Traffic lights, clumping and QBDs

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Abstract

In discrete time, ℓ-blocks of red lights are separated by ℓ-blocks of green lights. Cars arrive at random. We seek the distribution of maximum line length of idle cars, and justify conjectured probabilistic asymptotics algebraically for $2 \le \ell \le 3$ and numerically for $\ell \ge 4$.
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1 Introduction

Cars arrive at a traffic light according to a Bernoulli process: during each unit of time, one car arrives or no car arrive, respectively with probability p and $q = 1 - p$; we assume throughout that $p < q$. The traffic light alternates between being red and green during intervals of time of length ℓ. When the traffic light is red, arriving cars wait and form a line, when the traffic light is green, one car, if any are present, may pass through during each unit of time. If the line is empty at some time t while the light is green, and if a new car arrives, then the arriving car passes through immediately.

Assorted expressions emerge for this problem (Finch and Louchard [5, 6]). Let X_1, X_2, ... be a sequence of independent random variables satisfying

$$\begin{aligned}
&\mathbb{P}\{X_i = 1\} = p, \quad \mathbb{P}\{X_i = 0\} = q \quad \text{if } i \equiv 1, 2, \ldots, \ell \bmod 2\ell; \\
&\mathbb{P}\{X_i = 0\} = p, \quad \mathbb{P}\{X_i = -1\} = q \quad \text{if } i \equiv \ell + 1, \ell + 2, \ldots, 2\ell \bmod 2\ell.
\end{aligned}$$

Define $S_j = \max\{S_{j-1} + X_j, 0\}$ for all $j \ge 1$, with S_0 a non-negative integer. The quantity

$$M_T = \max_{0 \le j \le T} S_j$$

is the worst-case traffic congestion, as opposed to the average-case often cited.

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Only the circumstance when $\ell = 1$ is amenable to rigorous treatment, as far as is known. The Poisson clumping heuristic (Aldous [1]), while not a theorem, gives results identical to exact asymptotic expressions when such exist, and evidently provides excellent predictions otherwise. Consider an irreducible positive recurrent Markov chain with stationary distribution π. For sufficiently large k, the maximum of the chain satisfies

$$\mathbb{P}\left\{M_T < k\right\} \sim \exp\left(-\frac{\pi_k}{\mathbb{E}(C)}T\right) \tag{1}$$

as $T \to \infty$, where C is the sojourn time in k during a clump of nearby visits to k.

The traffic light process is a two-dimensional discrete-time Markov chain $\{S_t, \varphi_t\}_{t=0,1,2,\ldots}$ where $S_t \in \mathbb{N}$ is the length of the line at time t and $\varphi_t \in \{1, \ldots, 2\ell\}$ is an indicator of the state of the traffic light: it is red for $1 \le \varphi_t \le \ell$ and green for $\ell + 1 \le \varphi_t \le 2\ell$. We order the states in lexicographic order and decompose the state space $\mathcal{E} = \{(n, i) : n \ge 0, 1 \le i \le 2\ell\}$ into subsets of constant values of n: $\mathcal{E} = \cup_{n\ge0}\mathcal{E}_n$, with $\mathcal{E}_n = \{(n, 1), \ldots (n, 2\ell)\}$. We organise the transition matrix $\mathcal{P}_\ell$ in a conformant manner, so that it takes the block-structure of a Quasi-Birth-and-Death (QBD) process, that is,

$$\mathcal{P}_\ell = \begin{bmatrix} \mathcal{B} & \mathcal{A}_1 & & \\ \mathcal{A}_{-1} & \mathcal{A}_0 & \mathcal{A}_1 & \\ & \mathcal{A}_{-1} & \mathcal{A}_0 & \ddots \\ & & \ddots & \ddots \end{bmatrix} \tag{2}$$

where $\mathcal{B}$, $\mathcal{A}_{-1}$, $\mathcal{A}_0$, and $\mathcal{A}_1$ collect the transition probabilities from $\mathcal{E}_0$ to $\mathcal{E}_0$, and from $\mathcal{E}_n$ to $\mathcal{E}_{n-1}$, $\mathcal{E}_n$ and $\mathcal{E}_{n+1}$, for $n \ge 1$, respectively.

In detailed notation:

$$\begin{aligned} (\mathcal{A}_k)_{ij} &= \mathcal{P}[X_{t+1} = (n + k, j)|X_t = (n, i)] \qquad \text{for } n \ge 1, \\ (\mathcal{B})_{ij} &= \mathcal{P}[X_{t+1} = (0, j)|X_t = (0, i)]. \end{aligned}$$

Clearly, $\mathcal{B}$, $\mathcal{A}_{-1}$, $\mathcal{A}_0$, and $\mathcal{A}_1$ are matrices of size 2ℓ, and it is easy to verify that

$$\mathcal{A}_{-1} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & q & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q \\ q & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad \mathcal{A}_0 = \begin{bmatrix} \cdot & q & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \ddots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & p & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p \\ p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

$$\mathcal{A}_1 = \begin{bmatrix} \cdot & p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \ddots & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & p & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & p & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad \mathcal{B} = \mathcal{A}_{-1} + \mathcal{A}_0 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

We use this representation in Section 4 but we start by using an approximate representation, better suited for an exact analysis for small values of ℓ, as shown in Appendix 5 for $\ell = 2$ and in [5, 6] for $\ell = 2$ and 3. We parse the sequence $\{S_0, S_1, \ldots\}$ over intervals of length 2ℓ corresponding to cycles of ℓ red and ℓ green units of time and we define the Markov chain $\{Z_t\}$ with $Z_t = S_{2\ell t}$. For such a sub-walk, we need not keep track of φ_t; it is enough to decide on the value of φ_0. The transition probabilities are then determined by the proper product of the transition matrices of infinite size

$$U = \begin{bmatrix} q & p & \cdot & \cdot & \cdot & \\ \cdot & q & p & \cdot & \cdot & \\ \cdot & \cdot & q & p & \cdot & \\ \cdot & \cdot & \cdot & q & p & \\ \cdot & \cdot & \cdot & \cdot & q & \ddots \\ & & & & & \ddots \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \\ q & p & \cdot & \cdot & \cdot & \\ \cdot & q & p & \cdot & \cdot & \\ \cdot & \cdot & q & p & \cdot & \\ \cdot & \cdot & \cdot & q & p & \ddots \\ & & & & & \ddots \end{bmatrix},$$

where U is the one-step transition probability matrix when the traffic light is red and V is the matrix when the light is green.

For the first sub-walk, we assume that $\varphi_0 = 1$, and so the transition matrix of $\{Z_t\}$ is $P_\ell = U^\ell V^\ell$. We briefly consider in Appendix A the sub-walk with $\varphi_0 = \ell + 1$ and transition matrix $V^\ell U^\ell$. In detailed expression, $(P_\ell)_{n,n+j} = p_j$ for $n + j \geq 1$ and $(P_\ell)_{n,0} = p_{-n} + \cdots + p_{-j}$ for $n \leq \ell$, with

$$p_j = \binom{2\ell}{\ell + j} p^{\ell+j} q^{\ell-j}, \qquad -\ell \leq j \leq \ell \tag{3}$$

being the probability that the line in front of the street light increases (if $j > 0$) or decreases (if $j < 0$) by j cars during any cycle of length 2ℓ, For fixed $\ell \geq 1$, we have the following conjecture [5]:

$$\mathbb{P}\left\{M_T \leq \log_{q^2/p^2}(T) + h\right\} \sim \exp\left[-\frac{\chi_\ell(p)}{2\ell}\left(\frac{q^2}{p^2}\right)^{-h}\right], \tag{4}$$

