n}} dx.$$

Now let $u = \frac{x}{1-x}$, or $x = \frac{u}{1+u}$. Then $dx = \frac{du}{(1+u)^2}$, and we have

$$I(n) = 2 \int_0^1 \left(1 - \frac{u}{1+u} \right) (1+u^n)^{\frac{1}{n}} \frac{du}{(1+u)^2} = 2 \int_0^1 (1+u^n)^{\frac{1}{n}} \frac{du}{(1+u)^3}.$$

Writing $(1+u^n)^{\frac{1}{n}}$ as $\exp\left(\frac{1}{n} \log(1+u^n)\right)$ and expanding the exponential in series, we have

$$I(n) = 2 \int_0^1 \left(1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{1}{n} \log(1+u^n) \right)^k \right) \frac{du}{(1+u)^3}.$$

Now we can write (see [5, p. 351])

$$(\log(1+x))^k = k! \sum_{m=1}^{\infty} \frac{x^m}{m!} s(m, k), \quad (3)$$

where the $s(m, k)$ are (signed) Stirling numbers of the first kind. Hence

$$\begin{aligned} I(n) &= 2 \int_0^1 \frac{du}{(1+u)^3} + 2 \sum_{k=1}^{\infty} \int_0^1 k! \sum_{m=1}^{\infty} \frac{u^{mn} s(m, k)}{m! n^k k!} \frac{du}{(1+u)^3} \\ &= \frac{3}{4} + 2 \sum_{k=1}^{\infty} \frac{1}{n^k} \sum_{m=1}^{\infty} \frac{s(m, k)}{m!} \int_0^1 \frac{u^{mn}}{(1+u)^3} du. \end{aligned}$$

If we let $\zeta_r(i_1, \dots, i_k)$ denote the truncated multiple zeta value

$$\zeta_r(i_1, \dots, i_k) = \sum_{r \geq n_1 > n_2 > \dots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \dots n_k^{i_k}},$$

then we have the following relation, which is well-known although perhaps not in this notation (cf. [1]).

Lemma 1. For positive integers $m \geq k$,

$$s(m, k) = (-1)^{m-k} (m-1)! \zeta_{m-1}(\{1\}_{k-1}),$$

where $\{1\}_m$ means 1 repeated m times.

Proof. From the relation

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^n s(n, k) x^k$$

it follows that $s(n, k) = (-1)^{n-k} e_{n-k}(1, 2, \dots, n-1)$, where e_j is the j th elementary symmetric function. Divide by $(n-1)!$ to get

$$\frac{s(n, k)}{(n-1)!} = (-1)^{n-k} \frac{e_{n-k}(1, 2, \dots, n-1)}{(n-1)!} = (-1)^{n-k} e_{k-1} \left(1, \frac{1}{2}, \dots, \frac{1}{n-1}\right),$$

and the conclusion follows since evidently $\zeta_{n-1}(\{1\}_{n-k}) = e_{k-1} \left(1, \frac{1}{2}, \dots, \frac{1}{n-1}\right)$. \square

Thus

$$I(n) = \frac{3}{4} + 2 \sum_{k=1}^{\infty} \frac{1}{n^k} \sum_{m=1}^{\infty} (-1)^{m-k} \frac{\zeta_{m-1}(\{1\}_{k-1})}{m} \int_0^1 \frac{u^{mn}}{(1+u)^3} du. \quad (4)$$

If we write

$$\int_0^1 \frac{u^r}{(1+u)^3} du = \sum_{j=1}^{\infty} \frac{\beta_{j-1}}{r^j},$$

then the β_j can be computed explicitly as follows.

Lemma 2.

$$\beta_j = \frac{(-1)^j}{4} (E_{j+1}(-1) + E_{j+2}(-1)),$$

where the E_j are Euler polynomials.

Proof. Making the change of variable $u = e^{-t}$, we have

$$\int_0^1 \frac{u^r}{(1+u)^3} du = \int_0^{\infty} \frac{e^{-t}}{(1+e^{-t})^3} e^{-rt} dt.$$

By direct computation

$$\frac{e^{-t}}{(1+e^{-t})^3} = \frac{1}{4} \left[\frac{d^2}{dt^2} \left(\frac{2e^t}{1+e^{-t}} \right) - \frac{d}{dt} \left(\frac{2e^t}{1+e^{-t}} \right) \right].$$

The generating function of the Euler polynomials is defined by

$$\mathcal{E}(t, x) = \frac{2e^{tx}}{1+e^t} = \sum_{j \geq 0} E_j(x) \frac{t^j}{j!}. \quad (5)$$

Differentiating $\mathcal{E}(-t, -1)$ gives

$$\frac{d}{dt} \left(\frac{2e^t}{1+e^{-t}} \right) = - \sum_{n=0}^{\infty} (-1)^n E_{n+1}(-1) \frac{t^n}{n!}$$

and

$$\frac{d^2}{dt^2} \left(\frac{2e^t}{1+e^{-t}} \right) = \sum_{n=0}^{\infty} (-1)^n E_{n+2}(-1) \frac{t^n}{n!}.$$

Hence

$$\begin{aligned} \int_0^{\infty} \frac{e^{-t}}{(1+e^{-t})^3} e^{-rt} dt &= \sum_{n=0}^{\infty} \frac{(-1)^n}{4} (E_{n+2}(-1) + E_{n+1}(-1)) \int_0^{\infty} \frac{t^n}{n!} e^{-rt} dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{4} (E_{n+2}(-1) + E_{n+1}(-1)) \frac{1}{r^{n+1}}, \end{aligned}$$

from which the conclusion follows. \square

The well-known identity

$$E_n(x) + E_n(x+1) = 2x^n \quad (6)$$

gives $E_n(-1) = 2(-1)^n - E_n(0)$, so that

$$E_{j+1}(-1) + E_{j+2}(-1) = -E_{j+1}(0) - E_{j+2}(0).$$

But $E_n(0) = 0$ for n even, so we have

$$\beta_j = \begin{cases} \frac{1}{4} E_{j+2}(0), & \text{if } j \text{ is odd,} \\ -\frac{1}{4} E_{j+1}(0), & \text{if } j \text{ is even,} \end{cases}$$

or more succinctly $\beta_j = (-1)^{j+1} \frac{1}{4} E_{2\lfloor \frac{j+1}{2} \rfloor + 1}(0)$. If we set $a_n = \frac{1}{2} E_{2n+1}(0)$, then $2(-1)^{j-1} \beta_j = a_{\lfloor \frac{j+1}{2} \rfloor}$. The a_n can be written in terms of Bernoulli numbers as

$$a_n = \frac{(1 - 2^{2n+2}) B_{2n+2}}{2n + 2},$$

and we also have the exponential generating function

$$\sum_{n=0}^{\infty} a_n \frac{t^{2n+1}}{(2n+1)!} = -\frac{1}{2} \tanh \frac{t}{2}.$$

The first few a_j are

$$a_0 = -\frac{1}{4}, \quad a_1 = \frac{1}{8}, \quad a_2 = -\frac{1}{4}, \quad a_3 = \frac{17}{16}, \quad a_4 = -\frac{31}{4}, \quad a_5 = \frac{691}{8}, \quad a_6 = -\frac{5461}{4}.$$

Using Eq. (4) we can write

$$\begin{aligned} I(n) &= \frac{3}{4} + 2 \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{m-k} \frac{\zeta_{m-1}(\{1\}_{k-1}) \beta_{j-1}}{n^{j+k} m^{j+1}} \\ &= \frac{3}{4} + 2 \sum_{p=2}^{\infty} \frac{1}{n^p} \sum_{m=1}^{\infty} \sum_{k=1}^{p-1} (-1)^{m-k} \frac{\zeta_{m-1}(\{1\}_{k-1}) \beta_{p-k-1}}{m^{p-k+1}} \\ &= \frac{3}{4} + 2 \sum_{p=2}^{\infty} \frac{1}{n^p} \sum_{k=1}^{p-1} (-1)^k \beta_{p-k-1} \zeta(\overline{p-k+1}, \{1\}_{k-1}), \end{aligned}$$

from which we see that $I_0 = \frac{3}{4}$, $I_1 = 0$, and

$$I_p = 2 \sum_{k=1}^{p-1} (-1)^k \beta_{p-k-1} \zeta(\overline{p-k+1}, \{1\}_{k-1}) = 2 \sum_{j=2}^p (-1)^{p-j-1} \beta_{j-2} \zeta(\bar{j}, \{1\}_{p-j})$$

for $p \geq 2$. We have proved the following result.

