The Asymmetric Leader Election Algorithm with swedish stopping: A probabilistic analysis

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Dedicated to the memory of Philippe Flajolet.

Abstract

We study a leader election protocol that we call the Swedish leader election protocol. This name comes from a protocol presented by L. Bondesson, T. Nilsson, and G. Wikstrand in [1]. The goal is to select one among $n > 0$ players, by proceeding through a number of rounds. If there is only one player remaining, the protocol stops and the player is declared the leader. Otherwise, all remaining players flip a biased coin; with probability q the player survives to the next round, with probability $p = 1 - q$ the player loses (is killed) and plays no further ... unless all players lose in a given round (null round), so all of them play again. In the classical leader election protocol, any number of *null* rounds may take place, and with probability 1 some player will ultimately be elected. In the Swedish leader election protocol there is a maximum number τ of consecutive null rounds, and if the threshold is attained the protocol fails without declaring a leader.

In this paper, several parameters are asymptotically analyzed, among them: Success Probability, Number of rounds K_n, Number of null rounds T_n.

This paper is a companion paper to [7] where De-Poissonization was used, together with the Mellin transform. While this works fine as far as it goes, there are limitations, in particular of a computational nature. The approach chosen here is similar to earlier efforts of the same authors [8, 9, 10]. Identifying some underlying distributions as Gumbel (type) distributions, one can start with approximations at a very early stage and compute (at least in principle) all moments asymptotically. This is in contrast to [7] where only expected values were considered. In an appendix, it is shown that, whereever results are given in both papers, they coincide, although they are presented in different ways.

1 Introduction

We present a probabilistic analysis, based on an urn model, of a leader election protocol that we call the Swedish leader election protocol. This name comes from a protocol presented by L. Bondesson, T. Nilsson, and G. Wikstrand in [1]. The goal is to select one among $n > 0$ players, by proceeding through a number of rounds. If there is only one player remaining, the protocol stops and the player is declared the leader. Otherwise, all remaining players flip a biased coin; with probability q the player survives to the next round, with probability $p = 1 - q$ the player loses (is killed) and plays no further ... unless all players lose in a given round (null round), so all of them play again. In the classical leader election protocol, any number of *null* rounds may take place, and with probability 1 some player will ultimately be elected. In the Swedish leader election protocol there is a maximum number τ of consecutive null rounds, and if the threshold is attained the protocol fails without declaring a leader.

In this paper, several parameters are asymptotically analyzed, starting with n players (n large):

1. Success Probability.

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2. Number of rounds K_n.

3. Number of null rounds T_n. We say that a round is *null* if every active player tosses tails (all killed).

4. Number W_n, in case of failure, of players that were active at the last non-null round, the so-called *left-overs*.

5. Total number C_n of coins flipped.

2 Urn model

We will proceed as in [10], with urns numbered $1, 2, \ldots$.

Let us consider the model as a sequence of n geometric iid random variables (RVs), with distribution pq^{i-1}. Each RV corresponds to a ball thrown into urn i. We have the following properties:

- We have asymptotic independence of urns, for all events related to urn j containing $\mathcal{O}(1)$ balls. This is proved, by Poissonization-DePoissonization, in [9], [11] and [4] (in this paper for $p = 1/2$, but the proof is easily adapted). The error term is $\mathcal{O}(n^{-C})$ where C is a positive constant.

- We obtain asymptotic distributions of the interesting RVs. The number of balls in each urn is now Poisson-distributed with parameter $n(p/q)q^j$ in urn j containing $\mathcal{O}(1)$ balls. The asymptotic distibutions are related to Gumbel distributions functions or convergent series of such. The error term is $\mathcal{O}(n^{-1})$.

- Some summations now go to ∞. This is justified, for example, in [9].

- We have uniform integrability for the moments of our RVs. To show that the limiting moments are equivalent to the moments of the limiting distributions, we need a suitable rate of convergence. This is related to a uniform integrability condition (see Loève [6, Section 11.4]). For the kind of limiting distributions we consider here, the rate of convergence is analyzed in detail in [8] and [11]. The error term is $\mathcal{O}(n^{-C})$.

- Asymptotic expressions for the moments are obtained by Mellin transforms. The error term is $\mathcal{O}(n^{-C})$.

- $\Gamma(s)$ decreases exponentially in the direction $i\infty$:

  $$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}.$$

  Also, we have a "slow increase property" for all other functions we encounter. So inverting the Mellin transforms is easily justified.

If we compare the approach in this paper with other ones that appeared previously, then we can notice the following. Traditionally, one would stay with exact enumerations as long as possible, and only at a late stage move to asymptotics. Doing this, one would, in terms of asymptotics, carry many unimportant contributions around, which makes the computations quite heavy, especially when it comes to higher moments. Here, however, approximations are carried out as early as possible, and this allows for streamlined (and often automatic) computations of asymptotic distributions and higher moments.

The paper is organized as follows: Section 3 presents our main notations, Section 4 is devoted to the success probability, Section 5 to 11 give the asymptotic distribution and first two moments of the RVs of interest (all moments can be derived by the same technique): we compute the dominant and periodic part, both in the success and failure case. Section 12 contains some matrix expressions and Section 13 briefly presents the second model where we fail if we have τ null records, consecutive or not. Appendix A summarizes some definitions and identities. In [7], was presented an analytic treatment of the Swedish leader election protocol. In particular, the success probability and the dominant part of the mean of the following RV was computed: total number of rounds, total number of null rounds, number of left-overs. We prove, in Appendices B to E, the equivalence with the results given in this paper.

3 Notations

We will use several abbreviations for probabilities and moments in order to derive more compact expressions.

$$\begin{aligned}
n &:= \text{number of initial players}, n \text{ large},\\
\mathcal{P}(\lambda, u) &:= e^{-\lambda}\lambda^u/u!, \text{ (Poisson distribution)},\\
n^* &:= n\frac{p}{q},\\
Q &:= 1/q,\\
\log &:= \log_Q,\\
\eta &:= j - \log n^*,\\
L &:= \ln Q,\\
\{x\} &:= \text{ fractional part of } x,\\
\tilde{\alpha} &:= \alpha/L,\\
M &:= \log p,\\
\chi_l &:= \frac{2l\pi\mathbf{i}}{L},\\
K_i &:= \text{ total number of rounds, starting with } i \text{ players},\\
T_i &:= \text{ total number of null rounds, starting with } i \text{ players},\\
W_i &:= \text{ total number of leftovers, starting with } i \text{ players},\\
C_i &:= \text{ total number coins flipped, starting with } i \text{ players},\\
R_n &:= \mathbb{E}(K_n),\\
I_n &:= \mathbb{E}(T_n),\\
L_n &:= \mathbb{E}(W_n),\\
F_n &:= \mathbb{E}(C_n),\\
I &:= \text{ number of balls in the maximal non-empty urn},\\
J &:= \text{ either the position of the maximal non-empty urn, if it contains } I > 1 \text{ balls},\\
&\quad \text{or the position of the last non-empty urn \textit{before} the maximal non-empty urn},\\
&\quad \text{if the latter contains } I = 1 \text{ ball},\\
P_0(i,k) &:= \text{ Probability that, starting with } i \text{ players, we end after } k \text{ rounds},\\
\tilde{P}_0(i,k) &:= \text{ Probability that, starting with } i \text{ players, we end after } k \text{ rounds},\\
&\quad \text{given that the } i \text{ players were obtained in a null round, not preceded by another null round},\\
&\quad \text{this tilde notation will always have the same meaning in the sequel},\\
P_1(i,k,S) &:= \text{ Probability that, starting with } i \text{ players, we succeed after } k \text{ rounds},\\
\tilde{P}_1(i,k,S) &:= \text{ Probability that, starting with } i \text{ players, we succeed after } k \text{ rounds},\\
&\quad \text{given that the } i \text{ players were obtained in a null round, not preceded by another null round},\\
S_i = P_2(i,S) &:= \text{ Probability that, starting with } i \text{ players, we succeed},\\
\tilde{P}_2(i,S) &:= \text{ Probability that, starting with } i \text{ players, we succeed},\\
&\quad \text{given that the } i \text{ players were obtained in a null round},
\end{aligned}$$

$$
\begin{aligned}
P_3(i,r,F) &:= \text{ Probability that, starting with } i \text{ players, we fail,}\\
&\quad \text{with } r \text{ players remaining at the end (leftovers),}\\
\tilde{P}_3(i,r,F) &:= \text{ Probability that, starting with } i \text{ players, we fail,}\\
&\quad \text{with } r \text{ players remaining at the end (leftovers),}\\
&\quad \text{given that the } i \text{ players were obtained in a null round,}\\
P_4(r,F) &:= \text{ Probability that, starting with } n \text{ players, we fail,}\\
&\quad \text{with } r \text{ players remaining at the end (leftovers),}\\
S_n = P_5(S) &:= \text{ success probability, starting with } n \text{ players,}\\
P_6(F) &:= \text{ failure probability, starting with } n \text{ players} = 1 - P_5(S),\\
P_7(i,t,S) &:= \text{ Probability that, starting with } i \text{ players, we succeed with } t \text{ null rounds,}\\
\tilde{P}_7(i,t,S) &:= \text{ Probability that, starting with } i \text{ players, we succeed with } t \text{ null rounds,}\\
&\quad \text{given that the } i \text{ players were obtained in a null round,}\\
P_8(i,k,F) &:= \text{ Probability that, starting with } i \text{ players, we fail after } k \text{ rounds,}\\
\tilde{P}_8(i,k,F) &:= \text{ Probability that, starting with } i \text{ players, we fail after } k \text{ rounds,}\\
&\quad \text{given that the } i \text{ players were obtained in a null round,}\\
P_{10}(i,F) &:= \text{ Probability that, starting with } i \text{ players, we fail } = 1 - P_2(i,S),\\
\tilde{P}_{10}(i,F) &:= \text{ Probability that, starting with } i \text{ players, we fail } = 1 - \tilde{P}_2(i,S),\\
&\quad \text{given that the } i \text{ players were obtained in a null round,}\\
P_{11}(t,S) &:= \text{ Probability that, starting with } n \text{ players, we succeed with } t \text{ null rounds,}\\
P_{12}(t,F) &:= \text{ Probability that, starting with } n \text{ players, we fail with } t \text{ null rounds,}\\
P_{13}(i,t) &:= \text{ Probability that, starting with } i \text{ players, we end with } t \text{ null rounds,}\\
\tilde{P}_{13}(i,t) &:= \text{ Probability that, starting with } i \text{ players, we end with } t \text{ null rounds,}\\
&\quad \text{given that the } i \text{ players were obtained in a null round,}\\
R_{i,S} &:= \text{ mean number of rounds, starting with } i \text{ players, with success at the end,}\\
\tilde{R}_{i,S} &:= \text{ mean number of rounds, starting with } i \text{ players, with success at the end,}\\
&\quad \text{given that the } i \text{ players were obtained in a null round,}\\
R_{i,F} &:= \text{ mean number of rounds, starting with } i \text{ players, with failure at the end,}\\
\tilde{R}_{i,F} &:= \text{ mean number of rounds, starting with } i \text{ players, with failure at the end,}\\
&\quad \text{given that the } i \text{ players were obtained in a null round,}\\
I_{i,S} &:= \text{ mean number of null rounds, starting with } i \text{ players, with success at the end,}\\
\tilde{I}_{i,S} &:= \text{ mean number of null rounds, starting with } i \text{ players, with success at the end,}\\
&\quad \text{given that the } i \text{ players were obtained in a null round,}\\
I_{i,F} &:= \text{ mean number of null rounds, starting with } i \text{ players, with failure at the end,}\\
\tilde{I}_{i,F} &:= \text{ mean number of null rounds, starting with } i \text{ players, with failure at the end,}\\
&\quad \text{given that the } i \text{ players were obtained in a null round,}\\
L_{i,F} &:= \text{ mean number of leftovers, starting with } i \text{ players, with failure at the end,}\\
\tilde{L}_{i,F} &:= \text{ mean number of leftovers, starting with } i \text{ players, with failure at the end,}\\
&\quad \text{given that the } i \text{ players were obtained in a null round,}\\
F_{i,S} &:= \text{ mean number of coins flipped, starting with } i \text{ players, with success at the end,}\\
\tilde{F}_{i,S} &:= \text{ mean number of coins flipped, starting with } i \text{ players, with success at the end,}\\
&\quad \text{given that the } i \text{ players were obtained in a null round,}\\
F_{i,F} &:= \text{ mean number of coins flipped, starting with } i \text{ players, with failure at the end,}
\end{aligned}
$$

$$\tilde{F}_{i,F} := \text{ mean number of coins flipped, starting with } i \text{ players, with failure at the end, given that the } i \text{ players were obtained in a null round,}$$

$$\Sigma_1(i,\tau) := \sum_{v=0}^{\tau-1}(p^i)^v = \frac{1-p^{i\tau}}{1-p^i},$$

$$\Sigma_2(i,\tau) := \sum_{v=0}^{\tau-1}(p^i)^v(v+1) = \frac{p^{i\tau}(-1+\tau p^i-\tau)+1}{(1-p^i)^2},$$

$$\Sigma_3(i,\tau) := \sum_{v=0}^{\tau-1}(p^i)^v(v+1)^2 = \frac{p^{i\tau}(p^i+1-2\tau p^i+2\tau+\tau^2p^{2i}-2\tau^2p^i+\tau^2)-p^i+1}{(1-p^i)^3},$$

$$\Sigma_4(i,\tau) := \sum_{v=0}^{\tau-1}(p^i)^v v = \frac{p^{i\tau}(\tau p^i-\tau-p^i)+p^i}{(1-p^i)^2},$$

$$\Sigma_5(i,\tau) := \sum_{v=0}^{\tau-1}(p^i)^v v^2 = \frac{p^{i\tau}(p^i+2\tau p^i+\tau^2p^{2i}-2\tau^2p^i+\tau^2+p^{2i}-2\tau p^{2i})-p^{2i}+p^i}{(1-p^i)^3}.$$

Model 1 = $M1$ = we fail if we have τ consecutive null records,

Model 2 = $M2$ = we fail if we have τ null records, consecutive or not.

