Asymptotic Analysis of the Sums of Powers of Multinomial Coefficients: A Saddle Point Approach

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Dedicated to Daniel D. Bonar,
George R. Stibitz Distinguished Professor in Mathematics and Computer Science at Denison University,
a professor, mentor, and dear friend, in celebration of 50 years of sharing his love of Mathematics.

1 Introduction and Motivation

We define

$$a_{m,k}(n) := \sum_{i_1+i_2+\cdots+i_m=n} \binom{n}{i_1, i_2, \ldots, i_m}^k,$$

and

$$b_{m,k}(n) := \frac{a_{m,k}(n)}{(n!)^k}.$$

When we first encountered these integer sequences, we assumed that their asymptotic developments had been well studied, but after extensively checking the literature, we believe that only special cases of the asymptotics have been analyzed. Due to the fundamental nature of these integer sequences, we decided to make a comprehensive characterization of the asymptotic growth of these integer sequences, as $n \to \infty$, for any (fixed) positive values m and k.

These integer sequences are intimately connected with hypergeometric functions, as seen in equation (1). These integer sequences have also been of interest for a long time. The general family $a_{m,k}(n)$ appears, for instance, in Barrucand [10]. The asymptotic growth of $a_{2,k}(n)$ has been known for (almost) a century [14], and perhaps longer.

In the case $k = 1$, the values of $a_{m,k}(n)$ are simply the powers of m, namely, $a_{m,1}(n) = m^n$.

The $k = 2$ case has myriad interpretations. They are used in Proposition 1 of Borwein et al. [3] and in the discussion and remarks after the proposition is proved. They use the notation "$W_m(2n)$" for our sequences $a_{m,2}(n)$. Borwein and his co-authors point out that $a_{m,2}(n)$ is the number of abelian squares of length $2n$ constructed from an alphabet that has m letters (i.e., strings of the form $x_1, \ldots, x_n, x_{\sigma(1)}, \ldots, x_{\sigma(n)}$ where σ is a permutation of $1, \ldots, n$); also see [15] for more details about such abelian squares. For another application of the family of sequences $a_{m,2}(n)$, in number theory, see [2, p. 108]. The integers $a_{4,2}(n)$ are known as the Domb numbers; they enumerate the number of $2n$-step polygons on a diamond lattice; see [6] and also OEIS #A002895. The sequences $a_{m,2}(n)$ are also a key object of study in [17]. We close our discussion of the $k = 2$ case by noting

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that when $k = 2$ and $m = 2$, we have the sequence $a_{2,2}(n) = \binom{2n}{n}$, which is cited and utilized in hundreds of applications.

When $k = 3$ and $m = 2$, then $a_{2,3}(n)$ are known as the Franel numbers, which have dozens of applications; see OEIS #A000172 as a starting point for the vast literature on this sequence.

Many other notes and references could be given about the sequences $a_{m,k}(n)$ for small, fixed values of m and k, because these sequences are pervasive in the mathematical literature. For example, very recently, the sequences $a_{m,k}(n)$ were used in their full generality in a manuscript by Tao [16]; see especially Tao's Theorem 7 (and its proof), and Corollaries 2 and 3. It is infeasible for us to give a comprehensive discussion of all such appearances of this family of sequences. The On-Line Encyclopedia of Integer Sequences—and the myriad references therein—remains an excellent resource for finding applications of these sequences, for specific values of m and k.

To emphasize the generality of our results, we make a comparison with the known results:

Remark 1.1 *We summarize the special cases that were already proved in the literature.*

$k = 2$ case: The asymptotic growth of $a_{m,2}(n)$ was established in 2009 by Richmond and Shallit in [15]. They write "$f_m(n)$" for our sequences $a_{m,2}(n)$. Their Theorem 4 establishes the asymptotic growth of $a_{m,2}(n)$ for fixed m, as $n \to \infty$.

$m = 2$ case: The asymptotic growth of $a_{2,k}(n)$ was known as early as 1925, as obtained by Polya and Szegő [14, Problem 40, p. 55, of volume 1 of the English edition]. See also Farmer and Leth [7] and Wilson [18, equation (4.6) of Example 4.3] for more recent discussions.

Our Theorem 3.2 is a full generalization of all of these special cases, as we establish the asymptotic growth of $a_{m,k}(n)$ for any *positive fixed values m and k, as $n \to \infty$.*

2 Notation and Background

Since $\binom{n}{i_1,i_2,\ldots,i_m} = \frac{n!}{i_1!i_2!\cdots i_m!}$, it follows that

$$b_{m,k}(n) = \sum_{i_1+i_2+\cdots+i_m=n} \frac{1}{(i_1!)^k}\frac{1}{(i_2!)^k}\cdots\frac{1}{(i_m!)^k}.$$

Now we define

$$f_k(z) := \sum_{i=0}^{\infty} \frac{z^i}{(i!)^k},$$

and we observe that

$$\sum_{n=0}^{\infty} b_{m,k}(n) z^n = (f_k(z))^m.$$

The function $f_k(z)$ will be a central object of study, in our proof methodology.