$$\mathbb{E}(M_T) \sim \frac{\ln(T)}{\ln\left(\frac{q^2}{p^2}\right)} + \frac{\gamma + \ln\left(\frac{\chi_\ell(p)}{2\ell}\right)}{\ln\left(\frac{q^2}{p^2}\right)} + \frac{1}{2} + \varphi_\ell(T),$$

$$\mathbb{V}(M_T) \sim \frac{\pi^2}{6}\frac{1}{\ln\left(\frac{q^2}{p^2}\right)^2} + \frac{1}{12} + \psi_\ell(T)$$

as $T \to \infty$. The symbol γ denotes Euler's constant [4]; φ_ℓ and ψ_ℓ are periodic functions of $\log_{q^2/p^2}(T)$ with period 1 and of small amplitude; also

$$\chi_1(p) = \frac{p(q-p)^2}{q^3},$$

$$\chi_2(p) = \frac{\left[1+(q-p)\theta\right]^2(q-p)^2}{8q^6},$$

$$\chi_3(p) = \frac{\left[u+(q-p)^2\theta + \sqrt{2}(q-p)\sqrt{v+u\theta}\right]^2(q-p)^2}{48pq^9}$$

where

$$u = 1 - 2p + 6p^2 - 8p^3 + 4p^4, \qquad v = 1 + 6p^2 - 28p^3 + 54p^4 - 48p^5 + 16p^6,$$

$$\theta = \sqrt{1 + 4pq + 16p^2q^2}.$$

We give a brief justification of the case $\ell = 2$ in the Appendix; elaborate supporting algebraic details for $2 \leq \ell \leq 3$ are found in [6] — this is actually related to the Gumbel distribution function given by $\exp(-e^{-x})$. Many applications are given in Louchard and Prodinger [9].

A numerical approach is necessary for $\ell \geq 4$. We readily see that, like $\mathcal{P}_\ell$, the transition matrix P_ℓ has a QBD structure: if we group ℓ by ℓ the rows in P_ℓ, we have

$$P_\ell = \begin{bmatrix} B & A_1 & & \\ A_{-1} & A_0 & A_1 & \\ & A_{-1} & A_0 & \ddots \\ & & \ddots & \ddots \end{bmatrix} \tag{5}$$

with blocks A_{-1}, A_0, A_1 and B of size ℓ given by

$$\left[\begin{array}{c|c|c} A\_1 & A_0 & A_1 \end{array}\right] =$$

$$\left[\begin{array}{cccc|cccc|cccc} p_{-\ell} & p_{-\ell+1} & \cdots & p_{-1} & p_0 & p_1 & \cdots & p_{\ell-1} & p_l & & & \\ & p_{-\ell} & & & p_{-1} & p_0 & p_1 & & p_{\ell-1} & p_l & & \\ & & \ddots & \ddots & & \ddots & \ddots & \ddots & & \ddots & \ddots & \\ & & & p_{-\ell} & p_{-\ell+1} & & p_{-1} & p_0 & p_1 & & p_{\ell-1} & p_l \end{array}\right]$$

and

$$B = A_0 + A_{-1}\mathbf{1} \cdot \boldsymbol{e}_1^{\mathrm{T}} \tag{6}$$

where $\mathbf{1}$ and $\boldsymbol{e}_1$ are vectors of size ℓ, all components of $\mathbf{1}$ are equal to 1 and $\boldsymbol{e}_1^{\mathrm{T}} = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}$. This means that B is obtained by adding to the first column of A_0 the probability mass on each row of A_{-1}.

This allows us to use basic features of QBDs, as presented in Latouche and Ramaswami [8], an early reference being Neuts [10]. The theory is well established, and efficient numerical procedures are readily available. The paper is organised as follows. We give in the next section some background properties of QBDs and we analyse the stationary distribution of the Markov chain $\{Z_t\}$; in particular, we determine the decay rate of its stationary distribution. In Section 3, we analyse the expected sojourn times in clumps and obtain simple expressions for the asymptotics of π_k and $\mathbb{E}(C)$ in (1). We discuss in Section 4 the effect of parsing the sequence $\{S_t\}$ at epochs which are multiples of 2ℓ, and of working with $\{Z_t\}$, instead of using $\{S_t\}$ itself.

2 Stationary distribution of P_ℓ

We denote by $\boldsymbol{\pi}$ the stationary probability vector of P_ℓ: $\boldsymbol{\pi}^{\mathrm{T}} = \boldsymbol{\pi}^{\mathrm{T}} P_\ell$, $\boldsymbol{\pi}^{\mathrm{T}}\mathbf{1} = 1$. We partition $\boldsymbol{\pi}$ in a manner conformant with P_ℓ and we write

$$\boldsymbol{\pi}^{\mathrm{T}} = \begin{bmatrix}\boldsymbol{\pi}_0^{\mathrm{T}} & \boldsymbol{\pi}_1^{\mathrm{T}} & \boldsymbol{\pi}_2^{\mathrm{T}} & \ldots\end{bmatrix}$$

with

$$\boldsymbol{\pi}_k^{\mathrm{T}} = \begin{bmatrix}\pi_{\ell k} & \pi_{\ell k+1} & \ldots & \pi_{\ell(k+1)-1}\end{bmatrix} \qquad \text{for } k = 0, 1, \ldots$$

As $p < q$, the Markov chain is positive recurrent and

$$\boldsymbol{\pi}_k^{\mathrm{T}} = \boldsymbol{\pi}_0^{\mathrm{T}} R^k, \qquad k \geq 0, \tag{7}$$

where R is the unique non-negative matrix with eigenvalues in the open unit disk, solution of

$$R^2 A_{-1} + R(A_0 - I) + A_1 = 0$$

([10, Chapter 3], [8, Chapter 6]). In expanded form, (7) may be written as

$$\begin{bmatrix}\pi_{\ell k} & \pi_{\ell k+1}, \ldots, \pi_{\ell(k+1)-1}\end{bmatrix} = \begin{bmatrix}\pi_0 & \pi_1 & \ldots \pi_{\ell-1}\end{bmatrix} R^k, \qquad k \geq 0. \tag{8}$$

The boundary vector $\boldsymbol{\pi}_0$ is the unique solution of the linear system

$$\boldsymbol{\pi}_0^{\mathrm{T}}(B + RA_{-1} - I) = \mathbf{0} \tag{9}$$

$$\boldsymbol{\pi}_0^{\mathrm{T}}(I - R)^{-1}\mathbf{1} = 1. \tag{10}$$

There exist efficient and numerically stable algorithms to compute R (Bini *et al.* [2] and [3]) and for all practical purposes, R may be considered to be known, once A_{-1}, A_0 and A_1 are given.

The next lemma is stated without proof, details are in [8, Chapter 9].

Lemma 2.1 *The roots of the polynomial* $\det \Xi(z) = 0$*, with*

$$\Xi(z) = z^2 A_{-1} + z(A_0 - I) + A_1,$$

are

$$|z_1| \le |z_2| \le \cdots \le |z_{\ell-1}| < z_\ell < 1 = z_{\ell+1} < |z_{\ell+2}| \le \cdots \le |z_{2\ell}|,$$

and the roots z_1 *to* z_ℓ *are the eigenvalues of* R. □

Thus, R has a dominant eigenvalue z_ℓ of multiplicity 1, and it immediately results from (7) that

$$\boldsymbol{\pi}_k^{\mathrm{T}} = (\boldsymbol{\pi}_0^{\mathrm{T}} \boldsymbol{v}) \boldsymbol{u}^{\mathrm{T}} z_\ell^k + o(z_\ell^k) \qquad \text{asymptotically as } k \to \infty \tag{11}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are respectively the left- and right-eigenvector of R for the eigenvalue z_ℓ, normalised so that $\boldsymbol{u}^{\mathrm{T}} \boldsymbol{v} = 1$.