For any of the four RVs, we will denote by $\xi_i^{(2)}$, $\xi \in \{R, I, L, F\}$ the expectation of the square of the corresponding RVs.

Note that the maximal non-empty urn, if it contains $i \geq 2$ balls, corresponds to a null round. If the maximal non-empty urn contains $i = 1$ ball, the process is successful.

4 Success probability, M1.

We have the following recurrences:

$$P_2(1,S) = 1, \quad \tilde{P}_2(1,S) = 1,$$

$$P_2(i,S) = \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_2(\ell,S) = \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_2(\ell,S),\ i \geq 2,$$

$$\tilde{P}_2(i,S) = \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_2(\ell,S) = \Sigma_1(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_2(\ell,S)$$
$$= \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)}P_2(i,S),\ i \geq 2.$$

Explanation: we can have up to $\tau-1$ or $\tau-2$ null rounds (all killed), followed by ℓ survivors. Note that $\ell = i$ in the right-hand side leads to $P_2(i,S)$.

We also have from [10], (here and in the sequel $\sim$ always denotes $\sim_{n\to\infty}$),

$$\mathbb{P}(J=j, I=i) \sim \exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{e^{-L\eta i}}{i!},\ i \geq 2, \tag{1}$$

$$\mathbb{P}(J=j, I\geq 2, S) \sim f_1(\eta),$$

$$f_1(\eta) := \sum_{i=2}^{\infty}\exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{e^{-L\eta i}}{i!}\tilde{P}_2(i,S),$$

$$\mathbb{P}(J=j, I=1) \sim f_2(\eta),$$

$$f_2(\eta) := \exp\Big(-\frac{q}{p}e^{-L\eta}\Big)\frac{q}{p}e^{-L\eta}\left(1-\exp\left(-e^{-L\eta}\right)\right). \tag{2}$$

Explanation:

$$\mathbb{P}(J=j, I=i) \sim \mathcal{P}\Big(\frac{q}{p}e^{-L\eta}, 0\Big)\mathcal{P}\left(e^{-L\eta}, i\right),\ i \geq 2,$$
$$\mathbb{P}(J=j, I=1) \sim \mathcal{P}\Big(\frac{q}{p}e^{-L\eta}, 1\Big)\left[1-\mathcal{P}\left(e^{-L\eta}, 0\right)\right].$$

Note that the case $i \geq 2$ does not necessary lead to a success: urn J corresponds to the first null round, hence the multiplication by $\tilde{P}_2(i, S)$. On the other side, the case $i = 1$ does lead to a success: urn J corresponds to a round with one single player alive, which is immediately declared as the leader (there are no null rounds before).

This gives

$$\phi_1(\alpha) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_1(\eta) d\eta = \sum_{i=2}^{\infty} \frac{(1/p)^{-i+\tilde{\alpha}}}{Li!}\Gamma(i-\tilde{\alpha})\tilde{P}_2(i, S).$$

So the corresponding dominant part of $S_n := P_5(S)$ is given by

$$\phi_1(0) = \sum_{i=2}^{\infty} \frac{p^i}{Li}\frac{\Sigma_1(i, \tau-1)}{\Sigma_1(i, \tau)}P_2(i, S).$$

The corresponding periodic part is given by

$$\omega_{1,5} = \sum_{l \neq 0} \varphi_{1,5}(\chi_l)e^{-2l\pi\mathbf{i}\log n^*},$$

with

$$\varphi_{1,5}(\chi_l) = \phi_1(\alpha)\big|_{\alpha=-L\chi_l}.$$

We obtain

$$\varphi_{1,5}(\chi_l) = \sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\frac{\Sigma_1(i, \tau-1)}{\Sigma_1(i, \tau)}P_2(i, S).$$

Also,

$$\phi_2(\alpha) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_2(\eta) d\eta = \frac{1}{L}\left[\left(\frac{q}{p}\right)^{\tilde{\alpha}} - q\left(\frac{1}{p}\right)^{\tilde{\alpha}}\right]\Gamma(1-\tilde{\alpha}).$$

Hence

$$\phi_2(0) = \frac{p}{L} = \text{ Probability that the maximal non-empty urn contains one ball} \tag{3}$$
$$\omega_{2,5} = \sum_{l \neq 0} \varphi_{2,5}(\chi_l)e^{-2l\pi\mathbf{i}\log n^*},$$
$$\varphi_{2,5}(\chi_l) = \frac{1}{L}\left[\left(\frac{q}{p}\right)^{-\chi_l} - q\left(\frac{1}{p}\right)^{-\chi_l}\right]\Gamma(1+\chi_l) = \frac{p^{1+\chi_l}}{L}\Gamma(1+\chi_l).$$

And finally, with notations provided in the appendix we have the following result.

Theorem 4.1 *Related to success probability and the model M1, we have*

$$S_n = P_5(S) \sim \sum_{i=2}^{\infty} \frac{p^i}{Li}\frac{\Sigma_1(i, \tau-1)}{\Sigma_1(i, \tau)}P_2(i, S) + \frac{p}{L} + \sum_{l \neq 0} \varphi_{1,5}(\chi_l)e^{-2l\pi\mathbf{i}\log n^*} + \sum_{l \neq 0} \varphi_{2,5}(\chi_l)e^{-2l\pi\mathbf{i}\log n^*}$$
$$=: V_1 + \frac{p}{L} + \sum_{l \neq 0} \varphi_{1,5}(\chi_l)e^{-2l\pi\mathbf{i}\log n^*} + \sum_{l \neq 0} \varphi_{2,5}(\chi_l)e^{-2l\pi\mathbf{i}\log n^*}.$$

Of course, $P_6(F) = 1 - P_5(S)$. Note also that as $\tau \to \infty$, the dominant part gives

$$\sum_{i=2}^{\infty} \frac{p^i}{Li} + \frac{p}{L} = 1$$

as expected. For further use, we denote

$$\begin{aligned} \Pi_1 &:= \frac{p}{L} \text{ (one ball in the maximal non-empty urn)}, \\ \Pi_2(i) &:= \frac{p^i}{Li}, \\ Pd(S) &= V_1 + \frac{p}{L} \quad \text{(dominant part)}. \end{aligned}$$

5 Asymptotic distribution and moments of $K_n - \log n^*$, success case, M1.

In case of success, the moments of $K_n - \log n^*$ are computed as in [10], and expressed with some $\tilde{R}_{i,S}$, $\tilde{R}^{(2)}_{i,S}$, instead of $x_{i,S}$, $x^{(2)}_{i,S}$ used in [10]. They are computed as described in the sequel. Here and in the sequel, we give the first two moments. All moments could be computed, only with more (algebraic and Maple) efforts.

We use $f_1(\eta)$ as given by (4) of [10], with $\tilde{P}_1(i,k,S)$ instead of $P(i,k)$, i.e.,

$$f_1(\eta) = \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \exp\Big(-\frac{1}{p} e^{Lk} e^{-L\eta}\Big) \frac{e^{-L\eta i} e^{Lki}}{i!} \tilde{P}_1(i,k,S).$$

$f_2(\eta)$ is given by (2).

We have the following recurrences:

$$\begin{aligned} P_1(1,0,S) &= 1, \quad \tilde{P}_1(1,0,S) = 1, \\ P_1(1,\geq 1,S) &= 0, \quad \tilde{P}_1(1,\geq 1,S) = 0, \\ P_1(i,k,S) &= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_1(\ell, k-1-s, S), \ i \geq 2, \\ \tilde{P}_1(i,k,S) &= \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_1(\ell, k-1-s, S), \ i \geq 2. \end{aligned}$$

Explanation: we can have up to $\tau - 1$ or $\tau - 2$ null rounds (all killed), followed by ℓ survivors. This takes $s+1$ rounds already.

$$\begin{aligned} R_{i,S} &= \sum_k P_1(i,k,S)k = \sum_k \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_1(\ell,k-1-s,S)[k-1-s+s+1], \\ &= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_{\ell,S} + \sum_k \sum_{s=0}^{\tau-1} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_1(\ell,k-1-s,S) \\ &= \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_{\ell,S} + \Sigma_2(i,\tau) P_2(i,S)/\Sigma_1(i,\tau), \ R_{1,S} = 0, \end{aligned}$$

$$\begin{aligned} \tilde{R}_{i,S} &= \sum_k \tilde{P}_1(i,k,S)k = \sum_k \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_1(\ell,k-1-s,S)[k-1-s+s+1], \\ &= \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} R_{\ell,S} + \sum_k \sum_{s=0}^{\tau-2} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_1(\ell,k-1-s,S) \end{aligned}$$

$$= \Sigma_1(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_{\ell,S} + \Sigma_2(i,\tau-1)P_2(i,S)/\Sigma_1(i,\tau),\ \tilde{R}_{1,S}=0,$$

$$R^{(2)}_{i,S} = \sum_k P_1(i,k,S)k^2 = \sum_k\sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_1(\ell,k-1-s,S)[k-1-s+s+1]^2,$$

$$= \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R^{(2)}_{\ell,S} + \sum_k\sum_{s=0}^{\tau-1}(p^i)^s(s+1)^2\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_1(\ell,k-1-s,S)$$

$$+ 2\sum_{s=0}^{\tau-1}(p^i)^s(s+1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_{\ell,S}$$

$$= \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R^{(2)}_{\ell,S} + \Sigma_3(i,\tau)P_2(i,S)/\Sigma_1(i,\tau) + 2\Sigma_2(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_{\ell,S},\ R^{(2)}_{1,S}=0,$$

$$\tilde{R}^{(2)}_{i,S} = \sum_k \tilde{P}_1(i,k,S)k^2 = \sum_k\sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_1(\ell,k-1-s,S)[k-1-s+s+1]^2,$$

$$= \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R^{(2)}_{\ell,S} + \sum_k\sum_{s=0}^{\tau-2}(p^i)^s(s+1)^2\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_1(\ell,k-1-s,S)$$

$$+ 2\sum_{s=0}^{\tau-2}(p^i)^s(s+1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_{\ell,S}$$

$$= \Sigma_1(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R^{(2)}_{\ell,S} + \Sigma_3(i,\tau-1)P_2(i,S)/\Sigma_1(i,\tau)$$

$$+ 2\Sigma_2(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_{\ell,S},\ \tilde{R}^{(2)}_{1,S}=0.$$

Note that, in previous expressions, we could replace

$$\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_{\ell,S} \qquad \text{by} \qquad [R_{i,S}-\Sigma_2(i,\tau)P_2(i,S)/\Sigma_1(i,\tau)]/\Sigma_1(i,\tau).$$

To obtain the moments of $K_n-\log n^*$, we plug, *mutatis mutandis*, $\tilde{R}_{i,S}$, $\tilde{R}^{(2)}_{i,S}$ into the moments given in [10]. Note that to each value $I=i\geq 2$ corresponds $\tilde{P}_2(i,S)$ as explained in Section 4. Also $R_0(\chi_l)$ is no more null here and $\tilde{P}_2(1,S)=1$ by convention. This leads, with the quantities defined in the appendix, to

Theorem 5.1 *Asymptotic distribution and moments of $K_n-\log n^*$, success case, model M1.*

$$\mathbb{P}(K_n=j)\sim f_1(\eta)+f_2(\eta),$$

$$R_{n,S}-\log n^* = \mathbb{E}(K_n-\log n^*) \sim U_1 - MV_1 - \frac{V_2}{L} + \frac{p\gamma}{L^2} - \frac{1+pM}{L}$$

$$+ \sum_{l\neq 0}\left[T_1(\chi_l) - MR_0(\chi_l) - \frac{R_1(\chi_l)}{L} - \frac{\Gamma(1+\chi_l)}{L}\right]e^{-2l\pi\mathbf{i}\log n},$$

$$\mathbb{E}(K_n-\log n^*)^2 \sim U_2 - 2MU_1 - 2\frac{U_4}{L} + M^2S1 + 2M\frac{V_2}{L} + \frac{V_3+V_4}{L^2}$$

$$+ \frac{p(\pi^2/6+\gamma^2)}{L^3} - \frac{2\gamma(pM+1)}{L^2} + \frac{pM^2+2M+1}{L}$$

$$+ \sum_{l\neq 0}\left\{T_2(\chi_l) - 2MT_1(\chi_l) - 2\frac{T_3(\chi_l)}{L} + M^2R_0(\chi_l) + 2M\frac{R_1(\chi_l)}{L} + \frac{R_2(\chi_l)+R_3(\chi_l)}{L^2}\right.$$

$$+\Gamma(1+\chi_l)\left[2\frac{\psi(1+\chi_l)}{L^2}+\frac{1}{L}+2\frac{M}{L}\right]\Bigg\}e^{-2l\pi\mathbf{i}\log n}.$$

Note that the periodic component contains $\log n$ *in the exponent (and not* $\log n^*$*).*

To obtain the moments of $K_n - \log n^*$, *given success*, we simply divide the moments given in the theorem by $P_5(S)$.