We use the Pochhammer symbol, $(\alpha)_n := (\alpha)(\alpha+1)\cdots(\alpha+n-1)$. We also will follow the notation from R. A. Askey and A. B. Olde Daalhuis [1]. In equation (16.2.1), they define generalized hypergeometric series:

$${}_pF_q\left({a_1,\ldots,a_p \atop b_1,\ldots,b_q}; z\right) = \sum_{j=0}^{\infty} \frac{(a_1)_j\cdots(a_p)_j}{(b_1)_j\cdots(b_q)_j}\frac{z^j}{j!}.$$

We can frame the analysis of $f_k(z)$ in terms of hypergeometric series. For $k \geq 1$, we have

$$f_k(z) = {}_0F_{k-1}\left({\emptyset \atop 1,\ldots,1}; z\right). \tag{1}$$

For this interpretation of $f_k(z)$, we have $p = 0$, and therefore there are no a's. Regarding the b's, the sequence of 1's has length $k-1$, so $q = k-1$. We care about the asymptotic behavior of $f_k(z)$, so we naturally turn our attention to Section 16.11 of [1], by Askey and Daalhuis. Following their notation, in (16.11.3), their κ is equivalent to our k, and their ν is equal to $-(k-1)/2$. Unfortunately, however, we cannot use their formulation from (16.11.4) and (16.11.5), because in our situation, the b_j's are repeated; this causes the denominator of (16.11.5) to be zero. At alternative formulation must be used [5]. Curious readers can also compare with the notation from Paris and Kaminski [11, section 2.3], where many of the relevant details are explained. As further background reading for the interested reader, one might consider the earlier treatments found in [4, Section 12] and [12, Section 2.3].

Due to the key role of $f_k(z) = \sum_{i=0}^{\infty} \frac{z^i}{(i!)^k}$ in the proofs that will follow, we need to be mindful of the asymptotic behavior of $1/(n!)^k$ for large n. For this purpose, we replace the discrete $n!$ by the continuous $\Gamma(s+1)$, and we expand as a series. It is well known that there are unique constants $c_k^{(j)}$ such that

$$\frac{1}{(\Gamma(s+1))^k} \sim \frac{k^{ks+k/2}}{(2\pi)^{(k-1)/2}} \sum_{j=0}^{\infty} \frac{c_k^{(j)}}{\Gamma(ks + \frac{k+1}{2} + j)}, \qquad \text{as } s \to \infty.$$

As for the values of these constants, in the words of Paris and Kaminski, "their actual evaluation turns out to be the most difficult part of the theory." [11, p. 57] The first several values of $c_k^{(j)}$ are:

$$\begin{aligned}
c_k^{(0)} &= 1, \\
c_k^{(1)} &= \frac{k^2-1}{24}, \\
c_k^{(2)} &= \frac{(k^2-1)(k^2+23)}{(24^2)(2)}, \\
c_k^{(3)} &= \frac{(k^2-1)(k^4 - \frac{298}{5}k^2 + \frac{11237}{5})}{(24^3)(3!)}, \\
c_k^{(4)} &= \frac{(k^2-1)(k^6 - \frac{1887}{5}k^4 - \frac{241041}{5}k^2 + \frac{2482411}{5})}{(24^4)(4!)}, \\
c_k^{(5)} &= \frac{(k^2-1)(k^8 - 1060k^6 + \frac{1451274}{7}k^4 - \frac{220083004}{7}k^2 + \frac{1363929895}{7})}{(24^5)(5!)},
\end{aligned}$$

etc.

We also define

$$E_{k,r}(z) := \frac{\exp(kz^{1/k}e^{2\pi \mathbf{i} r/k})}{\sqrt{k}\,(2\pi)^{(k-1)/2} z^{(k-1)/(2k)} e^{2\pi \mathbf{i} r(k-1)/(2k)}} \sum_{j=0}^{\infty} \frac{c_k^{(j)}}{k^j z^{j/k} e^{2\pi \mathbf{i} rj/k}}.$$

Our use of $E_{k,r}(z)$ mirrors the role of $E(z)$ in equation 2.3.8 of [11].

3 Main Result

We characterize the asymptotic behavior of $a_{m,k}(n)$ and $b_{m,k}(n)$, as our key result.

Theorem 3.1 *For fixed $k \geq 2$ and m, as $n \to \infty$, the first-order asymptotic growth of $b_{m,k}(n)$ is*

$$b_{m,k}(n) \sim (em/n)^{kn} \sqrt{\frac{1}{2\pi n k^{m-1}} \left(\frac{m}{2\pi n}\right)^{(k-1)m}}. \tag{2}$$

Stirling's approximation is $n! \sim \sqrt{2\pi n}\,(n/e)^n$. Taking a power of k in Stirling's approximation, and then multiplying the result on both sides of equation (2), our theorem immediately yields the following corollary.