Theorem 2.2 *The maximum eigenvalue of* R *is* $z_\ell = \rho^{2\ell}$*, and its corresponding left eigenvector is*

$$\boldsymbol{u} = \begin{bmatrix} 1 & \rho^2 & \dots & \rho^{2(\ell-1)} \end{bmatrix}^{\mathrm{T}}, \tag{12}$$

where $\rho = p/q$*. Therefore,*

$$\pi_j = c(\ell, p) \rho^{2j} + o(\rho)^{2j} \tag{13}$$

asymptotically as $j \to \infty$*, with*

$$c(\ell, p) = \boldsymbol{\pi}_0^{\mathrm{T}} \boldsymbol{v}. \tag{14}$$

Proof Define the Markov chain $\{Y_0, Y_1, Y_2, \dots\}$ with transition matrix

$$P_2 = \begin{bmatrix} q(1+p) & p^2 & 0 & 0 & \dots \\ q^2 & 2pq & p^2 & 0 & \\ 0 & q^2 & 2pq & p^2 & \\ 0 & 0 & q^2 & 2pq & \\ \vdots & & & & \end{bmatrix}$$

and stationary distribution $\begin{bmatrix} \xi_0 & \xi_1 & \xi_2 & \dots \end{bmatrix}$. We have

$$\xi_k = \lim_{n \to \infty} \mathbb{P}[Y_n = k] = \xi_0 \rho^{2k}. \tag{15}$$

For fixed ℓ, define $Y_n^* = Y_{n\ell}$. The Markov chain $\{Y_0^*, Y_1^*, Y_2^*, \dots\}$ has the transition matrix

$$P_\ell^* = (P_2)^\ell = \begin{bmatrix} B^* & A_1 & & \\ A_{-1} & A_0 & A_1 & \\ & A_{-1} & A_0 & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

with the same matrices A_{-1}, A_0 and A_1 as P_ℓ, but with a different transition matrix B^* at the boundary. Thus, the stationary distribution $\boldsymbol{\zeta}$ of P_ℓ^* is such that $\boldsymbol{\zeta}_k^{\mathrm{T}} = \boldsymbol{\zeta}_0^{\mathrm{T}} R^k$ by (7), with the same matrix R.

As $\lim_{n\to\infty} \mathbb{P}[Y_{n\ell} = 2k] = \lim_{n\to\infty} \mathbb{P}[Y_n = 2k]$, P_ℓ^* has the same stationary distribution as P_2, and so we have simultaneously

$$\begin{aligned}\begin{bmatrix}\xi_{k\ell} & \dots & \xi_{((k+1)\ell-1}\end{bmatrix} &= \begin{bmatrix}\xi_{(k-1)\ell} & \dots & \xi_{k\ell-1}\end{bmatrix} R \\ &= \begin{bmatrix}\xi_{(k-1)\ell} & \dots & \xi_{k\ell-1}\end{bmatrix} \rho^{2\ell} \qquad \text{by (15).}\end{aligned}$$

We conclude that the maximal eigenvalue z_ℓ of R is equal to $\rho^{2\ell}$ and this, together with (8), proves (13). Furthermore, one easily concludes from the two equations above and from (15) that the left eigenvector of R is proportional to $\boldsymbol{u} = \begin{bmatrix}1 & \rho^2 & \dots & \rho^{2(\ell-1)}\end{bmatrix}$ as claimed. □

We do not have explicit expressions for the vectors $\boldsymbol{\pi}_0$ or $\boldsymbol{v}$ but we may use numerical procedures to obtain the constant $c(\ell, p)$.

3 Sojourn times

To implement the approximation (1) for the tail of the distribution of M_T, we need to define a clump E_k, that is, a set of states such that

$$[M_T < k] \;\equiv\; [S_k \cap [0, T]] \text{ empty},$$

where $S_k = \{t : S_t \in E_k\}$ is the usual absorbing set, and we need to estimate its sojourn time. As the transition matrix P_ℓ of $\{Z_t\}$ allows for jumps of size ℓ, we chose $E_k = \{k, k+1, \dots, k+\ell-1\}$. Next, following Aldous [1, Section B12], we approximate the expected sojourn time in the clump by the total expected sojourn time in E_k for the random walk on $(-\infty, +\infty)$ with jump size distribution $\{p_n : -\ell \le n \le \ell\}$; it is a Markov chain with transition matrix

$$\widetilde{P}_\ell = \begin{bmatrix} \ddots & & & & \\ & A_0 & A_1 & & \\ & A_{-1} & A_0 & A_1 & \\ & & A_{-1} & A_0 & \\ & & & & \ddots \end{bmatrix}. \tag{16}$$

As $p < q$, the Markov chain is transient and drifts to $-\infty$, so that the total expected number of visits w_{ij} to state j, starting from state i, during the whole history of the process is finite for all i and j. We denote by W the matrix with entries $W_{i,j} = w_{k+i,k+j}$ for $0 \le i, j \le \ell - 1$, independently of k.

To determine the matrix W, we need two matrices, G and H, well-known in the theory of QBDs. The matrix G is the unique stochastic solution of the equation

$$A_{-1} + (A_0 - I)G + A_1 G^2 = 0. \tag{17}$$

Its physical meaning is that G_{ij} is the probability that, starting from state $\ell k+i$, the Markov chain visits $\ell(k-1)+j$ before any other state with index $s<\ell k$, independently of k. The matrix H is similar to G in the reverse direction: for $0 \leq i,j \leq \ell-1$, H_{ij} is the probability that, starting from state $\ell k+i$, the Markov chain visits $\ell(k+1)+j$ before any other state with index $s>\ell k+\ell-1$, independently of k. It is the unique sub-stochastic solution of the equation

$$A_{-1}H^2+(A_0-I)H+A_1=0. \tag{18}$$

One easily verifies that

$$\begin{aligned} W &= I+(A_0+A_1G+A_{-1}H)+(A_0+A_1G+A_{-1}H)^2+\cdots \\ &= \sum_{\nu\geq 0}(A_0+A_1G+A_{-1}H)^\nu \\ &= (I-(A_0+A_1G+A_{-1}H))^{-1}. \end{aligned} \tag{19}$$

The next lemma provides us with the justification for (4).

Lemma 3.1 *For large values of* k*,*

$$\mathbb{P}[M_T<k] \sim \exp(-\frac{\chi_\ell(p)}{2\ell}T\rho^{2k}) \tag{20}$$

with

$$\chi_\ell(p)=c(\ell,p)\,\boldsymbol{u}^{\mathrm{T}}(I-(A_0+A_1G+A_{-1}H))\mathbf{1}.$$

Proof The proof immediately follows from [1, Eq. B12a]: the first passage time τ_k to E_k is approximately exponential with parameter $\lambda_k=\boldsymbol{\lambda}_k^{\mathrm{T}}\mathbf{1}$, where $\boldsymbol{\lambda}_k$ is the solution to the linear system $\boldsymbol{\lambda}_k^{\mathrm{T}}W=\boldsymbol{y}^{\mathrm{T}}$ with $\boldsymbol{y}^{\mathrm{T}}=\begin{bmatrix}\pi_k & \pi_{k+1} & \cdots & \pi_{k+\ell-1}\end{bmatrix}$. The factor $1/(2\ell)$ in the right-hand side of (20) is required because λ_k is determined by the Markov chain (5) and each unit of time there represents a full cycle of size 2ℓ of the traffic light.