6 Asymptotic distribution and moments of T_n (null rounds), with success, M1.

We have the following recurrences:

$$\begin{aligned}
P_7(1,0,S) &= 1, \quad \tilde{P}_7(1,0,S) = 1,\\
P_7(1,\geq 1,S) &= 0, \quad \tilde{P}_7(1,\geq 1,S) = 0,\\
P_7(i,t,S) &= \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_7(\ell,t-s,S),\ i\geq 2,\\
\tilde{P}_7(i,t,S) &= \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_7(\ell,t-1-s,S),\ i\geq 2.
\end{aligned}$$

Explanation: we can have up to $\tau-1$ or $\tau-2$ null rounds (all killed), followed by ℓ survivors. This leads to s or $s+1$ null rounds already.

$$\begin{aligned}
I_{i,S} &= \sum_t P_7(i,t,S)t = \sum_t\sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_7(\ell,t-s,S)[t-s+s],\\
&= \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}I_{\ell,S} + \sum_t\sum_{s=0}^{\tau-1}(p^i)^s s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_7(\ell,t-s,S)\\
&= \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}I_{\ell,S} + \Sigma_4(i,\tau)P_2(i,S)/\Sigma_1(i,\tau),\ I_{1,S}=0,\\
\tilde{I}_{i,S} &= \sum_t \tilde{P}_7(i,t,S)t = \sum_t\sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_7(\ell,t-1-s,S)[t-1-s+s+1],\\
&= \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}I_{\ell,S} + \sum_t\sum_{s=0}^{\tau-2}(p^i)^s(s+1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_7(\ell,t-1-s,S)\\
&= \Sigma_1(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}I_{\ell,S} + \Sigma_2(i,\tau-1)P_2(i,S)/\Sigma_1(i,\tau),\ \tilde{I}_{1,S}=1,\\
I^{(2)}_{i,S} &= \sum_t P_7(i,t,S)t^2 = \sum_t\sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_7(\ell,t-s,S)[t-s+s]^2,\\
&= \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}I^{(2)}_{\ell,S} + \sum_t\sum_{s=0}^{\tau-1}(p^i)^s s^2\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_7(\ell,t-1-s,S)\\
&+ 2\sum_{s=0}^{\tau-1}(p^i)^s s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}I_{\ell,S}\\
&= \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}I^{(2)}_{\ell,S} + \Sigma_5(i,\tau)P_2(i,S)/\Sigma_1(i,\tau) + 2\Sigma_4(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}I_{\ell,S},\ I^{(2)}_{1,S}=1,
\end{aligned}$$

$$\tilde{I}^{(2)}_{i,S} = \sum_t \tilde{P}_1(i,t,S)t^2 = \sum_t \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_7(\ell, t-1-s, S)[t-1-s+s+1]^2,$$

$$= \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I^{(2)}_{\ell,S} + \sum_t \sum_{s=0}^{\tau-2} (p^i)^s (s+1)^2 \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_7(\ell, t-1-s, S)$$

$$+ 2\sum_{s=0}^{\tau-2} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_{\ell,S}$$

$$= \Sigma_1(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I^{(2)}_{\ell,S} + \Sigma_3(i,\tau-1) P_2(i,S)/\Sigma_1(i,\tau)$$

$$+ 2\Sigma_2(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_{\ell,S},\ \tilde{I}^{(2)}_{1,S} = 0.$$

Again, in previous expressions, we could replace

$$\sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_{\ell,S} \quad \text{by} \quad [I_{i,S} - \Sigma_4(i,\tau) P_2(i,S)/\Sigma_1(i,\tau)]/\Sigma_1(i,\tau).$$

Next, with (1),

$$\mathbb{P}(J = j, T_n = t) \sim f_7(\eta, t),$$

$$f_7(\eta, t) = \sum_{i=2}^{\infty} \exp\Big(-\frac{1}{p} e^{-L\eta}\Big) \frac{e^{-L\eta i}}{i!} \tilde{P}_7(i,t,S).$$

Hence

$$\phi_7(\alpha, t) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_7(\eta, t) d\eta = \sum_{i=2}^{\infty} \frac{(1/p)^{-i+\tilde{\alpha}}}{Li!} \Gamma(i - \tilde{\alpha}) \tilde{P}_7(i,t,S).$$

Note that there are no null rounds if the maximal non-empty urn contains only 1 ball.

The dominant component of $P_{11}(t,S)$ is given by

$$\phi_7(0,t) = \sum_{i=2}^{\infty} \frac{(1/p)^{-i}}{Li!} \Gamma(i) \tilde{P}_7(i,t,S) = \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{P}_7(i,t,S),$$

and the periodic component by

$$\omega_{1,7}(t) = \sum_{l \neq 0} \varphi_7(\chi_l, t) e^{-2l\pi \mathbf{i} \log n^*},$$

with

$$\varphi_7(\chi_l, t) = \phi_7(\alpha, t)\big|_{\alpha = -L\chi_l}.$$

We obtain

$$\varphi_7(\chi_l, t) = \sum_{i=2}^{\infty} \frac{p^{i+\chi_l} \Gamma(i+\chi_l)}{Li!} \tilde{P}_7(i,t,S).$$

Hence we have the following theorem.

Theorem 6.1 *The asymptotic distribution of the number T_n of null rounds, with success, M1, is given by*

$$P_{11}(t,S) = \mathbb{P}(T_n = t) \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{P}_7(i,t,S) + \sum_{l \neq 0} \varphi_7(\chi_l, t) e^{-2l\pi \mathbf{i}\{\log n^*\}}.$$

The moments are given by

$$I_{n,S} \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{I}_{i,S} + \sum_{l\neq 0} \sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!} \tilde{I}_{i,S} e^{-2l\pi\mathbf{i}\{\log n^*\}},$$

$$I^{(2)}_{n,S} \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{I}^{(2)}_{i,S} + \sum_{l\neq 0} \sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!} \tilde{I}^{(2)}_{i,S} e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

Note that $T_n = \mathcal{O}(1)$.

7 Asymptotic distribution and moments of W_n (leftovers), with failure, M1.

We have the following recurrences:

$$\begin{aligned} P_3(r,r,F) &= (p^r)^\tau + \Sigma_1(r,\tau) q^r P_3(r,r,F), \text{ hence} \\ P_3(r,r,F) &= \frac{p^{r\tau}(1-p^r)}{1-p^r-q^r+q^r p^{r\tau}}, \\ \tilde{P}_3(r,r,F) &= (p^r)^{\tau-1} + \Sigma_1(r,\tau-1) q^r P_3(r,r,F). \end{aligned}$$

Explanation: we have r players alive at start or r players alive before the starting null round.

$$\begin{aligned} P_3(i,r,F) &= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=r}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_3(\ell,r,F) = \Sigma_1(i,\tau) \sum_{\ell=r}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_3(\ell,r,F), \ i>r, \ i\geq 2, \\ \tilde{P}_3(i,r,F) &= \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=r}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_3(\ell,r,F) = \Sigma_1(i,\tau-1) \sum_{\ell=r}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_3(\ell,r,F), \ i>r, \ i\geq 2, \\ &= \Sigma_1(i,\tau-1)/\Sigma_1(i,\tau) P_3(i,r,F), \ i>r, \ i\geq 2. \end{aligned}$$

Explanation: we can have up to $\tau-1$ or $\tau-2$ null rounds (all killed), followed by ℓ survivors.

$$\begin{aligned} L_{i,F} &= \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}} i + \sum_{r=0}^{i-1} P_3(i,r,F) r \\ &= \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}} i + \Sigma_1(i,\tau) \sum_{r=0}^{i-1} \sum_{\ell=r}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_3(\ell,r,F) r, \\ &= \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}} i + \Sigma_1(i,\tau) \left[\sum_{\ell=0}^{i-1} \binom{i}{\ell} q^\ell p^{i-\ell} \sum_{r=0}^{\ell} P_3(\ell,r,F) r + \sum_{r=0}^{i-1} q^i P_3(i,r,F) r \right] \\ &= \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}} i + \Sigma_1(i,\tau) \left[\sum_{\ell=0}^{i-1} \binom{i}{\ell} q^\ell p^{i-\ell} L_{\ell,F} + q^i \left[L_{i,F} - \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^i p^{i\tau}} i \right] \right], \\ \tilde{L}_{i,F} &= \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)} \sum_{r=0}^{i-1} P_3(i,r,F) r + \left[(p^i)^{\tau-1} + \Sigma_1(i,\tau-1) q^i P_3(i,i,F) \right] i, \end{aligned}$$

and similar equations for $L^{(2)}_{i,F}$, $\tilde{L}^{(2)}_{i,F}$.

Next, with (1),

$$\mathbb{P}(J=j, W_n=r) \sim f_3(\eta,r),$$

$$f_3(\eta,r) = \sum_{i=2}^{\infty} \exp\Big(-\frac{1}{p} e^{-L\eta}\Big) \frac{e^{-L\eta i}}{i!} \tilde{P}_3(i,r,F).$$

Hence

$$\phi_3(\alpha, r) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_3(\eta, r) d\eta = \sum_{i=2}^{\infty} \frac{(1/p)^{-i+\tilde{\alpha}}}{Li!} \Gamma(i - \tilde{\alpha}) \tilde{P}_3(i, r, F).$$

Note that there are no leftovers if the maximal non-empty urn contains only 1 ball.

The dominant component of $P_4(r, F)$ is given by

$$\phi_3(0, r) = \sum_{i=2}^{\infty} \frac{(1/p)^{-i}}{Li!} \Gamma(i) \tilde{P}_3(i, r, F) = \sum_{i=2}^{\infty} \frac{p^i}{Li} \Gamma(i) \tilde{P}_3(i, r, F),$$

and the periodic component by

$$\omega_{1,3}(r) = \sum_{l \neq 0} \varphi_3(\chi_l, r) e^{-2l\pi \mathbf{i} \log n^*},$$

with

$$\varphi_3(\chi_l, r) = \phi_3(\alpha)\big|_{\alpha = -L\chi_l}.$$

We obtain

$$\varphi_3(\chi_l, r) = \sum_{i=2}^{\infty} \frac{p^{i+\chi_l} \Gamma(i + \chi_l)}{Li!} \tilde{P}_3(i, r, F).$$

Hence

Theorem 7.1 *The asymptotic distribution of the number W_n of leftovers, with failure, in the model M1, is given by*

$$P_4(r, F) = \mathbb{P}(W_n = r) \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \frac{\Sigma_1(i, \tau - 1)}{\Sigma_1(i, \tau)} P_3(i, r, F) + \sum_{l \neq 0} \varphi_3(\chi_l, r) e^{-2l\pi \mathbf{i}\{\log n^*\}}.$$

The moments are given by

$$L_{n,F} \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{L}_{i,F} + \sum_{l \neq 0} \sum_{i=2}^{\infty} \frac{p^{i+\chi_l} \Gamma(i + \chi_l)}{Li!} \tilde{L}_{i,F} e^{-2l\pi \mathbf{i}\{\log n^*\}},$$

$$L^{(2)}_{n,F} \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{L}^{(2)}_{i,F} + \sum_{l \neq 0} \sum_{i=2}^{\infty} \frac{p^{i+\chi_l} \Gamma(i + \chi_l)}{Li!} \tilde{L}^{(2)}_{i,F} e^{-2l\pi \mathbf{i}\{\log n^*\}}.$$

Note that $W_n = \mathcal{O}(1)$.

8 Asymptotic distribution and moments of $K_n - \log n^*$, failure case, M1.

In case of failure, the moments of $K_n - \log n^*$ are computed as in [10], with some $\tilde{R}_{i,F}$, $\tilde{R}^{(2)}_{i,F}$, computed as follows. First we have

$$P_{10}(i, F) = 1 - P_2(i, S) = (p^i)^\tau + \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_{10}(\ell, F)$$

$$= (p^i)^\tau + \Sigma_1(i, \tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_{10}(\ell, F), \; i \geq 2, \; P_{10}(1, F) = 0.$$

Next the recurrences:

$$P_8(1, 0, F) = 0, \quad \tilde{P}_8(1, 0, F) = 0,$$

$$P_8(1,\geq 1,F)=0,\quad \tilde{P}_8(1,\geq 1,F)=0,$$

$$P_8(i,k,F)=(p^i)^\tau[\![k=\tau]\!]+\sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_8(\ell,k-1-s,F),\ i\geq 2,$$

$$\tilde{P}_8(i,k,F)=(p^i)^{\tau-1}[\![k=\tau-1]\!]+\sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_8(\ell,k-1-s,F),\ i\geq 2.$$

Explanation: we can have τ or $\tau-1$ null rounds (all killed) at start, leading to failure.

$$R_{i,F}=(p^i)^\tau\tau+\Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_{\ell,F}+\Sigma_2(i,\tau)[P_{10}(i,F)-(p^i)^\tau]/\Sigma_1(i,\tau),\ R_{1,F}=0,$$

$$\tilde{R}_{i,F}=(p^i)^{\tau-1}(\tau-1)+\Sigma_1(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_{\ell,F}$$
$$+\Sigma_2(i,\tau-1)[P_{10}(i,F)-(p^i)^\tau]/\Sigma_1(i,\tau),\ \tilde{R}_{1,F}=0,$$

and similar equations for $R^{(2)}_{i,F},\tilde{R}^{(2)}_{i,F}$. To obtain the moments of $K_n-\log n^*$, we plug $\tilde{R}_{i,F},\tilde{R}^{(2)}_{i,F}$ into the moments given in [10], *based only on* $f_1(\eta)$ as given by (4) of [10], with $\tilde{P}_8(i,k,F)$ instead of $P(i,k)$, i.e.,

$$f_1(\eta)=\sum_{k=1}^{\infty}\sum_{i=2}^{\infty}\exp\Big(-\frac{1}{p}e^{Lk}e^{-L\eta}\Big)\frac{e^{-L\eta i}e^{Lki}}{i!}\tilde{P}_8(i,k,F).$$

Indeed, a maximal non-empty urn with only 1 ball leads to a success. Note that to each value $I=i\geq 2$ corresponds $\tilde{P}_{10}(i,F)$. Also $R_0(\chi_l)$ is no more null here and $\tilde{P}_{10}(1,F)=0$ by convention. This gives

Theorem 8.1 *Asymptotic distribution and moments of* $K_n-\log n^*$*, failure case, model M1.*

$$\mathbb{P}(K_n=j)\sim f_1(\eta),$$

$$R_{n,F}-\log n^*=\mathbb{E}(K_n-\log n^*)\sim U_1-MV_1-\frac{V_2}{L}$$
$$+\sum_{l\neq 0}\left[T_1(\chi_l)-MR_0(\chi_l)-\frac{R_1(\chi_l)}{L}\right]e^{-2l\pi\mathbf{i}\log n},$$

$$\mathbb{E}(K_n-\log n^*)^2\sim U_2-2MU_1-2\frac{U_4}{L}+M^2V_1+2M\frac{V_2}{L}+\frac{V_3+V_4}{L^2}$$
$$+\sum_{l\neq 0}\Big\{T_2(\chi_l)-2MT_1(\chi_l)-2\frac{T_3(\chi_l)}{L}+M^2R_0(\chi_l)$$
$$+2M\frac{R_1(\chi_l)}{L}+\frac{R_2(\chi_l)+R_3(\chi_l)}{L^2}\Big\}e^{-2l\pi\mathbf{i}\log n}.$$

To obtain the moments of $K_n-\log n^*$, *given failure*, we simply divide the moments given in the theorem by $P_6(F)$.