Theorem 1. For $p \geq 2$,

$$I_p = (-1)^p \sum_{j=2}^p a_{\lfloor \frac{j+1}{2} \rfloor} \zeta(\bar{j}, \{1\}_{p-j}),$$

where $a_n = \frac{1}{2} E_{2n+1}(0) = (1 - 2^{2n+2}) B_{2n+2} / (2n + 2)$.

The first two cases are as follows.

$$I_2 = a_0\zeta(\bar{2}) = \frac{1}{8}\zeta(2)$$

$$I_3 = -a_0\zeta(\bar{2}, 1) - a_1\zeta(\bar{3}) = \frac{1}{4} \cdot \frac{\zeta(3)}{8} + \frac{1}{8} \cdot \frac{3}{4}\zeta(3) = \frac{1}{8}\zeta(3).$$

In all further computations, expressions for alternating multiple zeta values are simplified using the Multiple Zeta Value Data Mine [3]. By Theorem 1,

$$I_4 = a_0\zeta(\bar{2}, 1, 1) + a_1\zeta(\bar{3}, 1) + a_2\zeta(\bar{4}) = -\frac{1}{4}\zeta(\bar{2}, 1, 1) + \frac{1}{8}\zeta(\bar{3}, 1) + \frac{1}{8}\zeta(\bar{4}),$$

and since $\zeta(\bar{4}) = -\frac{7}{8}\zeta(4)$, $\zeta(\bar{2}, 1, 1) = -\frac{1}{16}\zeta(4) + \frac{1}{2}\zeta(\bar{3}, 1)$, this implies $I_4 = -\frac{3}{32}\zeta(4)$. Similarly,

$$I_5 = -a_0\zeta(\bar{2}, 1, 1, 1) - a_1\zeta(\bar{3}, 1, 1) - a_1\zeta(\bar{4}, 1) - a_2\zeta(\bar{5}) =$$

$$\frac{1}{4}\zeta(\bar{2}, 1, 1, 1) - \frac{1}{8}\zeta(\bar{3}, 1, 1) - \frac{1}{8}\zeta(\bar{4}, 1) + \frac{1}{4}\zeta(\bar{5}).$$

Now $\zeta(\bar{5}) = -\frac{15}{16}\zeta(5)$, and from [3]

$$\zeta(\bar{4}, 1) = -\frac{29}{32}\zeta(5) + \frac{1}{2}\zeta(2)\zeta(3)$$

$$\zeta(\bar{2}, \{1\}_3) = \frac{31}{64}\zeta(5) - \frac{1}{4}\zeta(2)\zeta(3) + \frac{1}{2}\zeta(\bar{3}, 1, 1),$$

giving the result $I_5 = -\frac{1}{8}\zeta(2)\zeta(3)$.

Here, without further details, are I_j for $j = 6, 7, 8, 9, 10, 11, 12$.

$$\begin{aligned}
I_6 &= \frac{83}{256}\zeta(6) - \frac{1}{16}\zeta(3)^2 \\
I_7 &= \frac{3}{16}\zeta(7) + \frac{27}{64}\zeta(3)\zeta(4) + \frac{3}{16}\zeta(2)\zeta(5) \\
I_8 &= -\frac{2533}{1536}\zeta(8) + \frac{3}{16}\zeta(3)\zeta(5) + \frac{5}{32}\zeta(2)\zeta(3)^2 \\
I_9 &= -\frac{5}{6}\zeta(9) - \frac{289}{128}\zeta(3)\zeta(6) - \frac{135}{64}\zeta(4)\zeta(5) - \frac{9}{8}\zeta(2)\zeta(7) + \frac{5}{96}\zeta(3)^3 \\
I_{10} &= \frac{293937}{20480}\zeta(10) - \frac{87}{32}\zeta(3)\zeta(7) - \frac{9}{16}\zeta(5)^2 - \frac{81}{64}\zeta(3)^2\zeta(4) - \frac{21}{16}\zeta(2)\zeta(3)\zeta(5) \\
I_{11} &= \frac{63}{8}\zeta(11) + \frac{58007}{3072}\zeta(3)\zeta(8) + \frac{5187}{256}\zeta(5)\zeta(6) + \frac{135}{8}\zeta(4)\zeta(7) + \frac{115}{12}\zeta(2)\zeta(9) \\
&\quad - \frac{13}{48}\zeta(2)\zeta(3)^3 - \frac{21}{32}\zeta(3)^2\zeta(5) \\
I_{12} &= -\frac{2095281645}{11321344}\zeta(12) + \frac{115}{12}\zeta(3)\zeta(9) + \frac{81}{8}\zeta(5)\zeta(7) + \frac{5765}{512}\zeta(3)^2\zeta(6) \\
&\quad + \frac{1323}{64}\zeta(3)\zeta(4)\zeta(5) + \frac{45}{4}\zeta(2)\zeta(3)\zeta(7) + \frac{45}{8}\zeta(2)\zeta(5)^2 - \frac{13}{192}\zeta(3)^4
\end{aligned}$$

In fact, the I_n are always rational polynomials in the ordinary zeta values $\zeta(i)$, in consequence of the following result.

Theorem 2. For $p \geq 2$,

$$I_p = \frac{(-1)^p}{2} \sum_{k=1}^{p-1} (-1)^k \zeta(k+1, \{1\}_{p-k-1}) \sum_{j=0}^{k-1} \binom{k-1}{j} a_{\lfloor \frac{p-1-j}{2} \rfloor}.$$

The proof makes use of an identity of Kölbig [8], which is phrased in terms of the integral

$$S_{n,p}(z) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{\log^{n-1}(t) \log^p(1-zt)}{t} dt.$$

But $S_{n,p}(z)$ can be written as a multiple zeta value if $z = 1$, and as an alternating multiple zeta value if $z = -1$. The key is the following result.