By (19),

$$\lambda_k^{\mathrm{T}}=\boldsymbol{y}^{\mathrm{T}}(I-(A_0+A_1G+A_{-1}H))\mathbf{1}$$

and by (12), 13),

$$\begin{bmatrix}\pi_k & \ldots & \pi_{k+\ell-1}\end{bmatrix}=c(\ell,p)\rho^{2k}\boldsymbol{u}^{\mathrm{T}}+o(\rho^{2k}),$$

so that

$$\mathbb{P}[M_T<k]\sim\exp(-c(\ell,p)\boldsymbol{u}^{\mathrm{T}}(I-(A_0+A_1G+A_{-1}H))\mathbf{1}\rho^{2k}\frac{T}{2\ell})$$

which we rewrite as (20). □

We illustrate on Figures 4, 5 and 6 the quality of the approximation given in Lemma 3.1 for the distribution of M_T. The red bars are the analytical approximation (20), the grey density is obtained by simulation of the random walk (5). The number of replications of the simulation is 40,000 and the parameters are $\ell=4$, $T=10^9$ and $p=0.35$, 0.40 and 0.45, respectively. The precision of the approximation is striking.

Remark 3.2 We have compared the values obtained for $c(\ell, p)$ from the numerical analysis to those obtained in [6] for $\ell = 2$ and 3. There were usually 14 to 15 identical significant digits, occasionally going down to 11 or up to 16. Calculations were done in MATLAB with the default 16-digits.

Remark 3.3 We conjecture that $\chi_\ell(p)/(2\ell) = q^2 c^2(\ell, p)$. This identity is proved formally in the appendix for $\ell = 2$ and in [6] for $\ell = 3$ and we have experimental evidence from our numerical analysis, the computed difference being of the order of 10^{-15}. However, we have not been able to show this analytically in all generality.

4 Detailed queue representation

We apply in this section the QBD theory directly to the original transition matrix $\mathcal{P}_\ell$ of (2). Its stationary probability vector $\boldsymbol{\gamma}$ is partitioned into subvectors $\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots$, with

$$\boldsymbol{\gamma}_k = \begin{bmatrix} \gamma_{k,1} & \ldots & \gamma_{k,2\ell} \end{bmatrix}$$

where the first index is the number of cars waiting in front of the traffic light and the second is the position of the traffic light in the cycle $R \cdots RG \cdots G$ of length 2ℓ. It is given by

$$\boldsymbol{\gamma}_k = \boldsymbol{\gamma}_0 \widehat{R}^k, \qquad k \geq 9, \tag{21}$$

where $\widehat{R}$ is the non-negative solution of $\widehat{R}^2 \mathcal{A}_{-1} + \widehat{R}(\mathcal{A}_0 - I) + \mathcal{A}_1 = 0$. We denote by $\widehat{z}_\ell$ its maximal eigenvalue and by $\widehat{\boldsymbol{u}}$ and $\widehat{\boldsymbol{v}}$ its corresponding left- and right-eigenvectors. Finally, we denote by $\widehat{\pi}_k = \boldsymbol{\gamma}_k^{\mathrm{T}} \mathbf{1}$ the stationary marginal probability that k cars are waiting.

Theorem 4.1 *The maximal eigenvalue of $\widehat{R}$ is $\widehat{z}_\ell = \rho^2$. The corresponding left eigenvector is*

$$\widehat{\boldsymbol{u}} = \begin{bmatrix} \rho^\ell & \rho^{\ell-1} & \cdots & \rho & 1 & \rho & \cdots & \rho^{\ell-1} \end{bmatrix}^{\mathrm{T}} \tag{22}$$

and

$$\widehat{\pi}_k = \widehat{c}(\ell, p)\rho^{2k} + o(\rho^{2k}) \tag{23}$$

asymptotically as $k \to \infty$, with

$$\widehat{c}(\ell, p) = \frac{1 - \rho^2}{2\ell} (\widehat{\boldsymbol{v}}^{\mathrm{T}} \mathbf{1})(\widehat{\boldsymbol{u}}^{\mathrm{T}} \mathbf{1}). \tag{24}$$

Proof By Lemma 2.1 applied to (2), $\widehat{z}_\ell$ is the largest root strictly less than 1 of the polynomial $\det \mathcal{A}(z)$, with

$$\mathcal{A}(z) = z^2\mathcal{A}_{-1} + z(\mathcal{A}_0 - I) + \mathcal{A}_0 = \begin{bmatrix} -z & p+zq & & & & & \\ & \ddots & \ddots & & & & \\ & & -z & p+zq & & & \\ & & & -z & z(p+zq) & & \\ & & & & \ddots & \ddots & \\ z(p+zq) & & & & & & -z \end{bmatrix}.$$

Simple calculations show that

$$\det \mathcal{A}(z) = z^\ell(z^\ell - (p+zq)^{2\ell}))$$

and that $\det \mathcal{A}(\rho^2) = 0$, so that ρ^2 is one of the eigenvalues of $\widehat{R}$. Furthermore, $\widehat{\boldsymbol{u}}^{\mathrm{T}}\mathcal{A}(\rho^2) = \boldsymbol{0}$, so that $\widehat{\boldsymbol{u}}$ is a left-eigenvector of $\widehat{R}$. since all components of $\widehat{\boldsymbol{u}}$ are non-negative, $\widehat{\boldsymbol{u}}$ is the Perron-Frobenius eigenvector of $\widehat{R}$ and ρ^2 is its maximal eigenvalue.

It results from (21) that

$$\boldsymbol{\gamma}_k = (\boldsymbol{\gamma}_0^{\mathrm{T}}\widehat{\boldsymbol{v}})\widehat{\boldsymbol{u}}\ \rho^{2k} + o(\rho^{2k})$$

and from [8, Lemma 6.3.2] that $\boldsymbol{\gamma}_0^{\mathrm{T}} = \boldsymbol{\alpha}^{\mathrm{T}}(I - R)$ where $\boldsymbol{\alpha}$ is the stationary probability vector of the matrix $\mathcal{A} = \mathcal{A}_{-1} + \mathcal{A}_0 + \mathcal{A}_1$. As $\mathcal{A}$ is doubly stochastic, $\boldsymbol{\alpha} = 1/(2\ell)\boldsymbol{1}$, and so

$$\begin{aligned}\widehat{\pi}_k &= \boldsymbol{\gamma}_k^{\mathrm{T}}\boldsymbol{1} \\ &= 1/(2\ell)\boldsymbol{1}^{\mathrm{T}}(I-R)\widehat{\boldsymbol{v}}\ \widehat{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{1}\rho^{2k} + o(\rho^{2k}) \\ &= (1-\rho^2)/(2\ell)\boldsymbol{1}^{\mathrm{T}}\widehat{\boldsymbol{v}}\ \widehat{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{1}\rho^{2k} + o(\rho^{2k})\end{aligned}$$

and this concludes the proof. □

We show on Figure 1 the two distributions. We clearly see the effect of the difference of behaviour when the queue is nearly empty. Nevertheless, comparing (23) with (13), we see that the distribution of the number of waiting cars has the same asymptotic decay.

We show on Figure 2 the distribution $\widehat{\pi}$ for different values of ℓ. One observes that the queue becomes stochastically much greater as ℓ increases. This is explained by the fact that the queue builds up during an interval of red light. Under normal circumstances, it will be served during the succeeding interval of green light, but it is possible that some cars remain when the light becomes red again, so that build-ups might accumulate for a while — despite of which, the *decay* of the distribution remains the same ρ^2 independently of ℓ.

Finally, to determine the tail of the distribution of $\widehat{M_T}$, we use the set $\widehat{E}_k = \{(k,i) : 1 \le i \le 2\ell\}$. The proof of Lemma 4.2 is similar to that of Lemma 3.1 and is omitted.