9 Asymptotic distribution and moments of T_n (null rounds), with failure, M1.

We have the recurrences:

$$P_9(1,0,F)=0,\quad \tilde{P}_9(1,0,F)=0,$$
$$P_9(1,\geq 1,F)=0,\quad \tilde{P}_9(1,\geq 1,F)=0,$$

$$P_9(i,t,F) = (p^i)^\tau [\![t=\tau]\!] + \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_9(\ell, t-s, F),$$
$$\tilde{P}_9(i,t,F) = (p^i)^{\tau-1} [\![t=\tau]\!] + \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_9(\ell, t-1-s, F).$$

Explanation: we can have τ or $\tau-1$ null rounds (all killed) at start, leading to failure.

$$\begin{aligned}
I_{i,F} &= \sum_t P_9(i,t,F)t = (p^i)^\tau \tau + \sum_t \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_9(\ell, t-s, F)[t-s+s] \\
&= (p^i)^\tau \tau + \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_{\ell,F} + \sum_t \sum_{s=0}^{\tau-1} (p^i)^s s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_9(\ell, t-s, F) \\
&= (p^i)^\tau \tau + \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_{\ell,F} + \Sigma_4(i,\tau)[P_{10}(i,F) - (p^i)^\tau]/\Sigma_1(i,\tau),\ I_{1,F} = 0, \\
\tilde{I}_{i,F} &= \sum_t \tilde{P}_9(i,t,F)t = (p^i)^{\tau-1} \tau + \sum_t \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_9(\ell, t-1-s, F)[t-1-s+s+1] \\
&= (p^i)^{\tau-1} \tau + \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_{\ell,F} + \sum_t \sum_{s=0}^{\tau-2} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_9(\ell, t-1-s, F) \\
&= (p^i)^{\tau-1} \tau + \Sigma_1(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_{\ell,F} + \Sigma_2(i,\tau-1)[P_{10}(i,F) - (p^i)^\tau]/\Sigma_1(i,\tau),\ \tilde{I}_{1,F} = 0.
\end{aligned}$$

Next, with (1),

$$\mathbb{P}(J=j, T_n=t) \sim f_9(\eta, t),$$
$$f_9(\eta,t) = \sum_{i=2}^{\infty} \exp\Big(-\frac{1}{p} e^{-L\eta}\Big) \frac{e^{-L\eta i}}{i!} \tilde{P}_9(i,t,F).$$

Hence

$$\phi_9(\alpha,t) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_9(\eta,t) d\eta = \sum_{i=2}^{\infty} \frac{(1/p)^{-i+\tilde{\alpha}}}{Li!} \Gamma(i-\tilde{\alpha}) \tilde{P}_9(i,t,F).$$

Note that there are no null rounds if the maximal non-empty urn contains only 1 ball.

The dominant component of $P_{12}(t,F)$ is given by

$$\phi_9(0,t) = \sum_{i=2}^{\infty} \frac{p^i}{Li!} \Gamma(i) \tilde{P}_9(i,t,F),$$

and the periodic component by

$$\omega_{1,9}(t) = \sum_{l \neq 0} \varphi_9(\chi_l, t) e^{-2l\pi \mathbf{i} \log n^*},$$

with

$$\varphi_9(\chi_l, t) = \phi_9(\alpha, t)\big|_{\alpha = -L\chi_l}.$$

We obtain

$$\varphi_9(\chi_l, t) = \sum_{i=2}^{\infty} \frac{p^{i+\chi_l} \Gamma(i+\chi_l)}{Li!} \tilde{P}_9(i,t,F).$$

Hence

Theorem 9.1 *The asymptotic distribution of the number T_n of null rounds, with failure, in the model M1, is given by*

$$P_{12}(t,F) = \mathbb{P}(T_n = t) \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{P}_9(i,t,F) + \sum_{l\neq 0} \varphi_9(\chi_l, t) e^{-2l\pi \mathbf{i}\{\log n^*\}}.$$

The moments are given by

$$I_{n,F} \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{I}_{i,F} + \sum_{l\neq 0}\sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!} \tilde{I}_{i,F} e^{-2l\pi \mathbf{i}\{\log n^*\}},$$

$$I^{(2)}_{n,F} \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{I}^{(2)}_{i,F} + \sum_{l\neq 0}\sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!} \tilde{I}^{(2)}_{i,F} e^{-2l\pi \mathbf{i}\{\log n^*\}}.$$

Note that $T_n = \mathcal{O}(1)$.

10 Asymptotic distribution and moments of C_n (coins flipped), with success, M1.

10.1 Case $I = 1$.

Note that, as explained in Section 4, this case *entails a success*. We will only deal here with the non-periodic part of our expressions. The maximal non-empty urn contains 1 ball and the position of the last non-empty urn *before* this maximal non-empty urn is denoted by J. Let us also denote by K the number of balls in urn J.

$$\mathbb{P}(J=j, K=k) \sim f_4(\eta, k), \ k \geq 1,$$

$$f_4(\eta,k) := \exp\Big(-\frac{q}{p}e^{-L\eta}\Big)\frac{q}{p}e^{-L\eta}\exp\left(-e^{-L\eta}\right)\frac{e^{-L\eta k}}{k!},$$

$$= \exp\Big(-\frac{1}{p}e^{-L\eta}\Big)\frac{q}{p}\frac{e^{-L\eta(k+1)}}{k!}.$$

Explanation:

$$\mathbb{P}(J=j, K=k) \sim \mathcal{P}\Big(\frac{q}{p}e^{-L\eta}, 1\Big)\mathcal{P}\left(e^{-L\eta}, k\right).$$

We have

$$\phi_4(\alpha,k) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_4(\eta,k) d\eta = \frac{q}{Lk!}\left(\frac{1}{p}\right)^{\tilde{\alpha}-k}\Gamma(1-\tilde{\alpha}+k),$$

$$\Pi_4(k) := \phi_4(0,k) = \frac{q}{L}p^k.$$

Note that

$$Z_1 := \sum_{k=1}^{\infty}\Pi_4(k) = \frac{p}{L} \equiv \Pi_1 \text{ (one ball in the maximal non-empty urn)}$$

which conforms to (3).

Let us denote by Δ the *difference* between the maximal non-empty urn (containing 1 ball) and J. We have

$$\mathbb{P}(J=j, I=1, \Delta=\delta) \sim f_8(\eta,\delta),$$

$$f_8(\eta,\delta) := \exp\Big(-\frac{q}{p}e^{-L\eta}\Big)e^{-L(\eta+\delta)}\left(1-\exp\left(-e^{-L\eta}\right)\right),$$

$$= \exp(-L\delta)\exp\Big(-\frac{q}{p}e^{-L\eta}\Big)e^{-L\eta}\left(1-\exp\left(-e^{-L\eta}\right)\right),$$

which shows that Δ is asymptotically independent of J.

Explanation:

$$\mathbb{P}(J=j, I=1, \Delta=\delta) \sim \mathcal{P}\left(e^{-L(\eta+1)},0\right)\mathcal{P}\left(e^{-L(\eta+1)},0\right)\dots$$
$$\dots\mathcal{P}\left(e^{-L(\eta+\delta)},1\right)\mathcal{P}\left(e^{-L(\eta+\delta+1)},0\right)\dots\left[1-\mathcal{P}\left(e^{-L\eta},0\right)\right].$$

We have

$$\begin{aligned}\phi_8(\alpha,\delta) &= \int_{-\infty}^{\infty} e^{\alpha\eta} f_8(\eta,\delta)d\eta \\ &= \exp(-L\delta)\frac{p}{Lq}\left[\left(\frac{q}{p}\right)^{\tilde{\alpha}} - q\left(\frac{1}{p}\right)^{\tilde{\alpha}}\right]\Gamma(1-\tilde{\alpha}), \\ \Pi_5(\delta) := \phi_8(0,\delta) &= e^{-L\delta}\frac{p^2}{Lq} = q^{\delta}\frac{p^2}{Lq}.\end{aligned}$$

Note that

$$\sum_{\delta=1}^{\infty}\Pi_5(\delta) = \frac{p}{L} \equiv \Pi_1$$

which again conforms to (3). We have

$$\begin{aligned}\mathbb{E}(\Delta) &= \frac{1}{L}, \\ \mathbb{E}(\Delta^2) &= \frac{1+q}{Lp}.\end{aligned}$$

However, note carefully that the player corresponding to $I=1$ is actually related to a *flipped coin in urn* J. So we must use a new RV $G=K+1$, $G\geq 2$, with distribution

$$\Pi_6(g) = \frac{q}{L}p^{g-1},\ g\geq 2$$

and

$$f_6(\eta,g) = \exp\left(-\frac{1}{p}e^{-L\eta}\right)\frac{q}{p}\frac{e^{-L\eta g}}{(g-1)!}.$$

We will also need

$$Z_5 = \mathbb{E}(G) := \sum_{g=2}^{\infty}\Pi_6(g)g = \frac{p(1+q)}{Lq}.$$

Later on, we will use the following variants:

$$\begin{aligned}&e^{-L\eta}f_6(\eta,g), \\ &e^{-2L\eta}f_6(\eta,g), \\ &\eta f_6(\eta,g), \\ &e^{-L\eta}\eta f_6(\eta,g), \\ &e^{-2L\eta}\eta f_6(\eta,g).\end{aligned}$$

These variants lead respectively to $\phi_{.}(0,g)$:

$$\begin{aligned}&\frac{p^g qg}{L}, \\ &\frac{qp^{g+1}g(g+1)}{L},\end{aligned}$$

$$
\begin{aligned}
& -\frac{p^{g-1}q[(g-1)\ln(p)+(g-1)\psi(g-1)+1]}{L^2(g-1)}, \\
& -\left[qp^g[2(g-1)+(g-1)^2\ln(p)+(g-1)^2\psi(g-1)+(g-1)\ln(p)\right. \\
& \left.+(g-1)\psi((g-1))+1]\right]/[L^2(g-1)], \\
& \Omega_{15}(g) \quad \text{is too long to be displayed here.}
\end{aligned}
$$

This leads to $Z_7, Z_8, Z_{11}, Z_{12}, Z_{10}, Z_{13}, Z_{15}$: we *simply sum on* $g \geq 2$. Indeed, the case $I = 1$ immediately leads to a success.

Now we will separate the contribution of urn J (containing G balls) from that of urns $< J$.

Let us denote by $S_\Gamma(j,i)$ the sum of $(n-i)$ iid RV $\Gamma(j)$, and $\Gamma(j)$ is a truncated geometric RV $< j$. As $\Sigma_0 := \sum_1^{j-1} pq^{l-1} = 1 - q^{j-1}$, we have (we give only the terms needed in the sequel)

$$
\begin{aligned}
E(j) &:= \mathbb{E}(\Gamma(j)) = \sum_1^{j-1} pq^{l-1}l/\Sigma_0 \sim \frac{1}{p} + q^{j-1} - jq^{j-1} - jq^{2(j-1)} + \mathcal{O}\big(q^{2(j-1)}\big), \\
E^{(2)}(j) &:= \mathbb{E}(\Gamma(j)^2) = \sum_1^{j-1} pq^{l-1}l^2/\Sigma_0 \sim \frac{1+q}{p^2} + q^{j-1}\frac{1+q}{p} - j\frac{2q}{p}q^{j-1} \\
&\quad - j^2q^{j-1} + \mathcal{O}\big(j^2q^{2(j-1)}\big).
\end{aligned}
$$

Note that, with $j = \eta + \log n^*$,

$$
q^j = e^{-L\eta}\frac{1}{n^*}.
$$

This leads, by carefully taking into account the *correlation* between J and I (we expand the mean up to the $\log n^*/n^*$ term and the square mean up to the $\log n^*$ term) to

$$
C_{n,1} \sim S_\Gamma(J,G) + JG, \tag{4}
$$

$$
\begin{aligned}
\mathbb{E}(C_{n,1}) &\sim \mathbb{E}(S_\Gamma(J,G) + JG), \\
&\sim \mathbb{E}\Big[(n-G)\frac{1}{p} + \frac{n}{q}\frac{e^{-L\eta}}{n^*} - \frac{n-G}{q}\frac{e^{-L\eta}}{n^*}(\log n^* + \eta) \\
&\quad - \frac{n}{q^2}\frac{e^{-2L\eta}}{n^{*2}}\log n^* + (\log n^* + \eta)G\Big] \\
&\sim \frac{n}{p}Z_1 - \frac{1}{p}Z_5 + \frac{n}{q}\frac{Z_7}{n^*} - \frac{n}{q}\frac{Z_7}{n^*}\log n^* + \frac{1}{q}\frac{Z_8}{n^*}\log n^* \\
&\quad - \frac{n}{q}\frac{Z_{10}}{n^*} - \frac{n}{q^2}\frac{Z_{11}}{n^{*2}}\log n^* + Z_5\log n^* + Z_{13},
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\mathbb{E}(C_{n,1}^2) &\sim \mathbb{E}((S_\Gamma(J,G) + JG)^2) \\
&\sim \mathbb{E}[n\mathbb{E}^{(2)}(J) + (n-G)(n-G-1)(\mathbb{E}(J))^2 \\
&\quad + 2\mathbb{E}[(n-G)E(J)JG] + \mathbb{E}[(\log n^* + \eta)^2G^2].
\end{aligned} \tag{6}
$$

10.2 Case $I > 1$.

First of all, we must compute the moments of $C_{i,S}$ and $\tilde{C}_{i,S}$. This gives

$$
F_{i,S} = \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}[si + i + F_{\ell,S}],\ F_{1,S} = 0,
$$

$$
\begin{aligned}
F^{(2)}_{i,S} &= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} \mathbb{E}[((s+1)i + C_{\ell,S})^2] \\
&= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} [((s+1)i)^2 + 2(s+1)iF_{\ell,S} + F^{(2)}_{\ell,S}],
\end{aligned}
$$

and similar expressions for $\tilde{F}_{i,S}$, $\tilde{F}^{(2)}_{i,S}$.