Lemma 3. If $|z| \leq 1$, then

$$\frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{\log^{n-1}(t) \log^p(1-zt)}{t} dt = \sum_{j_1 > j_2 > \dots > j_p \geq 1} \frac{z^{j_1}}{j_1^{n+1} j_2 \dots j_p}.$$

Proof. Since

$$\log(1 - zt) = - \sum_{i \geq 1} \frac{z^i t^i}{i} \quad \text{and} \quad \int_0^1 t^{m-1} \log^{n-1}(t) dt = \frac{(n-1)!}{m^n},$$

we have

$$\begin{aligned} & \int_0^1 \frac{\log^{n-1}(t) \log^p(1 - zt)}{t} dt \\ &= (-1)^p \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} \int_0^1 \frac{z^{i_1+\cdots+i_p} t^{i_1+\cdots+i_p-1} \log^{n-1}(t)}{i_1 i_2 \cdots i_p} dt \\ &= (-1)^p \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} \frac{(-1)^{n-1} (n-1)! z^{i_1+\cdots+i_p}}{i_1 i_2 \cdots i_p (i_1 + \cdots + i_p)^n}. \end{aligned}$$

By [6, Lemma 4.3], this is

$$(-1)^p \sum_{j_1 > j_2 > \cdots > j_p \geq 1} \frac{(-1)^{n-1} (n-1)! p! z^{j_1}}{j_1^{n+1} j_2 \cdots j_p}$$

and the conclusion follows. \square

It then follows from definitions that

$$S_{n,p}(1) = \zeta(n+1, \{1\}_{p-1}) \quad \text{and} \quad S_{n,p}(-1) = \zeta(\overline{n+1}, \{1\}_{p-1}).$$

In [8] Kölbig refers to $S_{n,p}(1)$ as $s_{n,p}$ and $S_{n,p}(-1)$ as $(-1)^p \sigma_{n,p}$; the result we need is [8, Theorem 3], which reads

$$\sum_{j=1}^n \binom{n+p-j-1}{p-1} \sigma_{j,n+p-j} + \sum_{j=1}^p \binom{n+p-j-1}{n-1} \sigma_{j,n+p-j} = s_{n,p}. \quad (7)$$

Proof of Theorem 2. Note that we can rewrite Theorem 1 as

$$I_p = \sum_{i=1}^{p-1} (-1)^i a_{\lfloor \frac{i}{2} \rfloor} \sigma_{i,p-i}$$

and Eq. (7) as

$$\sum_{i=1}^{p-1} \left(\binom{p-i-1}{p-j-1} + \binom{p-i-1}{j-1} \right) \sigma_{i,p-i} = s_{j,p-j}.$$

If we can find ρ_j so that

$$\begin{aligned} \sum_{j=1}^{p-1} \rho_j s_{j,p-j} &= \sum_{j=1}^{p-1} \sum_{i=1}^{p-1} \rho_j \left(\binom{p-i-1}{p-j-1} + \binom{p-i-1}{j-1} \right) \sigma_{i,p-i} \\ &= \sum_{i=1}^{p-1} \sigma_{i,p-i} \sum_{j=1}^{p-1} \rho_j \left(\binom{p-i-1}{p-j-1} + \binom{p-i-1}{j-1} \right) = I_p, \end{aligned}$$

i.e.,

$$\sum_{j=1}^{p-1} \rho_j \left(\binom{p-i-1}{p-j-1} + \binom{p-i-1}{j-1} \right) = (-1)^i a_{\lfloor \frac{i}{2} \rfloor} \quad (8)$$

for $i = 1, 2, \dots, p-1$, then I_p can be written in terms of the $s_{m,n}$. Now Eqs. (8) can be written

$$\sum_{j=1}^{p-i} \rho_{p-j} \binom{p-i-1}{j-1} + \sum_{j=1}^{p-i} \rho_j \binom{p-i-1}{j-1} = (-1)^i a_{\lfloor \frac{i}{2} \rfloor}, \quad 1 \leq i \leq p-1,$$

and if we make the condition $\rho_{p-j} = \rho_j$, this becomes

$$\sum_{j=1}^{p-i} \rho_j \binom{p-i-1}{j-1} = \frac{(-1)^i}{2} a_{\lfloor \frac{i}{2} \rfloor}, \quad 1 \leq i \leq p-1, \quad (9)$$

Restrict the system (9) to the last $\lfloor \frac{p}{2} \rfloor$ equations ($i = \lfloor \frac{p+1}{2} \rfloor, \dots, p-1$) and use binomial inversion to get

$$\rho_k = \frac{(-1)^{p+k}}{2} \sum_{j=0}^{k-1} a_{\lfloor \frac{p-j-1}{2} \rfloor} \binom{k-1}{j}, \quad 1 \leq k \leq \lfloor \frac{p}{2} \rfloor. \quad (10)$$

We claim that ρ_k so defined, if the definition is extended to $1 \leq k \leq p-1$, is also a solution of the first $\lfloor \frac{p-1}{2} \rfloor$ equations of (9). The conclusion then follows.

To prove the claim, it is enough to show that the extension of Eqn. (10) to $1 \leq k \leq p-1$ is consistent with the condition $\rho_{p-k} = \rho_k$, i.e., that

$$\frac{(-1)^k}{2} \sum_{j=0}^{p-k-1} a_{\lfloor \frac{p-j-1}{2} \rfloor} \binom{p-k-1}{j} = \frac{(-1)^{p+k}}{2} \sum_{j=0}^{k-1} a_{\lfloor \frac{p-j-1}{2} \rfloor} \binom{k-1}{j},$$

or, using the definition of a_n ,

$$\sum_{j=0}^{p-k-1} E_{2^{\lfloor \frac{p-j-1}{2} \rfloor + 1}}(0) \binom{p-k-1}{j} = (-1)^p \sum_{j=0}^{k-1} E_{2^{\lfloor \frac{p-j-1}{2} \rfloor + 1}}(0) \binom{k-1}{j}.$$

By considering the cases p odd and p even, we see this can be written

$$\sum_{j=0}^{p-k} E_{p-j}(0) \binom{p-k}{j} = (-1)^p \sum_{j=0}^k E_{p-j}(0) \binom{k}{j}.$$

The result then follows from taking $n = p - k$ in Lemma 4 below.

Lemma 4. *For nonnegative integers n, k ,*

$$\sum_{j=0}^n E_{k+j}(0) \binom{n}{j} = (-1)^{n+k} \sum_{j=0}^k E_{n+j}(0) \binom{k}{j}$$

Proof. Start with

$$\sum_{j=0}^n E_j(0) \binom{n}{j} = -E_n(0)$$

which follows from setting $x = 0$ in the identity (6). Since $E_n(0) = 0$ for n even, we can write this as

$$\sum_{j=0}^n E_j(0) \binom{n}{j} = (-1)^n E_n(0),$$

which is the case $k = 0$ of the conclusion. We can then use it as the base

case of a proof of the conclusion by induction on k . We have

$$\begin{aligned}
(-1)^{n+k+1} \sum_{j=0}^{k+1} E_{n+j}(0) \binom{k+1}{j} &= \\
(-1)^{n+k+1} \left[\sum_{j=1}^{k+1} E_{n+j}(0) \binom{k}{j-1} + \sum_{j=0}^k E_{n+j}(0) \binom{k}{j} \right] &= \\
(-1)^{n+k+1} \left[\sum_{j=0}^k E_{n+1+j}(0) \binom{k}{j} + \sum_{j=0}^k E_{n+j}(0) \binom{k}{j} \right] &= \\
\sum_{j=0}^{n+1} E_{k+j}(0) \binom{n+1}{j} - \sum_{j=0}^n E_{k+j}(0) \binom{n}{j} &= \sum_{j=1}^{n+1} E_{k+j}(0) \binom{n}{j-1} \\
&= \sum_{j=0}^n E_{k+1+j}(0) \binom{n}{j}.
\end{aligned}$$

□

Corollary 1. For $p \geq 2$, I_p is a rational polynomial in the $\zeta(i)$.