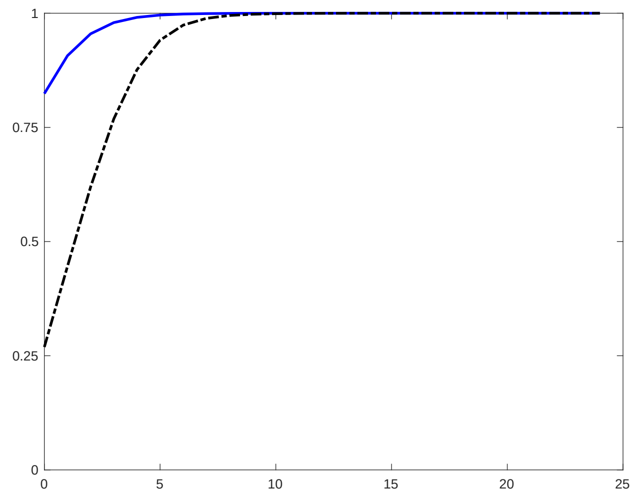

Figure 1: Cumulative probability distribution for the two models, the continuous blue line for random walk model, the dashed black line for the detailed model. The parameters are $\ell = 10$, $p = 0.4$.

Lemma 4.2 *For large values of* k*,*

$$\mathbb{P}[\widehat{M}_T < k] \sim \exp(-\frac{\widehat{\chi}_\ell(p)}{2\ell} T \rho^{2k}) \tag{25}$$

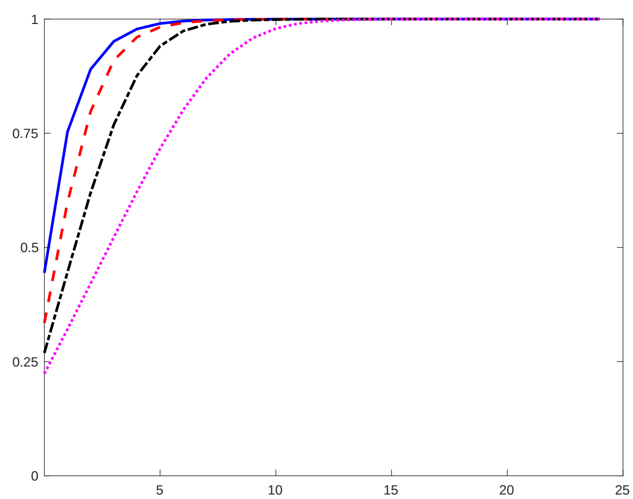

Figure 2: Cumulative probability distribution for the detailed model, for $p = 0.4$. The continuous blue line corresponds to $\ell = 1$, the red dashed line to $\ell = 5$, the black dot-dashed line to $\ell = 10$ and the magenta dotted line is for $\ell = 20$.

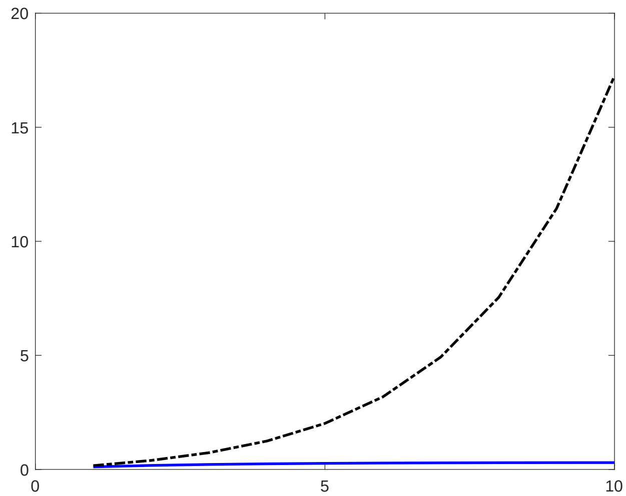

Figure 3: Values of $\chi_\ell(p)$ (continuous blue line), and $\widehat{\chi}_\ell(p)$ (dashed black line), for $p = 0.4$ and $\ell = 1$ to 10.

with

$$\widehat{\chi}_\ell(p) = (1-\rho^2)(\widehat{\boldsymbol{v}}^{\mathrm{T}}\mathbf{1})(\widehat{\boldsymbol{u}}^{\mathrm{T}}(I - (\mathcal{A}_0 + \mathcal{A}_1\widehat{G} + \mathcal{A}_{-1}\widehat{H}))\mathbf{1})$$

where $\widehat{G}$ *and* $\widehat{H}$ *are respectively the solutions of*

$$\mathcal{A}_{-1} + (\mathcal{A}_0 - I)\widehat{G} + \mathcal{A}_1\widehat{G}^2 = 0 \qquad \textit{and} \qquad \mathcal{A}_1 + (\mathcal{A}_0 - I)\widehat{H} + \mathcal{A}_{-1}\widehat{H}^2 = 0. \qquad \square$$

A comparison of $\chi_\ell(p)$ and $\widehat{\chi}_\ell(p)$ from Lemmas 3.1 and 4.2 show that they are very different. As an illustration, we give their values on Figure 3 for $p = 0.4$ and $\ell = 1$ to 10.

Remark 4.3 We have noticed that there appear to be a systematic ratio between the two: in all our numerical investigation we have observed that

$$|\frac{\chi_\ell(p)}{\widehat{\chi}_\ell(p)} - 1| < 2 \cdot 10^{-15}.$$

This leads us to conjecture that $\chi_\ell(p) = \widehat{\chi}_\ell(p)\rho^\ell$. Under this conjecture, we re-write (20) as

$$\mathbb{P}[M_T < k] \sim \exp(-\frac{\chi_\ell(p)}{2\ell} T \rho^{2k})$$

which indicates that

$$M_T = \widehat{M}_T - \ell/2 \tag{26}$$

asymptotically for large T and large values of M.

It is physically obvious that $\widehat{M}_T$ being the maximum taken over all times $t \leq T$ while M_T being the maximum taken at the end of cycles only, we should have $M_T \leq \widehat{M}_T$. The specific difference indicated in (26) is interesting but a formal proof eludes us so far.

5 Exact analysis: $\ell = 2$

We assume that $\varphi_0 = 1$ and we consider the process $\{Z_t\}$ with transition matrix

$$P_2 = U^2V^2 = \begin{pmatrix} \left(1+2p+3p^2\right)q^2 & 4p^3q & p^4 & 0 & 0 & \cdots \\ (1+3p)q^3 & 6p^2q^2 & 4p^3q & p^4 & 0 & \cdots \\ q^4 & 4pq^3 & 6p^2q^2 & 4p^3q & p^4 & \cdots \\ 0 & q^4 & 4pq^3 & 6p^2q^2 & 4p^3q & \cdots \\ 0 & 0 & q^4 & 4pq^3 & 6p^2q^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The equations for the stationary distribution are as follows:

$$\pi_0 = \left(1+2p+3p^2\right)q^2\pi_0 + (1+3p)q^3\pi_1 + q^4\pi_2, \tag{27}$$

$$\pi_1 = 4p^3q\pi_0 + 6p^2q^2\pi_1 + 4pq^3\pi_2 + q^4\pi_3 \tag{28}$$

$$\pi_j = p^4\pi_{j-2} + 4p^3q\pi_{j-1} + 6p^2q^2\pi_j + 4pq^3\pi_{j+1} + q^4\pi_{j+2} \qquad \text{for } j \ge 2. \tag{29}$$