Next, with (1),

$$
\begin{aligned}
\mathbb{P}(J = j, I = i) &\sim f_5(\eta, i), \\
f_5(\eta, i) &:= \exp\Big(-\frac{1}{p} e^{-L\eta}\Big) \frac{e^{-L\eta i}}{i!}, \\
\mathbb{P}(J = j) &\sim f_{10}(\eta), \\
f_{10}(\eta) &= \sum_{i=2}^{\infty} \exp\Big(-\frac{1}{p} e^{-L\eta}\Big) \frac{e^{-L\eta i}}{i!} = \exp\Big(-\frac{1}{p} e^{-L\eta}\Big) \left(\exp\left(-e^{-L\eta}\right) - 1 - e^{-L\eta}\right), \\
\phi_5(\alpha, i) &= \int_{-\infty}^{\infty} e^{\alpha\eta} f_5(\eta, i) d\eta = \frac{(1/p)^{-i+\tilde{\alpha}}}{Li!} \Gamma(i - \tilde{\alpha}), \\
\Pi_2(i) &:= \phi_5(0, i) = \frac{p^i}{Li}, \\
P_0 &:= \sum_{2}^{\infty} \Pi_2(i) = 1 - p/L.
\end{aligned}
$$

Later on, we will use the following variants:

$$
\begin{aligned}
&e^{-L\eta} f_5(\eta, i), \\
&e^{-2L\eta} f_5(\eta, i), \\
&\eta f_5(\eta, i), \\
&e^{-L\eta} \eta f_5(\eta, i), \\
&e^{-2L\eta} \eta f_5(\eta, i).
\end{aligned}
$$

These variants lead respectively to $\phi_.(0, i)$:

$$
\begin{aligned}
&\frac{p^i p}{L}, \\
&\frac{p^i p^2 (i+1)}{L}, \\
&-\frac{p^i [\ln(p) + \psi(i)]}{L^2 i}, \\
&-\frac{p^i p [i \ln(p) + i\psi(i) + 1]}{L^2 i}, \\
&-\frac{p^i p^2 [i^2 \ln(p) + i^2 \psi(i) + 2i + i \ln(p) + i\psi(i) + 1]}{L^2 i}.
\end{aligned}
$$

Multiplying by $\tilde{P}_2(i, S)$ and summing on $i \geq 2$, this leads to $V_7, V_5, V_8, V_{11}, V_{12}, V_{10}, V_{13}, V_{15}$. Indeed, the case $I > 1$ does *not* immediately lead to a success. Again we expand the mean up to the $\log n^*/n^*$ term and the square mean up to the $\log n^*$ term. We have

$$
\begin{aligned}
C_{n,2} &\sim S_\Gamma S(J, I) + JI + \tilde{C}_{I,S}, \qquad (7) \\
\mathbb{E}(C_{n,2}) &\sim \mathbb{E}(S_\Gamma(J, I) + JI + \tilde{F}_{i,S}) \\
&\sim \mathbb{E}(S_\Gamma(J, I) + JI) + U_1
\end{aligned}
$$

$$\sim \mathbb{E}\Big[(n-I)\frac{1}{p}+\frac{n}{q}\frac{e^{-L\eta}}{n^*}-\frac{n-I}{q}\frac{e^{-L\eta}}{n^*}(\log n^*+\eta)$$
$$-\frac{n}{q^2}\frac{e^{-2L\eta}}{n^{*2}}\log n^*+(\log n^*+\eta)I\Big]+U_1$$
$$\sim \frac{n}{p}V_1-\frac{1}{p}V_5+\frac{n}{q}\frac{V_7}{n^*}-\frac{n}{q}\frac{V_7}{n^*}\log n^*+\frac{1}{q}\frac{V_8}{n^*}\log n^*$$
$$-\frac{n}{q}\frac{V_{10}}{n^*}-\frac{n}{q^2}\frac{V_{11}}{n^{*2}}\log n^*+V_5\log n^*+V_{13}+U_1, \tag{8}$$

$$\begin{aligned}\mathbb{E}(C_{n,2}^2) &\sim \mathbb{E}((S_\Gamma(J,I)+JI)^2)+2\mathbb{E}[(S_\Gamma(J,I)+JI)\tilde{C}_{I,S}]\\ &\sim \mathbb{E}((S_\Gamma(J,I)+JI)^2)+2\mathbb{E}\left[\left[\frac{n}{p}-\frac{n}{q}\frac{e^{-L\eta}}{n^*}\log n^*+I\log n^*\right]\tilde{F}_{i,S}\right]\\ &\sim \mathbb{E}((S_\Gamma(J,I)+JI)^2)+2\left[\frac{n}{p}U_1-\frac{n}{q}\frac{U_3}{n^*}\log n^*+U_5\log n^*\right]\\ &\sim \mathbb{E}[n\mathbb{E}^{(2)}(J)+(n-I)(n-I-1)(\mathbb{E}(J))^2+2\mathbb{E}[(n-I)E(J)JI]\\ &+\mathbb{E}[(\log n^*+\eta)^2I^2]+2\left[\frac{n}{p}U_1-\frac{n}{q}\frac{U_3}{n^*}\log n^*+U_5\log n^*\right].\end{aligned}$$

10.3 General case.

The *total mean* is given by (we provide here only two terms)

$$\begin{aligned}\mathbb{E}(C_n) &\sim \mathbb{E}(C_{n,1})+\mathbb{E}(C_{n,2})\\ &\sim n\left(\frac{p}{L}+V_1\right)\frac{1}{p}+\left(-\frac{Z_7}{p}+Z_5-\frac{V_7}{p}+V_5\right)\log n^*.\end{aligned}$$

But the first term amounts to the mean of a sum of n GEOM pq^{l-1} RVs. Indeed, the GEOM pq^{l-1} RV has mean $\frac{1}{p}$, second moment $\frac{1+q}{p^2}$ and variance $\frac{q}{p^2}$. This is easy to explain: from (4) and (7), the correction $\tilde{C}_{I,S}$ is asymptotically $\mathcal{O}(1)$ and the correction $-\Delta$ is also asymptotically $\mathcal{O}(1)$. Similarly

$$\begin{aligned}\mathbb{E}(C_n^2) &\sim \mathbb{E}(C_{n,1}^2)+\mathbb{E}(C_{n,2}^2)\\ &\sim n^2\left(\frac{p}{L}+V_1\right)\frac{1}{p^2}+n\left(-\frac{2(Z_7-pZ_5)}{p^2}-\frac{2(V_7-pV_5)}{p^2}\right)\log n^*\end{aligned}$$

and the *variance* is finally given by (we must adequately condition on the dominant success probability $Pd(S):=\frac{p}{L}+V_1$)

$$\begin{aligned}\mathbb{V}(C_n) &\sim Pd(S)\left[\frac{\mathbb{E}(C_n^2)}{Pd(S)}-\left(\frac{\mathbb{E}(C_n)}{Pd(S)}\right)^2\right]\\ &\sim Pd(S)n\frac{q}{p^2}.\end{aligned}$$

So we obtain

Theorem 10.1 *The moments of C_n in case of success, in the model M1, are given by (with Maple, more terms could be provided, in particular the $\log^2 n^*$ and $\log n^*$ terms of the variance)*

$$\begin{aligned}\mathbb{E}(C_n) &\sim n\left(\frac{p}{L}+V_1\right)\frac{1}{p}+\left(-\frac{Z_7}{p}+Z_5-\frac{V_7}{p}+V_5\right)\log n^*,\\ \mathbb{V}(C_n) &\sim Pd(S)n\frac{q}{p^2}.\end{aligned}$$

Note again that the dominant term of the variance corresponds to a *sum of* n *iid GEOM* pq^{l-1} *RVs.* Intuitively, the asymptotic distribution should be Gaussian: again from (4) and (7), the correction $\tilde{C}_{I,S}$ is asymptotically $\mathcal{O}(1)$, but *not independent* of the dominant term and the correction $-\Delta$ is also asymptotically $\mathcal{O}(1)$, but *independent* of the dominant term. Actually we have the following theorem.

Theorem 10.2 *Conditioned on a success,*

$$\mathbb{P}\left[\frac{C_n - \mathbb{E}(C_n)}{\sqrt{\mathbb{V}(C_n)}} \leq x\right] \underset{n\to\infty}{\longrightarrow} \phi(x),$$

where $\phi(x)$ *denotes the Gaussian distribution function.*

Proof.

$$C_{n,S} = [\![I=1]\!][S_{\Gamma,1}(J,G) + JG] + [\![I>1]\!][S_{\Gamma,2}(J,I) + JI + \tilde{C}_{I,S}].$$

Here, $S_{\Gamma,1}, \eta_1$ are related to the case $I = 1$ and $S_{\Gamma,2}, \eta_2$ are related to the case $I > 1$. In the sequel, with some abuse of notation, $\mathcal{O}_V(1)$ will denote a RV, asymptotically independent of n, with finite moments.

$$C_{n,S} = \sum_j \left(\mathbb{P}[J=j, I=1][S_{\Gamma,1}(j,G) + jG] + \sum_{i\geq 2} \mathbb{P}[J=j, I=i]\left[[S_{\Gamma,2}(j,i) + ji]\tilde{P}_2(i,S) + \tilde{C}_{i,S}\right]\right).$$

We have, *conditioned on a success*, (we use the dominant success probability $Pd(S)$)

$$\begin{aligned}
\frac{C_{n,S}}{Pd(S)} &\sim \frac{\Pi_1}{Pd(S)} \sum_j \frac{\mathbb{P}[J=j,I=1]}{\Pi_1}[S_{\Gamma,1}(j,G) + jG] \\
&+ \frac{V_1}{Pd(S)} \sum_J \sum_{i\geq 2} \frac{\mathbb{P}[J=j,I=i]\tilde{P}_2(i,S)}{V_1}[S_{\Gamma,2}(j,i) + ji] \\
&+ \frac{V_1}{Pd(S)} \sum_J \sum_{i\geq 2} \frac{\mathbb{P}[J=j,I=i]}{V_1}\tilde{C}_{i,S} \\
&= \frac{\Pi_1}{Pd(S)} \sum_j \frac{\mathbb{P}[J=j,I=1]}{\Pi_1}[S_{\Gamma,1}(j,G) + jG] \\
&+ \frac{V_1}{Pd(S)} \sum_J \sum_{i\geq 2} \frac{\mathbb{P}[J=j,I=i]\tilde{P}_2(i,S)}{V_1}[S_{\Gamma,2}(j,i) + ji] + \mathcal{O}_V(1).
\end{aligned}$$

Again we will separate the contribution of urn J from that of urns $< J$. So, conditioning on $J = j$ and $\Gamma_k(j)$ denoting a sequence of iid truncated geometric RV $< j$,

$$\begin{aligned}
S_{\Gamma,1}(j,G) + jG &= S_{\Gamma,1}(j,0) - \sum_1^G \Gamma_k(j) + jG \\
&= S_{\Gamma,1}(j,0) + \mathcal{O}_V(1) + (\log n^* + \eta_1)G \\
&= S_{\Gamma,1}(j,0) + \mathcal{O}_V(1) + \log n^* \mathcal{O}_V(1),
\end{aligned}$$

and similarly for $S_{\Gamma,2}(j,i) + ji$. So

$$\begin{aligned}
\frac{C_{n,S}}{Pd(S)} &\sim \frac{\Pi_1}{Pd(S)} \sum_j \frac{\mathbb{P}[J=j,I=1]}{\Pi_1} S_{\Gamma,1}(j,0) \\
&+ \frac{V_1}{Pd(S)} \sum_J \sum_{i\geq 2} \frac{\mathbb{P}[J=j,I=i]\tilde{P}_2(i,S)}{V_1} S_{\Gamma,2}(j,0) + \mathcal{O}_V(1) + \log n^* \cdot \mathcal{O}_V(1).
\end{aligned}$$

Now we must show that $S_\Gamma(j,0)$ is asymptotically Gaussian. We could simply use Feller [2, example *IX,1,a* on triangular arrays], but we want an error estimation. We will provide the first terms of our expansions, but Maple "knows" more. The standard deviation of $\Gamma(j)$ will be denoted by $\sigma(j)$. We have

$$\Sigma_0 = 1 - \frac{e^{-L\eta}}{np},$$
$$E(j) \sim \frac{1}{p} - \frac{e^{-L\eta}(j-1)}{np},$$
$$\sigma(j) \sim \frac{\sqrt{q}}{p} - \frac{e^{-L\eta}(j-1)^2}{2n\sqrt{q}}.$$

Now the probability generating function (PGF) of $\Gamma(j)$ is given by

$$F(z) = \frac{\sum_1^{j-1} pq^{l-1}z^l}{\Sigma_0} = \frac{\frac{pz}{1-qz} - \frac{e^{-L\eta}z^j}{n(1-qz)}}{\Sigma_0},$$

and the PGF of $S_\Gamma(j,0)$ is given by $[F(z)]^n$. We will now use the Saddle point method (for a good introduction to this method, see Flajolet and Sedgewick [3, chapter VIII]). By Cauchy's theorem,

$$\mathbb{P}(S_\Gamma(j,0) = k) = \frac{1}{2\pi\mathbf{i}} \int_\Omega \frac{[F(z)]^n}{z^{k+1}} dz = \frac{1}{2\pi\mathbf{i}} \int_\Omega e^{H(z)} dz,$$

where Ω is inside the domain of analyticity of the integrand and encircles the origin and

$$H(z) = n\left[\ln\left[\frac{pz}{1-qz} - \frac{e^{-L\eta}z^j}{n(1-qz)}\right] - \ln[\Sigma_0]\right] - (k+1)\ln(z).$$

Set

$$H^{(i)} := \frac{d^i H}{dz^i}.$$

First we must find the solution of

$$H^{(1)}(\tilde{z}) = 0 \tag{9}$$

with smallest modulus.