Proof. For any positive integers n, m the multiple zeta value $\zeta(n+1, \{1\}_m)$ is a rational polynomial in the $\zeta(i)$, as follows from [2, Eq. (10)]. Then Theorem 2 implies the conclusion. □

3 Applications: convergence of norms

Let $U = \text{Uniform}[0, 1]$ denote a standard uniformly distributed random variable. Furthermore, for positive real n we define random variables Z_n by

$$Z_n = \|(U, 1 - U)\|_n = (U^n + (1 - U)^n)^{\frac{1}{n}}.$$

From the theory of norms we expect that the limit Z_∞ exists and

$$Z_\infty = \|(U, 1 - U)\|_\infty = \max\{U, 1 - U\}.$$

It is known that $\max\{U, 1 - U\} = \text{Uniform}[\frac{1}{2}, 1]$. It turns out that our previous considerations allow to refine this intuition. The integral $I(n)$ treated in detail before is exactly the expected value of Z_n . In the following we give asymptotic expansion of all positive real moments of Z_n .

Theorem 3. *The random variable Z_n , defined in terms of $U = \text{Uniform}[0, 1]$, converges for $n \rightarrow \infty$ in distribution and with convergence of all integer moments,*

$$Z_n = (U^n + (1 - U)^n)^{\frac{1}{n}} \rightarrow Z_\infty = \max\{U, 1 - U\},$$

For positive integer $s \geq 1$ we have

$$\begin{aligned} \mathbb{E}(Z_n^s) &= \frac{2(1 - \frac{1}{2^{s+1}})}{s + 1} + \\ &\sum_{p=2}^{\infty} \frac{(-1)^p}{n^p} \sum_{k=1}^{p-1} \frac{s^k}{s + 1} \sum_{j=1}^{s+1} \gamma_{s+1,j} (-1)^{j-1} E_{p-k+j-1}(0) \zeta(\overline{p+1-k}, \{1\}_{k-1}), \end{aligned}$$

where the values $\gamma_{s+1,j}$ are given by $\frac{(-1)^{j-1} \binom{s+1}{j}}{s!} = (-1)^{j-1} \zeta_s(\{1\}_{j-1})$.

For arbitrary positive real $s > 0$ we have

$$\begin{aligned} \mathbb{E}(Z_n^s) &= \frac{2(1 - \frac{1}{2^{s+1}})}{s + 1} + \sum_{p=2}^{\infty} \frac{(-1)^p}{n^p} \sum_{k=1}^{p-1} \frac{s^k}{s + 1} \zeta(\overline{p+1-k}, \{1\}_{k-1}) \\ &\times \sum_{\ell=1}^{p-k} (s + 1)^\ell B_{p-k,\ell}(E_1(0), \dots, E_{p-k-\ell+1}(0)), \end{aligned}$$

where $B_{n,k}(x_1, \dots, x_{n+1-k})$ denote the Bell polynomials.

A first by product of our moment expansions is a rate of convergence.

Corollary 2. *The distribution functions $F_n(x) = \mathbb{P}\{Z_n \leq x\}$ and $F_\infty(x) = \mathbb{P}\{Z_\infty \leq x\}$ satisfy*

$$\sup_{x \in \mathbf{R}} |F_n(x) - F_\infty(x)| \leq \frac{C}{n}.$$

We also can directly strengthen to almost-sure convergence.

Corollary 3. *The random variable $Z_n = (U^n + (1 - U)^n)^{\frac{1}{n}}$ converges almost surely to $Z_\infty = \max\{U, 1 - U\}$.*

Remark 1. We obtain in a similar way moment convergence of random variables

$$Z_n = (B^n + (1 - B)^n)^{\frac{1}{n}},$$

with B denoting a $Beta(\alpha, \beta)$ distributed random variable with real $\alpha, \beta > 0$, generalizing our results above (case $\alpha = \beta = 1$).

We note that

$$\mathbb{E}(Z_n^s) = \int_{\Omega} \left((U^n + (1-U)^n)^{\frac{1}{n}} \right)^s d\mathbb{P} = \int_0^1 \left(x^n + (1-x)^n \right)^{\frac{s}{n}} dx.$$

Proceeding as before we use the symmetry of the integrand.

$$\mathbb{E}(Z_n^s) = 2 \int_0^{\frac{1}{2}} (1-x)^s \left[1 + \left(\frac{x}{1-x} \right)^n \right]^{\frac{s}{n}} dx.$$

Substituting again $u = \frac{x}{1-x}$, or $x = \frac{u}{1+u}$, leads to

$$\mathbb{E}(Z_n^s) = 2 \int_0^1 \left(1 - \frac{u}{1+u} \right)^s (1+u^n)^{\frac{s}{n}} \frac{du}{(1+u)^2} = 2 \int_0^1 (1+u^n)^{\frac{s}{n}} \frac{du}{(1+u)^3}.$$

Writing $(1+u^n)^{\frac{s}{n}}$ as $\exp\left(\frac{s}{n} \log(1+u^n)\right)$ and expanding the exponential in series, we have

$$\mathbb{E}(Z_n^s) = 2 \int_0^1 \left(1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{s}{n} \log(1+u^n) \right)^k \right) \frac{du}{(1+u)^{s+2}}.$$

As before,

$$\begin{aligned} \mathbb{E}(Z_n^s) &= 2 \int_0^1 \frac{du}{(1+u)^{s+2}} + 2 \sum_{k=1}^{\infty} \int_0^1 k! \sum_{m=1}^{\infty} \frac{u^{mn} s(m,k) s^k}{m! n^k k!} \frac{du}{(1+u)^{s+2}} \\ &= \frac{2(1 - \frac{1}{2^{s+1}})}{s+1} + \sum_{k=1}^{\infty} \frac{s^k}{n^k} \sum_{m=1}^{\infty} \frac{(-1)^{m-k} \zeta_{m-1}(\{1\}_{k-1})}{m} \int_0^1 \frac{2u^{mn}}{(1+u)^{s+2}} du. \end{aligned}$$

It remains to expand the integral into powers of n . Make the substitution $u = e^{-t}$ and then integrate by parts:

$$\begin{aligned} \int_0^1 \frac{2u^{mn}}{(1+u)^{s+2}} du &= \int_0^{\infty} \frac{2e^{-t}}{(1+e^{-t})^{s+2}} e^{-mnt} dt = \\ &= -\frac{1}{2^s(s+1)} + \frac{nm}{s+1} \int_0^{\infty} \frac{2}{(1+e^{-t})^{s+1}} e^{-mnt} dt. \end{aligned}$$

We adapt the previous result for $s = 1$ using derivative polynomials. Changing the sign of the variable t in (5) and evaluation at $x = 0$ gives

$$\mathcal{E}(-t, 0) = \frac{2}{1+e^{-t}} = \sum_{j \geq 0} (-1)^j E_j(0) \frac{t^j}{j!}.$$

Thus, for our base function we choose the logistic function

$$f(t) = \frac{1}{2}\mathcal{E}(-t, 0) = \frac{1}{1 + e^{-t}}.$$

Lemma 5 (Derivative polynomials - logistic function). *For positive integer r the derivative $f_r(z) := \frac{d^{r-1}}{dt^{r-1}}f(t)$ can be written as a polynomial in f :*

$$f_r(z) = \sum_{j=1}^r c_{r,j} \cdot f(t)^j = \sum_{j=1}^r \frac{c_{r,j}}{(1 + e^{-t})^j}.$$

The numbers $c_{r,j}$ are explicitly given by

$$(-1)^{j-1}(j-1)! \left\{ \begin{matrix} r \\ j \end{matrix} \right\},$$

where $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is the number of ways to partition $\{1, 2, \dots, n\}$ into k nonempty subsets (Stirling number of the second kind). In particular, $c_{r,1} = 1$ and $c_{r,r} = (r-1)!(-1)^{r-1}$.