The generating function $F(z) = \sum_{j=2}^{\infty} \pi_j z^j$ may be expressed as

$$\begin{aligned} F(z) &= p^4z^2\sum_{j=2}^{\infty}\pi_{j-2}z^{j-2} + 4p^3qz\sum_{j=2}^{\infty}\pi_{j-1}z^{j-1} + 6p^2q^2\sum_{j=2}^{\infty}\pi_j z^j \\ &\quad + \frac{4pq^3}{z}\sum_{j=2}^{\infty}\pi_{j+1}z^{j+1} + \frac{q^4}{z^2}\sum_{j=2}^{\infty}\pi_{j+2}z^{j+2} \\ &= p^4z^2\left[F(z)+\pi_0+\pi_1 z\right] + 4p^3qz\left[F(z)+\pi_1 z\right] + 6p^2q^2F(z) \\ &\quad + \frac{4pq^3}{z}\left[F(z)-\pi_2z^2\right] + \frac{q^4}{z^2}\left[F(z)-\pi_2z^2-\pi_3z^3\right] \end{aligned}$$

and we conclude that

$$\begin{aligned} &\left[q^4 + 4pq^3z - \left(1-6p^2q^2\right)z^2 + 4p^3qz^3 + p^4z^4\right]F(z) \\ &\quad = -p^4z^4\left(\pi_0+\pi_1 z\right) - 4p^3qz^3\left(\pi_1 z\right) + 4pq^3z\left(\pi_2 z^2\right) + q^4\left(\pi_2z^2+\pi_3z^3\right). \end{aligned}$$

Replacing π_2 and π_3 by expressions in π_0 and π_1 from (27, 28), then cancelling the common factor $1-z$ between numerator and denominator, yields

$$F(z) = \frac{\left\{p^3(4-3p+pz)\pi_0 + \left[-1+6p^2-8p^3+3p^4+(4-3p)p^3z+p^4z^2\right]\pi_1\right\}z^2}{(q^2-p^2z)\left[q^2+(1+2pq)z+p^2z^2\right]}$$

hence

$$L = \lim_{z\to 1} F(z) = \frac{2p^3(1+q)\pi_0 - \left[2(q-p)-q^4\right]\pi_1}{2(q-p)}.$$

We observe three zeroes in the denominator $D(z)$ of $F(z)$. The first zero, of smallest modulus < 1, is negative and given by

$$z_1 = \frac{-1 - 2pq + \theta}{2p^2}$$

where $\theta = \sqrt{1 + 4pq}$. The second zero, of intermediate modulus, is positive and given by

$$z_2 = \frac{q^2}{p^2} = (\frac{1}{\rho})^2 > 1.$$

The third zero, of largest modulus > 1, is negative and given by

$$z_3 = \frac{-1 - 2pq - \theta}{2p^2}.$$

Finding the unknowns π_0 and π_1 is achieved by solving two simultaneous equations:

$$N(z_1) = 0,$$

found by substituting z_1 for z in the numerator $N(z)$ for $F(z)$ and setting this equal to zero, and

$$\pi_0 + \pi_1 + L = 1$$

which yields

$$\pi_0 = \frac{(q-p)\,(3 - 2p - \theta)}{2q^4},$$

$$\pi_1 = \frac{(q-p)\,[-1 - p - 2pq + (1+p)\theta]}{q^5}.$$

Thus we have a complete description of the stationary distribution. An exact expression for π_j is infeasible; therefore asymptotics as $j \to \infty$ are necessary. The second zero z_2 leads, by classical singularity analysis [7] to

$$A(p) = -\frac{N(z_2)}{z_2 D'(z_2)} = \frac{(q-p)\,[1 + (q-p)\theta]}{4q^4},$$

$$\pi_j \sim A(p) \left(\frac{p^2}{q^2}\right)^j$$

and this confirms (13). This is the expression that we shall employ in the clumping heuristic.

Consider the random walk on the integers with transition matrix $\widetilde{P}_2$ (see Equation (16)). For any i, the random walk jumps to $i + j$, $-2 \le j \le 2$with probability p_j defined in (3).

For nonzero j, let ν_j denote the probability that, starting from $-j$, the walker eventually hits 0. For $j = 0$, ν_0 is the probability that, starting from 0, the walker eventually returns to 0. Using

$$\nu_0 = p^4\nu_{-2} + 4p^3q\nu_{-1} + 6p^2q^2 + 4pq^3\nu_1 + q^4\nu_2 \tag{30}$$

$$\nu_j = p^4\nu_{j-2} + 4p^3q\nu_{j-1} + 6p^2q^2\nu_j + 4pq^3\nu_{j+1} + q^4\nu_{j+2}, \quad j \ge 1; \tag{31}$$

(in (31), ν_0 is replaced by 1). The generating function $\tilde{F}(z) = \sum_{j=1}^{\infty} \nu_j z^j$ is expressed as

$$\begin{aligned}
\tilde{F}(z) &= p^4 z^2 \sum_{j=1}^{\infty} \nu_{j-2} z^{j-2} + 4p^3 qz \sum_{j=1}^{\infty} \nu_{j-1} z^{j-1} + 6p^2 q^2 \sum_{j=1}^{\infty} \nu_j z^j \\
&\quad + \frac{4pq^3}{z} \sum_{j=1}^{\infty} \nu_{j+1} z^{j+1} + \frac{q^4}{z^2} \sum_{j=1}^{\infty} \nu_{j+2} z^{j+2} \qquad \text{by (31)}, \\
&= p^4 z^2 \left[\tilde{F}(z) + \nu_{-1} z^{-1} + 1\right] + 4p^3 qz \left[\tilde{F}(z) + 1\right] + 6p^2 q^2 \tilde{F}(z) \\
&\quad + \frac{4pq^3}{z} \left[\tilde{F}(z) - \nu_1 z\right] + \frac{q^4}{z^2} \left[\tilde{F}(z) - \nu_1 z - \nu_2 z^2\right].
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&(1-z)(q^2 - p^2 z)\left[q^2 + (1+2pq)z + p^2 z^2\right] \tilde{F}(z) \\
&\quad = -p^4 z^3 \nu_{-1} - p^4 z^4 - 4p^3 q z^3 + 4pq^3 z^2 \nu_1 + q^4 z \nu_1 \\
&\qquad + z^2 \left(\nu_0 - p^4 \nu_{-2} - 4p^3 q \nu_{-1} - 6p^2 q^2 - 4pq^3 \nu_1\right) \\
&\quad = z^2 \nu_0 + q^4 z \nu_1 - p^4 z^3 \nu_{-1} - 4p^3 q z^2 \nu_{-1} - p^4 z^2 \nu_{-2} - 6p^2 q^2 z^2 - 4p^3 q z^3 - p^4 z^4.
\end{aligned} \tag{32}$$

Only the first two of the four zeroes z_1, 1, z_2, z_3 are of interest. Let $\tilde{N}(z)$ denote the numerator for $\tilde{F}(z)$, that is, $\tilde{N}(z)$ is the expression on the right-hand side of (32). We have

$$\tilde{N}(z_1) = 0, \qquad \tilde{N}(1) = 0. \tag{33}$$

Using

$$\nu_{-j} = p^4 \nu_{-j-2} + 4p^3 q \nu_{-j-1} + 6p^2 q^2 \nu_{-j} + 4pq^3 \nu_{-j+1} + q^4 \nu_{-j+2}, \qquad j \geq 1;$$

we deduce that

$$\nu_{-j} = \nu_j \left(\frac{q^2}{p^2}\right)^j$$

since multiplying both sides of

$$\begin{aligned}
\nu_j \left(\frac{q^2}{p^2}\right)^j &= p^4 \nu_{j+2} \left(\frac{q^2}{p^2}\right)^{j+2} + 4p^3 q \nu_{j+1} \left(\frac{q^2}{p^2}\right)^{j+1} + 6p^2 q^2 \nu_j \left(\frac{q^2}{p^2}\right)^j \\
&\quad + 4pq^3 \nu_{j-1} \left(\frac{q^2}{p^2}\right)^{j-1} + q^4 \nu_{j-2} \left(\frac{q^2}{p^2}\right)^{j-2}
\end{aligned}$$