Set $\tilde{z} := z^* - \varepsilon$, where $z^* = \lim_{n\to\infty} \tilde{z}$. Here, it is easy to check that $z^* = 1$. Set $k = nE(j) + \sqrt{n}\sigma(j)x$, x fixed. We will soon see that $\varepsilon = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$, so we can expand z^j in $F(z)$ as

$$z^j = 1 - j\varepsilon + \frac{j(j-1)}{2}\varepsilon^2 + \cdots$$

Also $j = \log n^* + \eta$. This leads, to first order (keeping only the ε term in (9)), to

$$\varepsilon := \frac{-px}{\sqrt{nq}} + \mathcal{O}\left(\frac{\log n^*}{n}\right).$$

This shows that, asymptotically, ε is given by a series of powers of $n^{-1/2}$, where each coefficient is given by a series of powers of $\log n^*$. To obtain more precision, we set again $k = nE(j) + \sqrt{n}\sigma(j)x$, expand in powers of $n^{-1/2}$, and equate each coefficient to 0. We have, with $\tilde{z} := z^* - \varepsilon = 1 - \varepsilon$,

$$\mathbb{P}(S_\Gamma(j,0) = k) = \frac{1}{2\pi\mathbf{i}} \int_\Omega \exp\left[H(\tilde{z}) + H^{(2)}(\tilde{z})(z-\tilde{z})^2/2! + \sum_{l=3}^{\infty} H^{(l)}(\tilde{z})(z-\tilde{z})^l/l!\right] dz.$$

Note that the linear term vanishes. Set $z = \tilde{z} + \mathbf{i}\tau$. This gives

$$\mathbb{P}(S_\Gamma(j,0) = k) \sim \frac{1}{2\pi} \exp[H(\tilde{z})] \int_{-\infty}^{\infty} \exp\left[H^{(2)}(\tilde{z})(\mathbf{i}\tau)^2/2! + \sum_{l=3}^{\infty} H^{(l)}(\tilde{z})(\mathbf{i}\tau)^l/l!\right] d\tau. \tag{10}$$

Let us first analyze $H(\tilde{z})$. We obtain

$$H(\tilde{z}) = -x^2/2 + \mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big).$$

Also,

$$H^{(2)}(\tilde{z}) = n\frac{q}{p^2} + \mathcal{O}(\sqrt{n}\,),$$
$$H^{(4)}(\tilde{z}) = \mathcal{O}(n).$$

We can now compute (10), for instance by using the classical trick of setting

$$S^{(2)}(\tilde{z})(i\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(i\tau)^l/l! = -u^2/2.$$

Computing τ as a truncated series in u, this gives, by inversion,

$$\tau = \frac{u}{\sqrt{nq/p^2}} + u^2\mathcal{O}\Big(\frac{1}{n}\Big).$$

Setting $d\tau = \frac{d\tau}{du}du$, and integrating on $-\infty < u < \infty$, this gives

$$\frac{1}{\sqrt{2\pi nq/p^2}}\left[1 + \mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big)\right].$$

Finally (10) leads to

$$\mathbb{P}(S_\Gamma(j,0) = k) \sim \frac{1}{\sqrt{2\pi nq/p^2}} e^{-x^2/2}\left[1 + \mathcal{O}\Big(\frac{1}{\sqrt{n}}\Big)\right].$$

Now we consider

$$\mathbb{P}\left(\frac{\frac{C_{n,S}}{Pd(S)} - \frac{\mathbb{E}(C_{nS})}{Pd(S)}}{\sqrt{nq/p^2}} \le x\right)$$
$$\sim \mathbb{P}\left(\frac{\frac{\Pi_1}{Pd(S)}\sum_j \frac{\mathbb{P}[J=j,I=1]}{\Pi_1}[S_{\Gamma,1}(j,0) - nE_1(j)] + \frac{V_1}{Pd(S)}\sum_J\sum_{i\ge 2}\frac{\mathbb{P}[J=j,I=i]\tilde{P}_2(i,S)}{V_1}[S_{\Gamma,2}(j,0) - nE_2(j)]}{\sqrt{nq/p^2}} + \frac{\mathcal{O}_V(1) + \log n^*\mathcal{O}_V(1)}{\sqrt{nq/p^2}} \le x\right)$$
$$\sim \mathbb{P}\left(\frac{\frac{\Pi_1}{Pd(S)}\sum_j \frac{\mathbb{P}[J=j,I=1]}{\Pi_1}[S_{\Gamma,1}(j,0) - nE_1(j)]}{\sqrt{n}\sigma_1(j)} + \frac{\frac{V_1}{Pd(S)}\sum_J\sum_{i\ge 2}\frac{\mathbb{P}[J=j,I=i]\tilde{P}_2(i,S)}{V_1}[S_{\Gamma,2}(j,0) - nE_2(j)]}{\sqrt{n}\sigma_2(j)} \le x\right)$$

as

$$\frac{\sigma(j)}{\sqrt{q/p^2}} \underset{n\to\infty}{\longrightarrow} 1.$$

Now

$$\frac{\mathbb{V}(C_n)}{Pd(S)nq/p^2} \underset{n\to\infty}{\longrightarrow} 1,$$

which concludes the proof ■

See also Kalpathy et al. [5], for a leader election scheme which stops if $I > 1$. C_n is shown to be asymptotically Gaussian.

11 Distribution of C_n (number coins flipped), with failure, M1.

Only the case $I > 1$ matters here. Proceeding as before (we omit the details), we finally derive

Theorem 11.1 *The moments of C_n in case of failure, M1, are given by*

$$\begin{aligned}\mathbb{E}(C_n) &\sim nV_1\frac{1}{p} + \Big(-\frac{V_7}{p} + V_5\Big)\log n^*,\\ \mathbb{V}(C_n) &\sim V_1 n\frac{q}{p^2}.\end{aligned}$$

Again, the distribution should be asymptotically Gaussian, but we did not check the details.

12 Matrix expressions, M1.

We will give a few explicit matrix expressions for several quantities computed before. We will not use these expressions, but we only wanted to show that some compact relations can be written down in some cases.

12.1 Success probability

Let

$$\Pi[i,u] := \Sigma_1(i,\tau)\binom{i}{u}q^u p^{i-u},\ i,u \geq 2,$$

and

$$\varphi_1(i) := \Sigma_1(i,\tau)\binom{i}{1}q^1 p^{i-1},\ i \geq 2.$$

Then we have the expression

$$P_2(.,S) = \sum_{k=0}^{\infty}\Pi^k\varphi_1 = [I-\Pi]^{-1}\varphi_1.$$

Note that, to get some precision in $P_5(S)$, only finite matrices are necessary.

12.2 Number of rounds

Let

$$\varphi_2(i) := \Sigma_2(i,\tau)P_2(i,S)/\Sigma_1(i,\tau),\ i \geq 2.$$

Then we have the expression

$$R_{.,S} = \Pi x_{.,S} + \varphi_2 = [I-\Pi]^{-1}\varphi_2.$$

12.3 Number of null rounds

Let

$$\varphi_3(i) := V_4(i,\tau)P_2(i,S)/\Sigma_1(i,\tau),\ i \geq 2.$$

Then we have the expression

$$I_{.,S} = [I-\Pi]^{-1}\varphi_3.$$

12.4 Number of leftovers

Fix r. Let

$$\Pi_1[i,u] := \Sigma_1(i,\tau)\binom{i}{u}q^u p^{i-u},\ \ i,u>r,$$

and

$$\varphi_3(i) := \Sigma_1(i,\tau)\binom{i}{r}q^r p^{i-r}P_3(r,r,F).$$

Then we have the expression

$$P_3(.,r,F) = [I-\Pi_1]^{-1}\varphi_3.$$

13 Model 2

We will only briefly mention the modifications related to the main expressions. A supplementary last index will indicate how many null rounds are allowed before failure. Only the mean in the success case will be given, all other cases are similarly computed.

$$P_2(i,S,\tau) = \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_2(\ell,S,\tau-s),\ \ i\geq 2,$$

$$\tilde{P}_2(i,S,\tau) = \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_2(\ell,S,\tau-s-1) = P_2(i,S,\tau-1),$$

$$P_1(i,k,S,\tau) = \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_1(\ell,k-1-s,S,\tau-s),\ \ i\geq 2,$$

$$\tilde{P}_1(i,k,S,\tau) = \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_1(\ell,k-1-s,S,\tau-s-1) = P_1(i,k,S,\tau-1),$$

$$R_{i,S}(\tau) = \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_{\ell,S}(\tau-s) + \sum_{s=0}^{\tau-1}(p^i)^s(s+1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_2(\ell,S,\tau-s)$$

$$\tilde{R}_{i,S}(\tau) = R_{i,S}(\tau-1),$$

$$F_{i,S}(\tau) = \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}[si+i+F_{\ell,S}(\tau-s)],$$

$$\tilde{F}_{i,S}(\tau) = F_{i,S}(\tau-1),$$

$$P_7(i,t,S,\tau) = \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_7(\ell,t-s,S,\tau-s),\ \ i\geq 2,$$

$$\tilde{P}_7(i,t,S) = \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_7(\ell,t-1-s,S,\tau-s-1),\ \ i\geq 2,$$

$$I_{i,S}(\tau) = \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}I_{\ell,S}(\tau-s) + \sum_{s=0}^{\tau-1}(p^i)^s s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_2(\ell,S,\tau-s)$$

$$\tilde{I}_{i,S} = \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}I_{\ell,S}(\tau-s-1) + \sum_{s=0}^{\tau-2}(p^i)^s(s+1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_2(\ell,S,\tau-s-1),$$

$$P_3(r,r,F,\tau) = (p^r)^\tau + \Sigma_1(r,\tau)q^r P_3(r,r,F,\tau),$$

$$\tilde{P}_3(r,r,F,\tau) = P_3(r,r,F,\tau-1),$$

$$P_3(i,r,F,\tau) = \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=r}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_3(\ell,r,F,\tau-s), i>r,\ \ i\geq 2,$$

$$\tilde{P}_3(i,r,F,\tau) = \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=r}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_3(\ell,r,F,\tau-s-1) = P_3(i,r,F,\tau-1),$$

$$L_{i,F}(\tau) = \sum_{s=0}^{\tau-1}\sum_{\ell=r}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} \sum_{r=0}^{i-1} P_3(\ell,r,F,\tau-s)r + [(p^i)^\tau + q^i P_3(i,i,F,\tau)]i,$$

$$\tilde{L}_{i,F} = L_{i,F}(\tau-1).$$

Appendices.