Proof. In [7] a general theory of derivative polynomials is developed: if f is a function such that $f'(t) = P(f(t))$ for a polynomial function P , then evidently $f^{(n)}(t) = P_n(f(t))$ for polynomials P_n , and if we let

$$F(x, t) = \sum_{n \geq 0} \frac{t^n}{n!} P_n(x)$$

then [7, Theorem 1] gives

$$F(x, t) = f(f^{-1}(x) + t). \tag{11}$$

In the case $f(t) = (1 + e^{-t})^{-1}$, Eq. (11) gives

$$\sum_{n \geq 0} \frac{t^n}{n!} P_n(x) = \frac{x}{x + (1-x)e^{-t}} = \frac{xe^t}{1 + x(e^t - 1)} = xe^t \sum_{m=0}^{\infty} (-1)^m x^m (e^t - 1)^m.$$

Using the identity

$$(e^t - 1)^m = m! \sum_{p \geq m} \left\{ \begin{matrix} p \\ m \end{matrix} \right\} \frac{t^p}{p!},$$

this becomes

$$\begin{aligned} \sum_{n \geq 0} \frac{t^n}{n!} P_n(x) &= x e^t \sum_{m=0}^{\infty} (-1)^m x^m m! \sum_{p \geq m} \left\{ \begin{matrix} p \\ m \end{matrix} \right\} \frac{t^p}{p!} = \\ &= \sum_{q=0}^{\infty} \frac{t^q}{q!} \sum_{p=0}^{\infty} \frac{t^p}{p!} \sum_{m=0}^p (-1)^m x^{m+1} m! \left\{ \begin{matrix} p \\ m \end{matrix} \right\}. \end{aligned}$$

Extract the coefficient of $t^n/n!$ on both sides to get

$$\begin{aligned} P_n(x) &= \sum_{p=0}^n \binom{n}{p} (-1)^m x^{m+1} m! \left\{ \begin{matrix} p \\ m \end{matrix} \right\} = \sum_{m=0}^n (-1)^m x^{m+1} m! \sum_{p=m}^n \binom{n}{p} \left\{ \begin{matrix} p \\ m \end{matrix} \right\} \\ &= \sum_{m=0}^n (-1)^m x^{m+1} m! \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\}, \end{aligned}$$

where we used the identity [5, Eq. (6.15)] in the last step. The conclusion then follows. \square

Henceforth $c_{r,j}$ denotes the coefficients of the derivative polynomials discussed above.

Lemma 6. *Define $\gamma_{s+1,r}$ as the solutions of the triangular linear system of equations*

$$\begin{pmatrix} c_{1,1} & c_{2,1} & \cdots & c_{s+1,1} \\ 0 & c_{2,2} & \cdots & c_{s+1,2} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & c_{s+1,s+1} \end{pmatrix} \cdot \begin{pmatrix} \gamma_{s+1,1} \\ \gamma_{s+1,2} \\ \vdots \\ \gamma_{s+1,s+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Then, $\gamma_{s+1,r}$ is given by

$$\gamma_{s+1,r} = \frac{(-1)^{r-1} \begin{bmatrix} s+1 \\ r \end{bmatrix}}{s!} = (-1)^{r-1} \zeta_s(\{1\}_{r-1}).$$

and

$$\frac{2}{(1+e^{-t})^{s+1}} = \sum_{k \geq 0} \frac{t^k}{k!} \sum_{j=1}^{s+1} (-1)^{k+j-1} \gamma_{s+1,j} E_{k+j-1}(0).$$

Proof. The system of linear equations can be expressed as

$$\sum_{r=j}^{s+1} c_{r,j} \gamma_{s+1,r} = \delta_{j,s+1}, \quad 1 \leq j \leq s+1.$$

Hence,

$$(-1)^{j-1} (j-1)! \sum_{r=j}^{s+1} \left\{ \begin{matrix} r \\ j \end{matrix} \right\} \gamma_{s+1,r} = \delta_{j,s+1}.$$

By the inversion relationships between Stirling numbers we directly observe that

$$\gamma_{s+1,r} = \frac{(-1)^{r-1} \left[\begin{matrix} s+1 \\ r \end{matrix} \right]}{s!}.$$

By Lemma 1 we obtain the second expression. \square

Remark 2. The generalized Euler polynomials $E_n^{(r)}(x)$, $r \in \mathbb{N}$, are defined by the generating function

$$\mathcal{E}_r(t, x) = \left(\frac{2}{1+e^t} \right)^r e^{xt} = \sum_{k \geq 0} E_k^{(r)}(x) \frac{t^k}{k!},$$

see [12]. The result above implies the formula

$$E_k^{(r)}(0) = 2^{r-1} \sum_{j=1}^r (-1)^{j-1} \gamma_{r,j} E_{k+j-1}(0),$$

also leading to a new formula for $E_k^{(r)}(x)$. Cf.

$$E_k^{(r)}(0) = \frac{2^{r-1}}{(r-1)!} \sum_{j=0}^r s(r, j) (-1)^{r+j} E_{k+j-1}(0)$$

which follows from [10] and gives an alternative derivation of the $\gamma_{r,j}$.

Proof. By our previous result

$$\frac{1}{(1+e^{-t})^{s+1}} = \sum_{j=1}^{s+1} \gamma_{s+1,j} \frac{d^{j-1}}{dt^{j-1}} f(t),$$

where $f(t) = \frac{1}{2}\mathcal{E}(-t, 0) = \frac{1}{1+e^{-t}}$. Then

$$\begin{aligned} \frac{2}{(1+e^{-t})^{s+1}} &= \sum_{j=1}^{s+1} \gamma_{s+1,j} \frac{d^{j-1}}{dt^{j-1}} 2f(t) \\ &= \sum_{j=1}^{s+1} \gamma_{s+1,j} \frac{d^{j-1}}{dt^{j-1}} \sum_{k \geq 0} (-1)^k E_k(0) \frac{t^k}{k!} \\ &= \sum_{j=1}^{s+1} \gamma_{s+1,j} \sum_{k \geq 0} (-1)^{k+j-1} E_{k+j-1}(0) \frac{t^k}{k!}. \end{aligned}$$

□

Lemma 6 implies that

$$\int_0^\infty \frac{2}{(1+e^{-t})^{s+1}} e^{-nmt} dt = \sum_{j=1}^{s+1} \gamma_{s+1,j} \sum_{k \geq 0} (-1)^{k+j-1} E_{k+j-1}(0) \frac{1}{m^{k+1} n^{k+1}}.$$

Furthermore

$$\begin{aligned} \mathbb{E}(Z_n^s) &= \frac{2(1 - \frac{1}{2^{s+1}})}{s+1} + \sum_{k=1}^\infty \frac{s^k}{n^k} \sum_{m=1}^\infty (-1)^{m-k} \frac{\zeta_{m-1}(\{1\}_{k-1})}{m} \\ &\quad \times \left(-\frac{1}{2^s(s+1)} + \frac{1}{s+1} \sum_{j=1}^{s+1} \gamma_{s+1,j} \sum_{\ell \geq 0} (-1)^{\ell+j-1} E_{\ell+j-1}(0) \frac{1}{m^\ell n^\ell} \right). \end{aligned}$$