by p^{2j}/q^{2j} gives an identity. Replacing $q^4 \nu_2$ by $p^4 \nu_{-2}$ in our Equation (30) for ν_0 gives

$$\nu_0 = 2p^4 \nu_{-2} + 4p^3 q \nu_{-1} + 6p^2 q^2 + 4pq^3 \nu_1. \tag{34}$$

Also, replacing $q^2\nu_1$ by $p^2\nu_{-1}$ in Equations (33, 34) reduces the number of variables to three. The simultaneous solution is

$$\nu_0 = \frac{-1+2p+8p^2-8p^3+(q-p)^2\theta}{4pq},$$

$$\nu_{-1} = \frac{1-8p^2+16p^3-8p^4-(q-p)\theta}{8p^3q},$$

$$\nu_{-2} = \frac{-1-2p+12p^2-24p^4+24p^5-8p^6+(q-p)(1+2p-4p^2)\theta}{8p^5q}$$

yielding

$$\nu_1 = \frac{1-8p^2+16p^3-8p^4-(q-p)\theta}{8pq^3}$$

in particular. The root z_2 may be used to obtain the asymptotics of ν_j for j large.

Readers might be tempted to use the level k as the absorbing set S. But the maximum could be above k without ever touching level k because of the transition p^4. So we must use as S the levels k and $k+1$: no maximum can be above $k+1$ without touching at least one of the levels k or $k+1$. In the revised notation, this leads to the absorbing set $\Omega = \{0, -1\}$.

An idea of Aldous [1] now comes crucially into play. The rate λ of clumps of visits to Ω is equal to $\lambda_0 + \lambda_{-1}$ where parameters λ_0 and λ_{-1} are solutions of the system

$$\lambda_0 + \lambda_{-1}\nu_{-1} = (1-\nu_0)\,\pi_j,$$

$$\lambda_0\nu_1 + \lambda_{-1} = (1-\nu_0)\,\pi_{j+1} \sim \frac{p^2}{q^2}(1-\nu_0)\,\pi_j.$$

In words, for nonzero j, the ratio $\nu_j/(1-\nu_0)$ is the expected sojourn time in $\{0\}$, given that the walk started at $-j$. The total clump rate is consequently

$$\lambda \sim \frac{(q-p)\,[1+(q-p)\theta]}{2q^2}\pi_j$$

in association with the transition matrix U^2V^2, that is, the sub-walk for $\varphi_0 = 1$.

The total clump rate for $\varphi_0 = 3$, that is, the transition matrix V^2U^2, can similarly be shown to be

$$\lambda' \sim \frac{(q-p)\,[1+(q-p)\theta]}{2p^2}\pi_j$$

and this particular sub-walk clearly contains the full walk maximum. Of course, the four sub-walk maxima are not independent.

Note, if $j = \log_{q^2/p^2}(n) + h + 1$, we have

$$\left(\frac{q^2}{p^2}\right)^j = n\left(\frac{q^2}{p^2}\right)^{h+1}$$

thus

$$\pi_j \frac{n}{4} \sim \frac{A(p)}{4}\left(\frac{p^2}{q^2}\right)^j n = \frac{A(p)}{4}\left(\frac{q^2}{p^2}\right)^{-(h+1)} = \frac{(q-p)\,[1+(q-p)\theta]}{16q^4}\,\frac{p^2}{q^2}\left(\frac{q^2}{p^2}\right)^{-h}.$$

By the clumping heuristic, the desired exponential argument is

$$\frac{\lambda'}{\pi_j}\cdot \pi_j \frac{n}{4} \sim \frac{(q-p)^2\,[1+(q-p)\theta]^2}{32q^6}\left(\frac{q^2}{p^2}\right)^{-h} = \frac{\chi_2(p)}{4}\left(\frac{q^2}{p^2}\right)^{-h}$$

as was to be shown.

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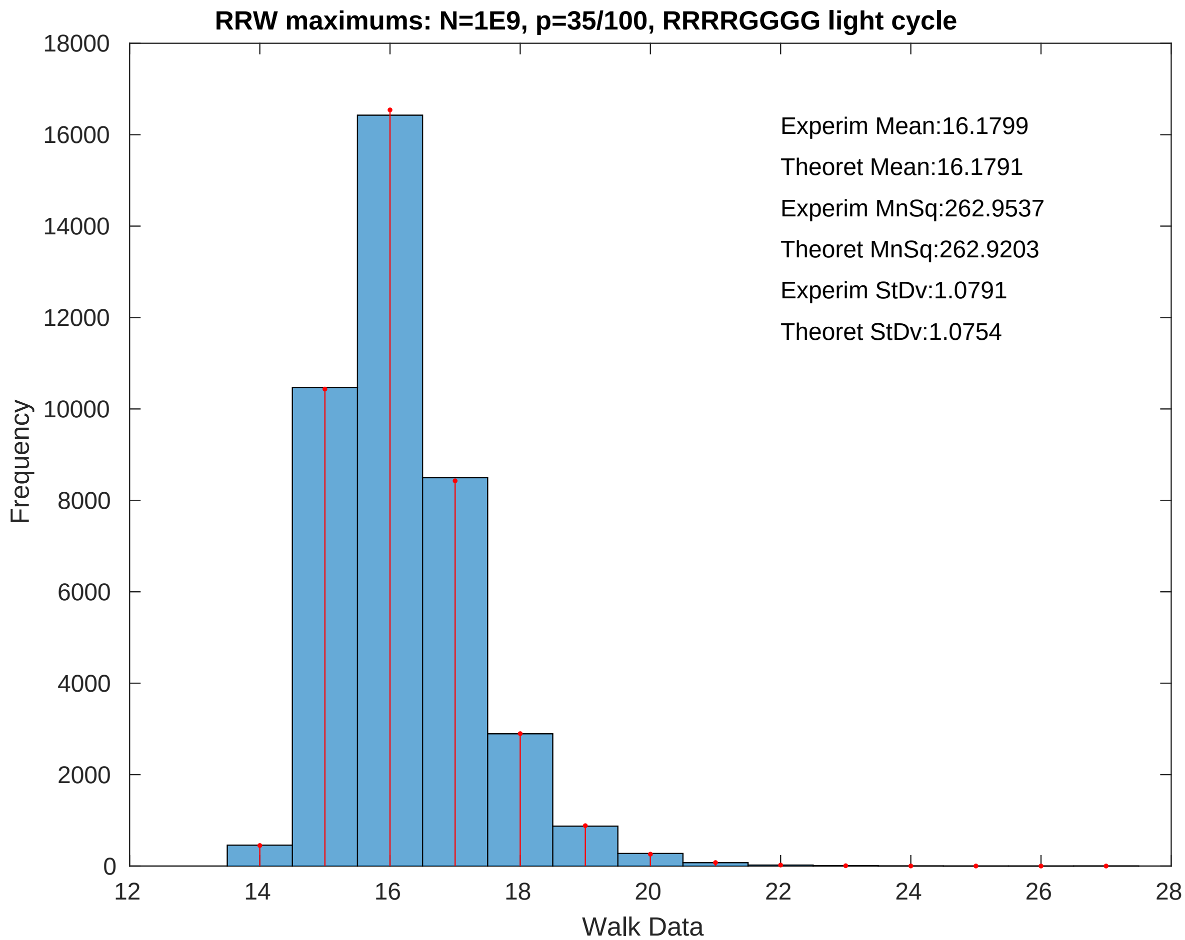

Figure 4: Comparison of the heuristic distribution of M_T and its density obtained by simulation. The number of replications is $40,000$, the parameters are $T = 10^9$, $\ell = 4$ and $p = 0.35$.