A Some definitions and identities

Here, $\xi \in \{R, I, L, F\}$. ξ_i must be replaced by $\xi_{i,S}$ or $\xi_{i,F}$ depending on the case we consider. Also $\tilde{P}(i)$ must be replaced by $\tilde{P}_2(i,S)$ or $\tilde{P}_{10}(i,F)$, respectively. Note that, compared with [10], we use here $\Pi_2(i) = \frac{p^i}{Li}$ instead of $\frac{p^i}{i}$, for $V_1, \ldots, V_5$. Also we have $\tilde{P}_2(1,S) = 1, \tilde{P}_{10}(1,F) = 0$.

$$V_1 := \sum_{i=2}^{\infty} \frac{p^i}{Li}\tilde{P}(i),$$

$$V_2 := \sum_{i=2}^{\infty} \frac{p^i \psi(i)}{Li}\tilde{P}(i),$$

$$V_3 := \sum_{i=2}^{\infty} \frac{p^i \psi(i)^2}{Li}\tilde{P}(i),$$

$$V_4 := \sum_{i=2}^{\infty} \frac{p^i \psi(1,i)}{Li}\tilde{P}(i),$$

$$V_5 := \sum_{i=2}^{\infty} \frac{p^i i}{Li}\tilde{P}(i),$$

$$V_7 := \sum_{i=2}^{\infty} \frac{p^i p}{L}\tilde{P}(i) = pV_1,$$

$$V_8 := \sum_{i=2}^{\infty} \frac{p^i p i}{L}\tilde{P}(i) = pV_5,$$

$$V_{10} := \sum_{i=2}^{\infty} -\frac{p^i p[i\ln(p) + i\psi(i) + 1]}{L^2 i}\tilde{P}(i),$$

$$V_{11} := \sum_{i=2}^{\infty} \frac{p^i p^2(i+1)}{L}\tilde{P}(i) = p^2V_5 + p^2V_1,$$

$$V_{12} := \sum_{i=2}^{\infty} -\frac{p^i[\ln(p) + \psi(i)]}{L^2 i}\tilde{P}(i),$$

$$V_{13} := \sum_{i=2}^{\infty} -\frac{p^i[\ln(p) + \psi(i)]i}{L^2 i}\tilde{P}(i),$$

$$V_{15} := \sum_{i=2}^{\infty} -\frac{p^i p^2[-i^2\ln(p) - i^2\psi(i) - 2i + i\ln(p) - i\psi(i) + 1]}{L^2 i}\tilde{P}(i),$$

$$V_{16} := \sum_{i=2}^{\infty} -\frac{p^i p[i\ln(p) + i\psi(i) + 1]i}{L^2 i}\tilde{P}(i),$$

$$R_0(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!}\Gamma(i+\chi_l)\tilde{P}(i),$$

$$R_1(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!}\Gamma(i+\chi_l)\psi(i+\chi_l)\tilde{P}(i),$$

$$R_2(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!}\Gamma(i+\chi_l)\psi(1, i+\chi_l)\tilde{P}(i),$$

$$R_3(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!}\Gamma(i+\chi_l)\psi^2(i+\chi_l)\tilde{P}(i),$$

$$T_1(\chi_l) := \sum_{i=2}^{\infty} \frac{p^i}{Li!}\xi_i\Gamma(i+\chi_l),$$

$$T_2(\chi_l) := \sum_{i=2}^{\infty} \frac{p^i}{Li!}\xi_i^{(2)}\Gamma(i+\chi_l),$$

$$T_3(\chi_l) := \sum_{i=2}^{\infty} \frac{p^i}{Li!}\xi_i\psi(i+\chi_l)\Gamma(i+\chi_l),$$

$$U_1 := \sum_{i=2}^{\infty} \frac{p^i\xi_i}{Li},$$

$$U_2 := \sum_{i=2}^{\infty} \frac{p^i\xi_i^{(2)}}{Li},$$

$$U_3 := \sum_{i=2}^{\infty} \frac{pp^i\xi_i}{L},$$

$$U_4 := \sum_{i=2}^{\infty} \frac{p^i\xi_i\psi(i)}{i},$$

$$U_5 := \sum_{i=2}^{\infty} \frac{p^i i\xi_i}{Li},$$

$$Z_7 := \frac{p^2[2 - 3p + p^2]}{q^2 L},$$

$$Z_8 := \sum_{g=2}^{\infty} \frac{p^g q g^2}{L} = \frac{p^2[4 - 3p + p^2]}{q^2 L},$$

$$Z_{11} := \frac{2p^3(p^2 - 3p + 3)}{Lq^2},$$

$$Z_{12} := \sum_{g=2}^{\infty} -\frac{p^{g-1}q[(g-1)\ln(p) + (g-1)\psi(g-1) + 1]}{L^2(g-1)},$$

$$Z_{10} := \sum_{g=2}^{\infty} -\frac{qp^g[2(g-1) + (g-1)^2\ln(p) + (g-1)^2\psi((g-1)) + (g-1)\ln(p) + (g-1)\psi((g-1)) + 1]}{L^2(g-1)},$$

$$Z_{13} := \sum_{g=2}^{\infty} -\frac{p^{g-1}q[(g-1)\ln(p) + (g-1)\psi((g-1)) + 1]g}{L^2(g-1)},$$

$$Z_{15} := \sum_{g=2}^{\infty} \Omega_{15}(g).$$

B Success Probability

We show here that, where the results are given both, here, and in [7], they coincide. First, we look at Theorem 4.1. The constant is given by

$$\frac{1}{L}\sum_{k\geq 2} S_k \frac{p^k}{k}\frac{1-p^{(\tau-1)k}}{1-p^{\tau k}} + \frac{p}{L},$$

and it should coincide with

$$\frac{1}{L}\Big(qp^{\tau} + \sum_{k\geq 1}\frac{S_k}{k}\Big(p^k - \frac{q^k p^{\tau k}}{(1-q^{\tau+1})^k}\Big)\Big).$$

We use the notation S_n from [7] for the quantity $P_2(n,S)$ from this paper. We have the recursion

$$\frac{1-p^k}{1-p^{\tau k}} S_k = \sum_{j=1}^{k}\binom{k}{j} p^{k-j} q^j S_j.$$

Therefore

$$\begin{aligned}
\sum_{k\geq 2} S_k \frac{p^k}{k}\frac{1-p^{(\tau-1)k}}{1-p^{\tau k}} &= \sum_{k\geq 2} S_k \frac{p^k}{k}\frac{1-p^{\tau k}+p^{\tau k}-p^{(\tau-1)k}}{1-p^{\tau k}} \\
&= \sum_{k\geq 2} S_k \frac{p^k}{k} - \sum_{k\geq 2} S_k \frac{p^{\tau k}}{k}\frac{1-p^k}{1-p^{\tau k}} \\
&= \sum_{k\geq 2} S_k \frac{p^k}{k} - \sum_{k\geq 2} \frac{p^{\tau k}}{k}\sum_{j=1}^{k}\binom{k}{j} p^{k-j} q^j S_j \\
&= \sum_{k\geq 1} S_k \frac{p^k}{k} - p + p^{\tau} q - \sum_{j\geq 1}\frac{q^j S_j}{j}\sum_{k\geq j} p^{\tau k}\binom{k-1}{j-1} p^{k-j} \\
&= \sum_{k\geq 1} S_k \frac{p^k}{k} - p + p^{\tau} q - \sum_{j\geq 1}\frac{q^j S_j}{j}\frac{p^{\tau j}}{(1-p^{\tau+1})^j},
\end{aligned}$$

which is the desired formula after trivial modifications.

For the Fourier coefficients, we have to prove that

$$p\Gamma(\chi_l+1) + \sum_{k\geq 2} S_k \frac{p^k}{k!}\frac{1-p^{(\tau-1)k}}{1-p^{\tau k}}\Gamma(\chi_l+k) = qp^{\tau}\Gamma(\chi_l+1) + \sum_{k\geq 1}\frac{S_k}{k!}\Gamma(\chi_l+k)\Big(p^k - \frac{q^k p^{\tau k}}{(1-q^{\tau+1})^{\chi_l+k}}\Big),$$

which is done in a similar way:

$$\begin{aligned}
&\sum_{k\geq 2} S_k \frac{p^k}{k!}\frac{1-p^{(\tau-1)k}}{1-p^{\tau k}}\Gamma(\chi_l+k) = \sum_{k\geq 2} S_k \frac{p^k}{k!}\frac{1-p^{\tau k}+p^{\tau k}-p^{(\tau-1)k}}{1-p^{\tau k}}\Gamma(\chi_l+k) \\
&\quad= \sum_{k\geq 1} S_k \frac{p^k}{k!}\Gamma(\chi_l+k) - p\Gamma(\chi_l+1) - \sum_{k\geq 2}\frac{p^{\tau k}}{k!}\Gamma(\chi_l+k)\sum_{j=1}^{k}\binom{k}{j} p^{k-j} q^j S_j \\
&\quad= \sum_{k\geq 1} S_k \frac{p^k}{k!}\Gamma(\chi_l+k) - p\Gamma(\chi_l+1) - \sum_{j\geq 1}\frac{q^j S_j}{j!}\sum_{k\geq j} p^{\tau k}\Gamma(\chi_l+k)\frac{1}{(k-j)!}p^{k-j} + qp^{\tau}\Gamma(\chi_l+1) \\
&\quad= \sum_{k\geq 1} S_k \frac{p^k}{k!}\Gamma(\chi_l+k) - p\Gamma(\chi_l+1) + qp^{\tau}\Gamma(\chi_l+1) - \sum_{j\geq 1}\frac{q^j S_j}{j!}\frac{\Gamma(\chi_l+j)p^{\tau j}}{(1-p^{\tau+1})^{\chi_j+k}},
\end{aligned}$$

which is the formula.

C Total number of rounds

Next we turn to the nonfluctuating part of R_n. We must use the total number of rounds: $\tilde{R}_i = \tilde{R}_{i,S} + \tilde{R}_{i,F}$. Now from Theorems 5.1 and 8.1, we have

$$R_n - \log n^* = \mathbb{E}(K_n - \log n^*) \sim \bar{U}_1 - M\bar{V}_1 - \frac{\bar{V}_2}{L} + \frac{p\gamma}{L^2} - \frac{1+pM}{L}$$
$$+ \sum_{l\neq 0}\left[\bar{T}_1(\chi_l) - M\bar{R}_0(\chi_l) - \frac{\bar{R}_1(\chi_l)}{L} - \frac{\Gamma(1+\chi_l)}{L}\right]e^{-2l\pi \mathbf{i}\log n}$$

where, now,

$$\begin{aligned}
\bar{T}_1(\chi_l) &:= \sum_{i=2}^{\infty}\frac{p^i}{Li!}\tilde{R}_i\Gamma(i+\chi_l),\\
\bar{U}_1 &:= \sum_{i=2}^{\infty}\frac{p^i\tilde{R}_i}{Li},\\
\bar{R}_0(\chi_l) &:= 0,\\
\bar{R}_1(\chi_l) &:= \Gamma(\chi_l),\\
\bar{V}_1 &:= \frac{L-p}{L},\\
\bar{V}_2 &:= \frac{L}{2} - \frac{\gamma(L-p)}{L}
\end{aligned}$$

We have the following recurrences, with P_0 now combining S and F,

$$P_0(1,0) = 1, \tilde{P}_0(1,0) = 1,$$
$$P_0(i,k) = (p^i)^\tau [\![k=\tau]\!] + \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_0(\ell,k-1-s),\ i\geq 2,$$
$$\tilde{P}_0(i,k) = (p^i)^{\tau-1}[\![k=\tau-1]\!] + \sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_0(\ell,k-1-s),\ i\geq 2.$$

$$\begin{aligned}
R_i &= \sum_k P_0(i,k)k = (p^i)^\tau\tau + \sum_k\sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_0(\ell,k-1-s)[k-1-s+s+1],\\
&= (p^i)^\tau\tau + \sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell + \sum_k\sum_{s=0}^{\tau-1}(p^i)^s(s+1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_0(\ell,k-1-s)\\
&= (p^i)^\tau\tau + \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell + \Sigma_2(i,\tau)\left[1-(p^i)^\tau\right]/\Sigma_1(i,\tau),\\
&= \frac{1-p^{i\tau}}{1-p^i}\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell + \frac{1-p^{i\tau}}{1-p^i},\ R_1 = 0,
\end{aligned}$$

$$\begin{aligned}
\tilde{R}_i &= \sum_k \tilde{P}_0(i,k)k = (p^i)^{\tau-1}(\tau-1) + \Sigma_1(i,\tau-1)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}R_\ell,\\
&+ \Sigma_2(i,\tau-1)\left[1-(p^i)^\tau\right]/\Sigma_1(i,\tau)\\
&= (p^i)^{\tau-1}(\tau-1) + \Sigma_1(i,\tau-1)\left[R_i - \Sigma_2(i,\tau)\left[1-(p^i)^\tau\right]/\Sigma_1(i,\tau) - (p^i)^\tau\right]/\Sigma_1(i,\tau)\\
&+ \Sigma_2(i,\tau-1)\left[1-(p^i)^\tau\right]/\Sigma_1(i,\tau)
\end{aligned}$$

$$= R_i \frac{1 - p^i - p^{i(\tau-1)} + p^{i\tau}}{(1 - p^{i\tau})(1 - p^i)}$$
$$= R_i \frac{1 - p^{i(\tau-1)}}{1 - p^{i\tau}}, \tilde{R}_1 = 0.$$

In [7], the following recursion is derived:

$$R_n(\tau) = \sum_{j=1}^{n} \binom{n}{j} p^{n-j} q^j R_j(\tau) + p^n R_n(\tau - 1) + 1, \quad \tau > 0,\ n \geq 2,$$

and the interest is in $R_n = R_n(\tau)$. We write

$$R_n(\tau) = K_n(\tau) + p^n R_n(\tau - 1) = K_n(\tau) + p^n(K_n(\tau) + R_n(\tau - 2)) = \cdots = \frac{1 - p^{\tau n}}{1 - p^n} K_n(\tau).$$

We find the recursion

$$R_n = \frac{1 - p^{\tau n}}{1 - p^n} \sum_{j=1}^{n} \binom{n}{j} p^{n-j} q^j R_j + \frac{1 - p^{\tau n}}{1 - p^n}.$$