Setting $t = 0$ in Lemma 6 we get

$$\frac{1}{2^s} = \sum_{j=1}^{s+1} (-1)^{j-1} \gamma_{s+1,j} E_{j-1}(0).$$

Consequently, the first summand cancels and we get

$$\begin{aligned}
\mathbb{E}(Z_n^s) &= \frac{2(1 - \frac{1}{2^{s+1}})}{s+1} + \\
&\sum_{k=1}^{\infty} \frac{s^k}{n^k} \sum_{m=1}^{\infty} (-1)^{m-k} \frac{\zeta_{m-1}(\{1\}_{k-1})}{m(s+1)} \sum_{j=1}^{s+1} \gamma_{s+1,j} \sum_{\ell \geq 1} (-1)^{\ell+j-1} E_{\ell+j-1}(0) \frac{1}{m^\ell n^\ell} \\
&= \frac{2(1 - \frac{1}{2^{s+1}})}{s+1} + \\
&\sum_{p=2}^{\infty} \frac{(-1)^p}{n^p} \sum_{k=1}^{p-1} \frac{s^k}{s+1} \sum_{j=1}^{s+1} \gamma_{s+1,j} (-1)^{j-1} E_{p-k+j-1}(0) \zeta(\overline{p-k+1}, \{1\}_{k-1})
\end{aligned}$$

by changing the order of summation.

Concerning arbitrary positive real $s > 0$ we have to proceed in a slightly different way. Let $B_{n,k}(x_1, \dots, x_{n-k+1})$ denote the k th Bell polynomial defined by

$$B_{n,k}(x_1, \dots, x_{n-k+1}) = \sum_{\substack{\sum_{\ell=1}^{n-k+1} j_\ell = k \\ \sum_{\ell=1}^{n-k+1} \ell j_\ell = n}} \frac{n!}{j_1! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}. \quad (12)$$

We have

$$\begin{aligned}
\frac{2}{(1 + e^{-t})^{s+1}} &= (\mathcal{E}(-t, 0))^{s+1} = (1 + (\mathcal{E}(-t, 0) - 1))^{s+1} = \\
\sum_{j \geq 0} \frac{(s+1)^j}{j!} (\mathcal{E}(-t, 0) - 1)^j &= \sum_{j \geq 0} \frac{\sum_{\ell=1}^j (s+1)^\ell B_{j,\ell}(E_1(0), \dots, E_{j-\ell+1}(0))}{j!} (-1)^j t^j.
\end{aligned}$$

Consequently,

$$\int_0^\infty \frac{2}{(1 + e^{-t})^{s+1}} e^{-mnt} dt = \sum_{j \geq 0} (-1)^j \frac{\sum_{\ell=1}^j (s+1)^\ell B_{j,\ell}(E_1(0), \dots, E_{j-\ell+1}(0))}{(mn)^{j+1}}.$$

Finally,

$$\begin{aligned}
\mathbb{E}(Z_n^s) &= \frac{2(1 - \frac{1}{2^{s+1}})}{s+1} + \sum_{k=1}^{\infty} \frac{s^k}{n^k} \sum_{m=1}^{\infty} (-1)^{m-k} \frac{\zeta_{m-1}(\{1\}_{k-1})}{m} \\
&\quad \times \frac{1}{s+1} \sum_{j \geq 1} (-1)^j \frac{\sum_{\ell=1}^j (s+1)^\ell B_{j,k}(E_1(0), \dots, E_{j-\ell+1}(0))}{(mn)^j} \\
&= \frac{2(1 - \frac{1}{2^{s+1}})}{s+1} + \sum_{p=2}^{\infty} \frac{(-1)^p}{n^p} \sum_{k=1}^{p-1} \frac{s^k}{s+1} \zeta(p+1-k, \{1\}_{k-1}) \\
&\quad \times \sum_{\ell=1}^{p-k} (s+1)^\ell B_{p-k,\ell}(E_1(0), \dots, E_{p-k-\ell+1}(0)).
\end{aligned}$$

Proof of Corollary 2. We use the general version of the Berry-Esseen inequality [4]:

$$\sup_{x \in \mathbf{R}} |F(x) - G(x)| \leq c_1 \int_{-T}^T \left| \frac{\phi_F(t) - \phi_G(t)}{t} \right| dt + c_2 \sup_{x \in \mathbf{R}} \left(G(x + \frac{1}{T}) - G(x) \right).$$

From our moment expansion

$$\mathbb{E}(Z_n^s) = \mathbb{E}(Z_\infty^s) + \mathcal{O}\left(\frac{s\zeta(2)}{2^{s+2}n^2}\right),$$

we obtain for the characteristic functions $\phi_n(t) = \mathbb{E}(e^{itZ_n})$ and $\phi_\infty(t) = \mathbb{E}(e^{itZ_\infty})$

$$\frac{|\phi_n(t) - \phi_\infty(t)|}{|t|} \leq \frac{C_1}{n^2}.$$

Choosing $T = n$ this gives a $\frac{1}{n}$ bound for the integral. We get $\sup_{x \in \mathbf{R}} (G(x + \frac{1}{T}) - G(x)) \leq \frac{C_2}{n}$ leading to the stated result. \square

Proof of Corollary 3. By the Markov inequality we have

$$\mathbb{P}\{|Z_n - Z_\infty| > \frac{1}{\ell}\} \leq \ell^2 \mathbb{E}((Z_n - Z_\infty)^2) = \ell^2 (\mathbb{E}(Z_n^2) + \mathbb{E}(Z_\infty^2) - 2\mathbb{E}(Z_n Z_\infty)).$$

The random variables Z_n and Z_∞ are defined in terms of the same uniform distribution and we readily obtain the expansion of

$$\mathbb{E}(Z_n Z_\infty) = \int_0^1 (x^n + (1-x)^n)^{\frac{1}{n}} \cdot \max\{x, 1-x\} dx = \frac{2}{3} \cdot \frac{7}{8} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

leading to $\mathbb{P}\{|Z_n - Z_\infty| > \frac{1}{\ell}\} \leq C \cdot \frac{\ell^2}{n^2}$. Let

$$E_{n,\ell} = \left\{ \omega \in \Omega : |Z_n - Z_\infty| > \frac{1}{\ell} \right\}, \quad n \in \mathbb{N}, \quad \ell > 0.$$

We have

$$\sum_{n \geq 1} \mathbb{P}\{E_{n,\ell}\} \leq \sum_{n \geq 1} \frac{C\ell^2}{n^2} < \infty.$$

Let $E_\ell = \limsup E_{n,\ell}$. By the Borel-Cantelli Lemma we have $\mathbb{P}(E_\ell) = 0$ for any $\ell > 0$, giving the stated result. \square

3.1 Independent uniformly distributed random variables

Let U_j denote mutually independent standard uniformly distributed random variables, $1 \leq j \leq r$ with $r \geq 2$. Further, let \mathbf{U} denote the random vector

$$\mathbf{U} = (U_1, \dots, U_r).$$

Let Z_n be defined as

$$Z_n = \|\mathbf{U}\|_n = (U_1^n + U_2^n + \dots + U_r^n)^{\frac{1}{n}}$$

A folklore result states that any order statistic for uniform distributions is Beta-distributed. In particular,

$$Z_\infty = \|\mathbf{U}\|_\infty = B(r, 1).$$

We are interested in the asymptotics of Z_n as $n \rightarrow \infty$ and derive asymptotics of the moments

$$I_s = \mathbb{E}(Z_n^s) = \int_{[0,1]^r} (x_1^n + \dots + x_r^n)^{\frac{s}{n}} d(x_1, \dots, x_r).$$