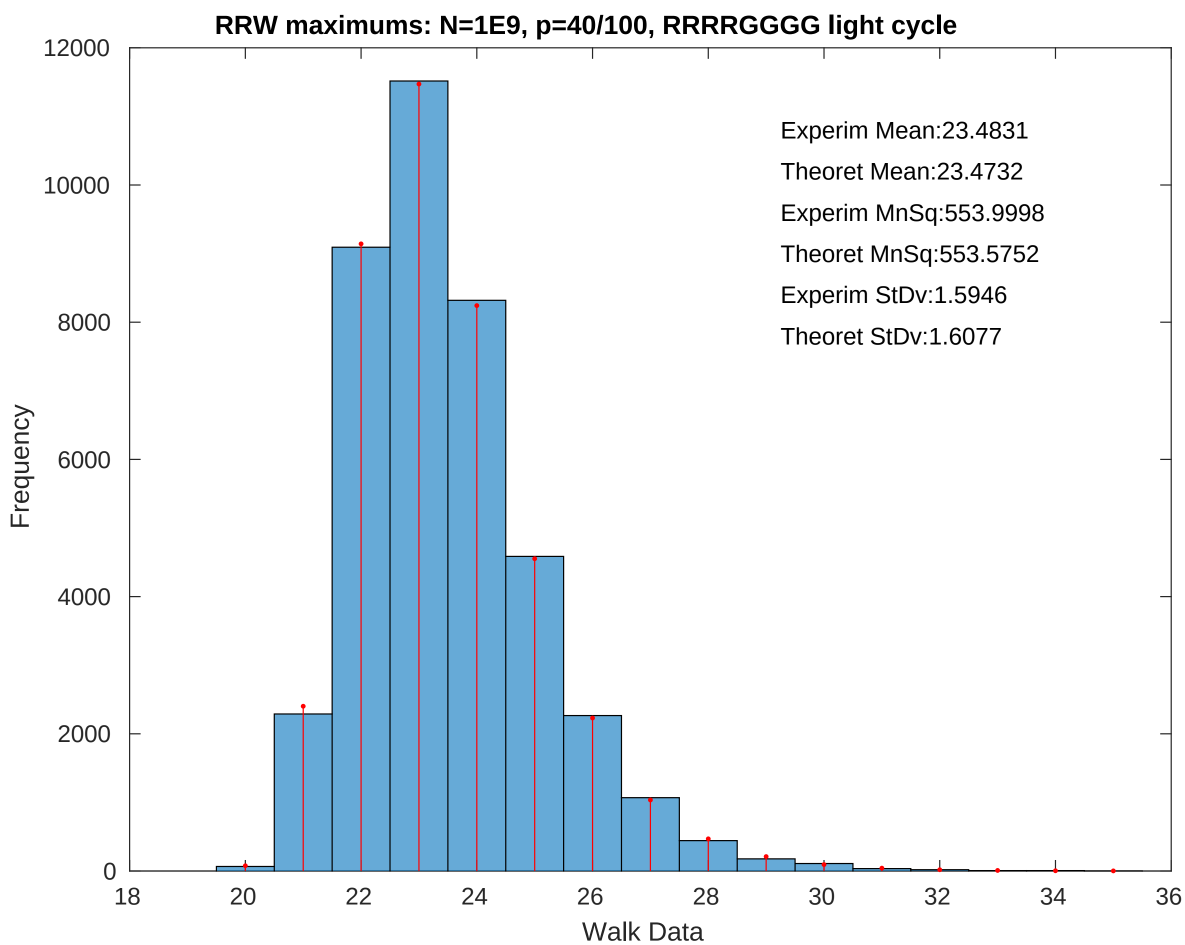

Figure 5: Comparison of the heuristic distribution of M_T and its density obtained by simulation. The number of replications is $40,000$, the parameters are $T = 10^9$, $\ell = 4$ and $p = 0.40$.

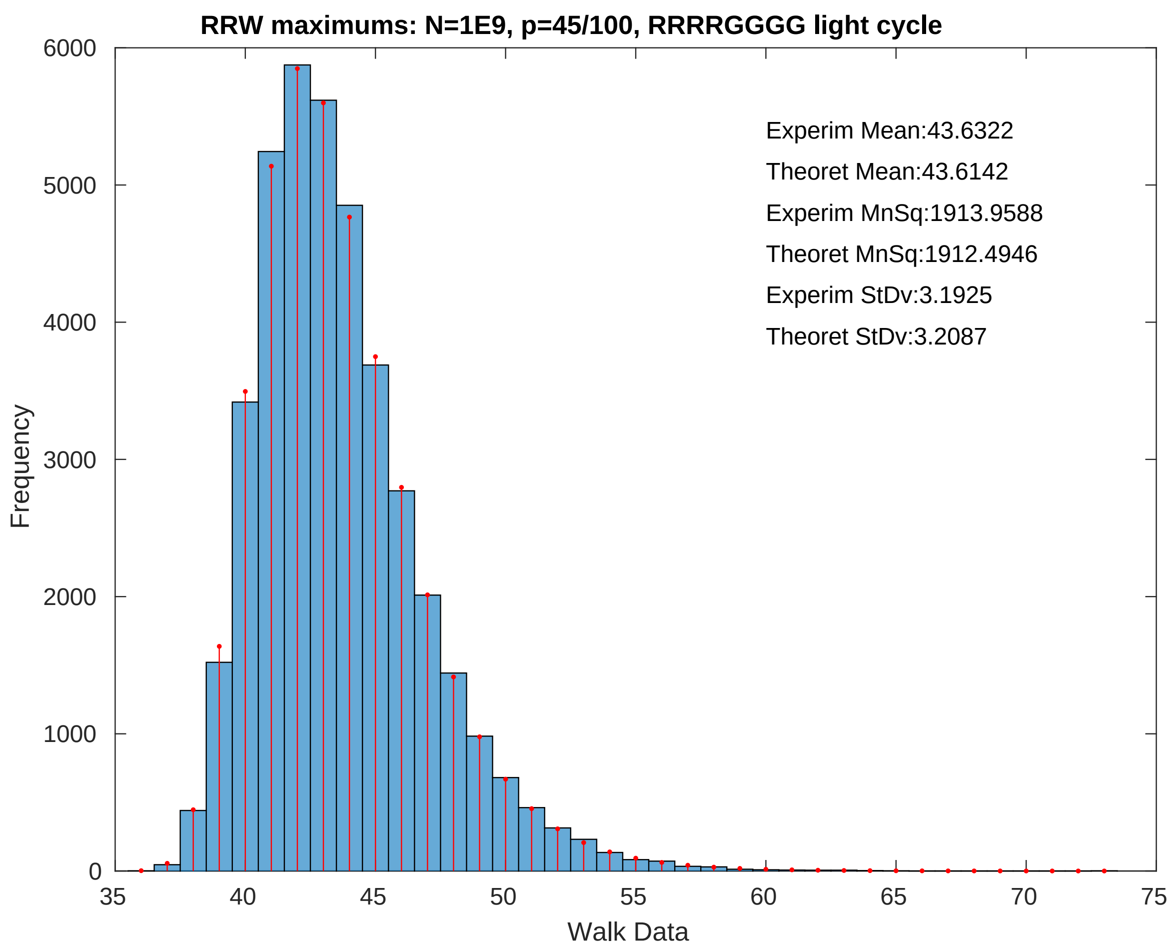

Figure 6: Comparison of the heuristic distribution of M_T and its density obtained by simulation. The number of replications is $40,000$, the parameters are $T = 10^9$, $\ell = 4$ and $p = 0.45$.