This coincides with the recursion given here. Now

$$\begin{aligned}
&\bar{U}_1 - M\bar{V}_1 - \frac{\bar{V}_2}{L} + \frac{p\gamma}{L^2} - \frac{1 + pM}{L} \\
&\quad = \sum_{i=2}^{\infty} \frac{p^i \tilde{R}_i}{Li} - \frac{\ln p}{L} \frac{L - p}{L} - \frac{1}{L}\Big(\frac{L}{2} - \frac{\gamma(L - p)}{L}\Big) + \frac{p\gamma}{L^2} - \frac{1}{L} - \frac{p \ln p}{L^2} \\
&\quad = \frac{1}{L} \sum_{i=2}^{\infty} \frac{p^i(1 - p^{i(\tau-1)})}{i} \frac{R_i}{1 - p^{i\tau}} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L} \\
&\quad = \frac{1}{L} \sum_{i=2}^{\infty} \frac{p^i(1 - p^{i\tau} + p^{i\tau} - p^{i(\tau-1)})}{i} \frac{R_i}{1 - p^{i\tau}} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L} \\
&\quad = \frac{1}{L} \sum_{i=2}^{\infty} \frac{p^i R_i}{i} - \frac{1}{L} \sum_{i=2}^{\infty} \frac{p^{i\tau}}{i} \frac{R_i(1 - p^i)}{1 - p^{i\tau}} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L} \\
&\quad = \frac{1}{L} \sum_{i=2}^{\infty} \frac{p^i R_i}{i} - \frac{1}{L} \sum_{i=1}^{\infty} \frac{p^{i\tau}}{i} \sum_{j=1}^{i} \binom{i}{j} p^{i-j} q^j R_j - \frac{1}{L} \sum_{i=1}^{\infty} \frac{p^{i\tau}}{i} + \frac{p^\tau}{L} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L} \\
&\quad = \frac{1}{L} \sum_{i=2}^{\infty} \frac{p^i R_i}{i} - \frac{1}{L} \sum_{j \geq 1} \frac{q^j R_j}{j} \sum_{i \geq j} p^{i\tau} \binom{i-1}{j-1} p^{i-j} + \frac{1}{L} \ln(1 - p^\tau) + \frac{p^\tau}{L} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L} \\
&\quad = \frac{1}{L} \sum_{i=2}^{\infty} \frac{p^i R_i}{i} - \frac{1}{L} \sum_{j \geq 1} \frac{q^j R_j}{j} \frac{p^{j\tau}}{(1 - p^{\tau+1})^j} + \frac{1}{L} \ln(1 - p^\tau) + \frac{p^\tau}{L} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}.
\end{aligned}$$

Thus the constant term in the asymptotic expansion of $R_n - \log n$ is

$$\begin{aligned}
&\frac{\ln p}{L} + 1 + \frac{1}{L} \sum_{i=2}^{\infty} \frac{p^i R_i}{i} - \frac{1}{L} \sum_{j \geq 1} \frac{q^j R_j}{j} \frac{p^{j\tau}}{(1 - p^{\tau+1})^j} + \frac{1}{L} \ln(1 - p^\tau) + \frac{p^\tau}{L} - \frac{\ln p}{L} - \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L} \\
&= \frac{1}{L} \sum_{i \geq 2} \frac{p^i R_i}{i} - \frac{1}{L} \sum_{j \geq 2} \frac{q^j R_j}{j} \frac{p^{j\tau}}{(1 - p^{\tau+1})^j} + \frac{1}{L} \ln(1 - p^\tau) + \frac{p^\tau}{L} + \frac{1}{2} + \frac{\gamma}{L} - \frac{1}{L}.
\end{aligned}$$

This is the expansion that was also obtained in [7].

D Total number of null rounds

We have the recurrences:

$$
\begin{aligned}
&P_{13}(1,0) = 1, \quad \tilde{P}_{13}(1,1) = 1,\\
&P_{13}(i,t) = (p^i)^\tau [\![t=\tau]\!] + \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_{13}(\ell, t-s),\\
&\tilde{P}_{13}(i,t) = (p^i)^{\tau-1} [\![t=\tau]\!] + \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_{13}(\ell, t-1-s).
\end{aligned}
$$

$$
\begin{aligned}
I_i = \sum_t P_{13}(i,t)t &= (p^i)^\tau \tau + \sum_t \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_{13}(\ell, t-s)[t-s+s]\\
&= (p^i)^\tau \tau + \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell + \sum_t \sum_{s=0}^{\tau-1} (p^i)^s s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_{13}(\ell, t-s),\\
&= (p^i)^\tau \tau + \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell + \Sigma_4(i,\tau)[1-(p^i)^\tau]/\Sigma_1(i,\tau)\\
&= \frac{1-p^{i\tau}}{1-p^i} \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell + \frac{p^i\left(1-p^{i\tau}\right)}{1-p^i}, \quad I_1 = 0.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{I}_i = \sum_t \tilde{P}_{13}(i,t)t &= (p^i)^{\tau-1} \tau + \sum_t \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_{13}(\ell, t-1-s)[t-1-s+s+1]\\
&= (p^i)^{\tau-1} \tau + \sum_{s=0}^{\tau-2} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} I_\ell + \sum_t \sum_{s=0}^{\tau-2} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_{13}(\ell, t-1-s)\\
&= (p^i)^{\tau-1} \tau + \Sigma_1(i,\tau-1)\left[I_i - (p^i)^\tau \tau - \Sigma_4(i,\tau)[1-(p^i)^\tau]/\Sigma_1(i,\tau)\right]/\Sigma_1(i,\tau)\\
&\qquad\qquad + \Sigma_2(i,\tau-1)[1-(p^i)^\tau]/\Sigma_1(i,\tau)\\
&= \frac{1-p^i+p^{i(\tau+1)}-p^{i\tau}}{(1-p^{i\tau})(1-p^i)} + \frac{1-p^i-p^{i(\tau-1)}+p^{i\tau}}{(1-p^{i\tau})(1-p^i)} I_i\\
&= 1 + \frac{1-p^{i(\tau-1)}}{1-p^{i\tau}} I_i, \quad \tilde{I}_1 = 1.
\end{aligned}
$$

The mean is given by

$$
I_n \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{I}_i + \sum_{l \neq 0} \sum_{i=2}^{\infty} \frac{p^{i+\chi_l} \Gamma(i+\chi_l)}{Li!} \tilde{I}_i e^{-2l\pi \mathbf{i} \{\log n^*\}}.
$$

Indeed, I_n from the paper [7] satisfies the same recursion as here, after unwinding it as shown in the previous example.

And now we look at the nonfluctuating part in the asymptotic expansion of the mean:

$$
\begin{aligned}
\sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{I}_i &= \frac{1}{L} \sum_{n \geq 2} \frac{p^n}{n} \left(1 + \frac{1-p^{n(\tau-1)}}{1-p^{n\tau}} I_n\right)\\
&= \frac{1}{L}(-\ln(1-p)-p) + \frac{1}{L} \sum_{n \geq 2} \frac{p^n}{n} \frac{1-p^{n\tau}+p^{n\tau}-p^{n(\tau-1)}}{1-p^{n\tau}} I_n
\end{aligned}
$$

$$
\begin{aligned}
&= 1 - \frac{p}{L} + \frac{1}{L}\sum_{n\ge 2}\frac{p^n}{n}I_n - \frac{1}{L}\sum_{n\ge 2}\frac{p^{n\tau}}{n}\frac{1-p^n}{1-p^{n\tau}}I_n \\
&= 1 - \frac{p}{L} + \frac{1}{L}\sum_{n\ge 1}\frac{p^n}{n}I_n - \frac{1}{L}\sum_{n\ge 2}\frac{p^{n\tau}}{n}\left(\sum_{1\le j\le n}\binom{n}{j}q^j p^{n-j}I_j + p^n\right) \\
&= 1 - \frac{p}{L} + \frac{1}{L}\sum_{n\ge 1}\frac{p^n}{n}I_n - \frac{1}{L}\sum_{n\ge 1}\frac{p^{n(\tau+1)}}{n} + \frac{p^{\tau+1}}{L} - \frac{1}{L}\sum_{n\ge 1}\frac{p^{n\tau}}{n}\sum_{1\le j\le n}\binom{n}{j}q^j p^{n-j}I_j \\
&= 1 - \frac{p}{L} + \frac{1}{L}\ln(1-p^{\tau+1}) + \frac{p^{\tau+1}}{L} + \frac{1}{L}\sum_{n\ge 1}\frac{p^n}{n}I_n - \frac{1}{L}\sum_{j\ge 1}\frac{q^j I_j}{j}\sum_{n\ge j}\binom{n-1}{j-1}p^{n\tau}p^{n-j} \\
&= 1 - \frac{p}{L} + \frac{1}{L}\ln(1-p^{\tau+1}) + \frac{p^{\tau+1}}{L} + \frac{1}{L}\sum_{n\ge 1}\frac{p^n}{n}I_n - \frac{1}{L}\sum_{j\ge 1}\frac{q^j I_j}{j}\frac{p^{j\tau}}{(1-\tau^{\tau+1})^j}.
\end{aligned}
$$

This is the expression given in [7].

E Total number of leftovers

We have here only the failure case. This gives (we drop the F index)

$$
\begin{aligned}
L_i &= \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^ip^{i\tau}}i + \Sigma_1(i,\tau)\left[\sum_{\ell=0}^{i-1}\binom{i}{\ell}q^\ell p^{i-\ell}L_\ell + q^i\left[L_i - \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^ip^{i\tau}}i\right]\right] \\
\tilde{L}_i &= \left[p^{i(\tau-1)} + \frac{1-p^{i(\tau-1)}}{1-p^i}q^i\frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^ip^{i\tau}}\right]i + \frac{1-p^{i(\tau-1)}}{1-p^{i\tau}}\left[L_i - \frac{p^{i\tau}(1-p^i)}{1-p^i-q^i+q^ip^{i\tau}}i\right] \\
&= \frac{1-p^{i(\tau-1)}}{1-p^{i\tau}}L_i + \frac{p^{i(\tau-1)}(1-p^i)}{1-p^{i\tau}}i.
\end{aligned}
$$

The mean is given by

$$
L_n \sim \sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{L}_i + \sum_{l\ne 0}\sum_{i=2}^{\infty}\frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{L}_i e^{-2l\pi\mathbf{i}\{\log n^*\}}.
$$

The recursion derived in [7] is

$$
L_n = \frac{1-p^{n\tau}}{1-p^n}\sum_{j=1}^{n}\binom{n}{j}p^{n-j}q^jL_j + np^{n\tau}.
$$

Here, we have

$$
\begin{aligned}
L_n &= \frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}}n + \frac{1-p^{n\tau}}{1-p^n}\left[\sum_{j=1}^{n-1}\binom{n}{j}q^jp^{n-j}L_j + q^n\left(L_n - \frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}}n\right)\right] \\
&= \frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}}n + \frac{1-p^{n\tau}}{1-p^n}\left[\sum_{j=1}^{n}\binom{n}{j}q^jp^{n-j}L_j - q^n\frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}}n\right] \\
&= \frac{p^{n\tau}(1-p^n)}{1-p^n-q^n+q^np^{n\tau}}n\left[1 - \frac{1-p^{n\tau}}{1-p^n}q^n\right] + \frac{1-p^{n\tau}}{1-p^n}\sum_{j=1}^{n}\binom{n}{j}q^jp^{n-j}L_j \\
&= p^{n\tau}n + \frac{1-p^{n\tau}}{1-p^n}\sum_{j=1}^{n}\binom{n}{j}q^jp^{n-j}L_j,
\end{aligned}
$$

and hence we do have the same recursion.

Now we turn to the nonfluctuating part of the mean:

$$\sum_{i=2}^{\infty}\frac{p^i}{Li}\tilde{L}_i=\sum_{i\ge 2}\frac{p^i}{Li}\left[\frac{1-p^{i(\tau-1)}}{1-p^{i\tau}}L_i+\frac{p^{i(\tau-1)}(1-p^i)}{1-p^{i\tau}}i\right]$$
$$=\frac{1}{L}\sum_{i\ge 2}\frac{p^i}{i}\frac{1-p^{i\tau}+p^{i\tau}-p^{i(\tau-1)}}{1-p^{i\tau}}L_i+\frac{1}{L}\sum_{i\ge 2}\frac{p^{i\tau}(1-p^i)}{1-p^{i\tau}}$$
$$=\frac{1}{L}\sum_{i\ge 2}\frac{p^i}{i}L_i-\frac{1}{L}\sum_{i\ge 2}\frac{p^{i\tau}}{i}\left[\sum_{j=1}^{i}\binom{i}{j}p^{i-j}q^jL_j+\frac{1-p^i}{1-p^{i\tau}}ip^{i\tau}\right]+\frac{1}{L}\sum_{i\ge 2}\frac{p^{i\tau}(1-p^i)}{1-p^{i\tau}}$$
$$=\frac{1}{L}\sum_{i\ge 2}\frac{p^i}{i}L_i-\frac{1}{L}\sum_{i\ge 2}\frac{p^{i\tau}}{i}\sum_{j=1}^{i}\binom{i}{j}p^{i-j}q^jL_j-\frac{1}{L}\sum_{i\ge 2}\frac{1-p^i}{1-p^{i\tau}}p^{2i\tau}+\frac{1}{L}\sum_{i\ge 2}\frac{p^{i\tau}(1-p^i)}{1-p^{i\tau}}$$
$$=\frac{1}{L}\sum_{i\ge 2}\frac{p^i}{i}L_i-\frac{1}{L}\sum_{j\ge 1}\frac{q^jL_j}{j}\sum_{i\ge j}p^{i\tau}\binom{i-1}{j-1}p^{i-j}+\frac{1}{L}\sum_{i\ge 2}p^{i\tau}(1-p^i)$$
$$=\frac{1}{L}\sum_{i\ge 2}\frac{p^i}{i}L_i-\frac{1}{L}\sum_{j\ge 1}\frac{q^jL_j}{j}\frac{q^{j\tau}}{(1-p^{\tau+1})^j}+\frac{1}{L}\frac{1}{1-p^{\tau}}-\frac{1}{L}\frac{1}{1-p^{\tau+1}}-qp^{\tau},$$

which is the same expression as in [7].

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