The special case $r = 2$, $Z_n = (U_1^n + U_2^n)^{\frac{1}{n}}$ is the direct counterpart of our earlier results for $(U^n + (1-U)^n)^{\frac{1}{n}}$. Our asymptotic series involves for $r \geq 2$ multiple zeta values. Interestingly, for $r \geq 3$ variants of multiple zeta values and Euler sums appear. Let $\zeta_r^*(i_1, \dots, i_k)$ denote the truncated multiple zeta star value

$$\zeta_r^*(i_1, \dots, i_k) = \sum_{r \geq n_1 \geq n_2 \geq \dots \geq n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \dots n_k^{i_k}}, \quad (13)$$

and $\zeta_r^*(i_1, \dots, i_k; x_1, \dots, x_k)$ denote the truncated weighted multiple zeta star value

$$\zeta_r^*(i_1, \dots, i_k; x_1, \dots, x_k) = \sum_{r \geq n_1 \geq n_2 \geq \dots \geq n_k \geq 1} \frac{x_1^{n_1} \dots x_k^{n_k}}{n_1^{i_1} n_2^{i_2} \dots n_k^{i_k}}. \quad (14)$$

Then $\zeta_r^*(i_1, \dots, i_k; \{1\}_k)$ is the ordinary zeta value $\zeta_r^*(i_1, \dots, i_k)$, and

$$\zeta_r^*(\{1\}_k; \{1\}_{k-1}, 2) = \sum_{r \geq n_1 \geq n_2 \geq \dots \geq n_k \geq 1} \frac{2^{n_k}}{n_1 n_2 \dots n_k} = \sum_{n_1=1}^r \frac{1}{n_1} \sum_{n_2=1}^{n_1} \frac{1}{n_2} \dots \sum_{n_k=1}^{n_{k-1}} \frac{2^{n_k}}{n_k}.$$

Theorem 4. *The random variable $Z_n = \|\mathbf{U}\|_n$ converges to $Z_\infty = B(r, 1)$ with convergence of all positive integer moments.*

$$\mathbb{E}(Z_n^s) = \frac{r}{r-1+s} \left(1 - \frac{s(r-1)}{n^2} \zeta(\bar{2}) + \mathcal{O}\left(\frac{1}{n^3}\right) \right).$$

In particular, for $r = 2$ and $Z_n = (U_1^n + U_2^n)^{\frac{1}{n}}$ we have the exact representation

$$\mathbb{E}(Z_n^s) = \frac{2}{1+s} \left(1 + \sum_{p \geq 2} \frac{(-1)^p}{n^p} \sum_{\ell=0}^{p-2} s^{p-\ell-1} (-\zeta(\overline{\ell+2}, \{1\}_{p-\ell-2})) \right).$$

For $r = 3$ we have the exact representation

$$\begin{aligned} \mathbb{E}(Z_n^s) &= \frac{3}{2+s} \left(1 + \sum_{p \geq 2} \frac{2(-1)^p}{n^p} \sum_{\ell=0}^{p-2} s^{p-\ell-1} (-\zeta(\overline{\ell+2}, \{1\}_{p-\ell-2})) \right) + \frac{3}{2+s} \sum_{k=1}^{\infty} \frac{s^k}{n^k} \sum_{\ell_1, \ell_2 \geq 0} \frac{(-1)^{\ell_1+\ell_2}}{n^{\ell_1+\ell_2+2}} \\ &\times \left[\sum_{i=1}^{\ell_1+1} \binom{i+\ell_2-1}{\ell_2} \sum_{m=1}^{\infty} \frac{(-1)^{m-k} \zeta_{m-1}(\{1\}_{k-1}) \left(\zeta_m^*(\{1\}_{\ell_1+2-i}; \{1\}_{\ell_1+1-i}, 2) - \zeta_m^*(\{1\}_{\ell_1+2-i}) - \frac{1}{m^{\ell_1+2-i}} \right)}{m^{1+i+\ell_2}} \right. \\ &\left. + \sum_{i=1}^{\ell_2+1} \binom{i+\ell_1-1}{\ell_1} \sum_{m=1}^{\infty} \frac{(-1)^{m-k} \zeta_{m-1}(\{1\}_{k-1}) \left(\zeta_m^*(\{1\}_{\ell_2+2-i}; \{1\}_{\ell_2+1-i}, 2) - \zeta_m^*(\{1\}_{\ell_2+2-i}) - \frac{1}{m^{\ell_2+2-i}} \right)}{m^{1+i+\ell_1}} \right]. \end{aligned}$$

3.2 Exact representations

First, we decompose the hypercube into r parts according to the maximum of the x_i :

$$[0, 1]^r = \bigcup_{i=1}^r \{x_i \in [0, 1], \quad 0 \leq x_j \leq x_i, \quad j \in \{1, \dots, r\} \setminus \{i\}\}.$$

These parts are not disjoint, but their intersection is of measure zero. By the symmetry of I_s we get

$$I_s = r \int_0^1 \left(\int_{[0, x_r]^{r-1}} (x_1^n + x_2^n + \cdots + x_r^n)^{\frac{s}{n}} d(x_1, \dots, x_{r-1}) \right) dx_r.$$

We use the substitution $x_j = x_r u_j$, $dx_j = x_r du_j$ to obtain

$$I_s = r \int_0^1 x_r^{r-1} \left(\int_{[0,1]^{r-1}} (x_r^n u_1^n + x_r^n u_2^n + \cdots + x_r^n u_{r-1}^n + 1)^{\frac{s}{n}} d\mathbf{u} \right) dx_r.$$

This implies that the integrals can be separated:

$$\begin{aligned} I_s &= r \int_0^1 x_r^{r-1+s} dx_r \cdot \int_{[0,1]^{r-1}} (1 + u_1^n + \cdots + u_{r-1}^n)^{\frac{s}{n}} d\mathbf{u} \\ &= \frac{r}{r-1+s} \cdot \int_{[0,1]^{r-1}} (1 + u_1^n + \cdots + u_{r-1}^n)^{\frac{s}{n}} d\mathbf{u}. \end{aligned}$$

In order to derive an asymptotic expansion of the remaining integral we use the $\exp - \log$ representation:

$$(1 + u_1^n + \cdots + u_{r-1}^n)^{\frac{s}{n}} = \exp\left(\frac{s}{n} \ln(1 + u_1^n + \cdots + u_{r-1}^n)\right) = 1 + \sum_{k=1}^{\infty} \frac{s^k}{n^k k!} \ln^k(1 + u_1^n + \cdots + u_{r-1}^n).$$

Using Eq. (3), this implies

$$I_s = \frac{r}{r-1+s} \cdot \left(1 + \sum_{k=1}^{\infty} \int_{[0,1]^{r-1}} \sum_{m=1}^{\infty} \frac{s^k s(m, k)}{n^k m!} (u_1^n + \cdots + u_{r-1}^n)^m d\mathbf{u} \right),$$

where (as above) $s(m, k)$ denotes the signed Stirling numbers of the first kind. Then using Lemma 1, we have

$$I_s = \frac{r}{r-1+s} \cdot \left(1 + \sum_{k=1}^{\infty} \frac{s^k (-1)^k}{n^k} \sum_{m=1}^{\infty} \frac{(-1)^m \zeta_{m-1}(\{1\}_{k-1})}{m} \int_{[0,1]^{r-1}} (u_1^n + \cdots + u_{r-1}^n)^m d\mathbf{u} \right).$$

In order to evaluate the remaining integral we substitute $u_j = e^{-t_j}$ and obtain

$$\int_{[0,1]^{r-1}} (u_1^n + \cdots + u_{r-1}^n)^m d\mathbf{u} = \int_{[0,\infty)^{r-1}} e^{-t_1 - \cdots - t_{r-1}} (e^{-t_1 n} + \cdots + e^{-t_{r-1} n})^m dt.$$

We expand the exponentials and use the multinomial theorem. By the symmetry of the integrand and the fact

$$\int_0^\infty u^p e^{-ku} du = \frac{p!}{k^{p+1}}$$

we obtain

$$\begin{aligned} & \int_{[0,\infty)^{r-1}} e^{-t_1 - \dots - t_{r-1}} (e^{-t_1 n} + \dots + e^{-t_{r-1} n})^m dt \\ &= \sum_{a=1}^{r-1} \binom{r-1}{a} \sum_{\substack{j_1 + \dots + j_a = m \\ j_i \geq 1}} \binom{m}{j_1, \dots, j_a} \sum_{\ell_1, \dots, \ell_a \geq 0} \frac{(-1)^{\ell_1 + \dots + \ell_a}}{n^{\ell_1 + \dots + \ell_a + a} j_1^{\ell_1 + 1} \dots j_a^{\ell_a + 1}}. \end{aligned}$$

For $r = 2$ there is only a single summand and we get

$$\int_{[0,\infty)} e^{-t} e^{-tnm} dt = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(nm)^{\ell+1}}.$$

Changing summation gives the desired result. For $r = 3$ we get

$$\begin{aligned} & \int_{[0,\infty)^2} e^{-t_1 - t_2} (e^{-t_1 n} + e^{-t_2 n})^m d(t_1, t_2) = \\ & 2 \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(nm)^{\ell+1}} + \sum_{\ell_1, \ell_2 \geq 0} \frac{(-1)^{\ell_1 + \ell_2}}{n^{\ell_1 + \ell_2 + 2}} \sum_{j=1}^{m-1} \binom{m}{j} \frac{1}{j^{\ell_1 + 1} (m-j)^{\ell_2 + 1}}. \end{aligned}$$

In order to simplify the arising sums we use a classical partial fraction decomposition, which appears already in [11],

$$\frac{1}{j^a (m-j)^b} = \sum_{i=1}^a \frac{\binom{i+b-2}{b-1}}{m^{i+b-1} j^{a+1-i}} + \sum_{i=1}^b \frac{\binom{i+a-2}{a-1}}{m^{i+a-1} (m-j)^{b+1-i}}, \quad (15)$$

Thus,

$$\begin{aligned} & \sum_{\ell_1, \ell_2 \geq 0} \frac{(-1)^{\ell_1 + \ell_2}}{n^{\ell_1 + \ell_2 + 2}} \sum_{j=1}^{m-1} \binom{m}{j} \frac{1}{j^{\ell_1 + 1} (m-j)^{\ell_2 + 1}} \\ &= \sum_{\ell_1, \ell_2 \geq 0} \frac{(-1)^{\ell_1 + \ell_2}}{n^{\ell_1 + \ell_2 + 2}} \left(\sum_{i=1}^{\ell_1 + 1} \frac{\binom{i+\ell_2-1}{\ell_2}}{m^{i+\ell_2}} \sum_{j=1}^{m-1} \binom{m}{j} \frac{1}{j^{\ell_1 + 2 - i}} + \sum_{i=1}^{\ell_2 + 1} \frac{\binom{i+\ell_1-1}{\ell_1}}{m^{i+\ell_1}} \sum_{j=1}^{m-1} \binom{m}{j} \frac{1}{j^{\ell_2 + 2 - i}} \right). \end{aligned}$$

Lemma 7. For positive integers r, m we have

$$\sum_{j=1}^m \binom{m}{j} \frac{1}{j^r} = \zeta_m^*(\{1\}_r; \{1\}_{r-1}, 2) - \zeta_m^*(\{1\}_r).$$

Proof. We use induction with respect to r . For $r = 1$ we have

$$\begin{aligned} \sum_{j=1}^m \binom{m}{j} \frac{1}{j} &= \int_0^1 \frac{(1+t)^m - 1}{t} dt = \int_1^2 \frac{t^m - 1}{t-1} dt = \\ &= \int_1^2 (t^{m-1} + t^{m-2} + \dots + t + 1) dt = \sum_{k=1}^m \frac{2^k}{k} - H_m = \zeta_m^*(1; 2) - \zeta_m^*(1). \end{aligned}$$

Assuming the result for $r - 1$,

$$\begin{aligned} \sum_{j=1}^m \binom{m}{j} \frac{1}{j^r} &= \sum_{j=1}^m \sum_{k=1}^m \binom{k-1}{j-1} \frac{1}{j^r} = \sum_{k=1}^m \sum_{j=1}^k \binom{k-1}{j-1} \frac{1}{j^r} = \sum_{k=1}^m \frac{1}{k} \sum_{j=1}^k \binom{k}{j} \frac{1}{j^{r-1}} \\ &= \sum_{k=1}^m \frac{1}{k} \left(\zeta_m^*(\{1\}_{r-1}; \{1\}_{r-2}, 2) - \zeta_m^*(\{1\}_{r-1}) \right) = \zeta_m^*(\{1\}_r; \{1\}_{r-1}, 2) - \zeta_m^*(\{1\}_r). \end{aligned}$$

□

This gives

$$\begin{aligned} &\sum_{\ell_1, \ell_2 \geq 0} \frac{(-1)^{\ell_1 + \ell_2}}{n^{\ell_1 + \ell_2 + 2}} \sum_{j=1}^{m-1} \binom{m}{j} \frac{1}{j^{\ell_1 + 1} (m-j)^{\ell_2 + 1}} \\ &= \sum_{\ell_1, \ell_2 \geq 0} \frac{(-1)^{\ell_1 + \ell_2}}{n^{\ell_1 + \ell_2 + 2}} \left[\sum_{i=1}^{\ell_1 + 1} \frac{\binom{i + \ell_2 - 1}{\ell_2}}{m^{i + \ell_2}} \left(\zeta_m^*(\{1\}_{\ell_1 + 2 - i}; \{1\}_{\ell_1 + 1 - i}, 2) - \zeta_m^*(\{1\}_{\ell_1 + 2 - i}) - \frac{1}{m^{\ell_1 + 2 - i}} \right) \right. \\ &\quad \left. + \sum_{i=1}^{\ell_2 + 1} \frac{\binom{i + \ell_1 - 1}{\ell_1}}{m^{i + \ell_1}} \left(\zeta_m^*(\{1\}_{\ell_2 + 2 - i}; \{1\}_{\ell_2 + 1 - i}, 2) - \zeta_m^*(\{1\}_{\ell_2 + 2 - i}) - \frac{1}{m^{\ell_2 + 2 - i}} \right) \right]. \end{aligned}$$

4 Outlook and Acknowledgments

It seems that similar phenomena appear when discussing random variables $Z_n = \|(X_1, \dots, X_n)\|_n$, where the X_i are i.i.d. random variables.

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