Theorem 3.2 *For fixed $k \geq 2$ and m, as $n \to \infty$, the first-order asymptotic growth of $a_{m,k}(n)$ is*

$$a_{m,k}(n) \sim m^{kn} \sqrt{\frac{m^{k-1}}{k^{m-1}} \Big(\frac{m}{2\pi n}\Big)^{(k-1)(m-1)}}. \tag{3}$$

4 Proofs

Our goal is to treat the functions under study as complex-valued objects, and then to use analytic methods—in particular, the saddle point method—to retrieve asymptotic information about $b_{m,k}(n)$. For a very readable discussion about such methods, we suggest Chapter VIII of Flajolet and Sedgewick [8].

We have

$$b_{m,k}(n) = \frac{1}{2\pi \mathbf{i}} \int_{\Omega} \frac{(f_k(z))^m}{z^{n+1}}\, dz, \tag{4}$$

where Ω is a closed contour in the counterclockwise direction about the origin. To use the saddle point method, we focus on a contour that is a circle at a large, fixed distance from the origin. So we use $\Omega := \{z = \rho e^{\mathbf{i}\theta} \mid -\pi \leq \theta \leq \pi\}$; in particular, we use a contour Ω that only depends on the choice of ρ. Paris and Kaminski [11, p. 58] prove that

$$f_k(z) \sim \sum_{r=-P}^{P} E_{k,r}(z),$$

as $|z| \to \infty$. As in Paris and Kaminski's exposition, "P is chosen such that $2P+1$ is the smallest odd integer satisfying $2P+1 > \frac{1}{2}\kappa$." In our case, $\kappa = k$. Moreover, an elementary calculation shows that, for our analysis, the relevant value of P is exactly $P = 2\lfloor (k+2)/8 \rfloor + 1$.

Lemma 4.1 *Consider $r \neq 0$ with $-P \leq r \leq P$. Let $z = \rho e^{\mathbf{i}\theta}$ where $-\pi \leq \theta \leq \pi$. Then we have*

$$|E_{k,r}(\rho e^{\mathbf{i}\theta})| = O\Big(\frac{\exp\big(k\rho^{1/k}\cos(\pi/k)\big)}{\rho^{(k-1)/(2k)}}\Big)$$

as $\rho \to \infty$.

Proof. We consider the exponential term $\exp(kz^{1/k}e^{2\pi\mathbf{i}r/k})$ in the numerator of $E_{k,r}(z)$. Since $z = \rho e^{\mathbf{i}\theta}$, it follows that

$$\exp\big(kz^{1/k}e^{2\pi\mathbf{i}r/k}\big) = \exp\big(k\rho^{1/k}e^{\mathbf{i}(\theta+2\pi r)/k}\big).$$

Now taking the modulus, we have

$$\big|\exp\big(kz^{1/k}e^{2\pi\mathbf{i}r/k}\big)\big| = \big|\exp\big(k\rho^{1/k}\cos\left((\theta+2\pi r)/k\right)\big)\big|.$$

Since $r \neq 0$ and $-P \leq r \leq P$, then we have $0 < |r| < k$. Only usage the fact that $0 < |r| < k$, it follows immediately from basic trigonometry that $\cos((\theta + 2\pi r)/k) \leq \cos(\pi/k)$ for all $-\pi \leq \theta \leq \pi$. So it follows that

$$\big|\exp\big(kz^{1/k}e^{2\pi\mathbf{i}r/k}\big)\big| \leq \exp\big(k\rho^{1/k}\cos(\pi/k)\big).$$

Now the lemma follows immediately, by inspecting the modules of $E_{k,r}(\rho e^{\mathbf{i}\theta})$. ∎

Also, when $z = \rho e^{\mathbf{i}\theta}$ with $\theta = \pm\pi$, we observe that

$$\exp\left(k\rho^{1/k}\cos(\pi/k)\right) = \exp(kz^{1/k}).$$

It follows by Lemma 4.1 that, for $r \neq 0$ with $-P \leq r \leq P$, and for $-\pi \leq \theta \leq \pi$, we have

$$\left|E_{k,r}(\rho e^{\mathbf{i}\theta})\right| = O\left(\left|E_{k,0}(z)\right|\right), \qquad \text{where the RHS uses } z = e^{\mathbf{i}\theta} \text{ with } \theta = \pm\pi.$$

In other words, $|E_{k,r}(\rho e^{\mathbf{i}\theta})|$ (for any z on Ω) is dominated by the value of $|E_{k,0}(z)|$ with z at either endpoint of Ω, i.e., with z chosen according to $\theta = \pm\pi$.

Moreover, the value of $|E_{k,0}(\rho e^{\mathbf{i}\theta})|$ increases monotonically (for fixed ρ) as θ decreases from π down to 0. Similarly, the value of $|E_{k,0}(\rho e^{\mathbf{i}\theta})|$ increases monotonically (for fixed ρ) as θ increases from $-\pi$ up to 0. We will use saddle point analysis in the discussion below. When considering

$$\frac{1}{2\pi\mathbf{i}}\int_{\Omega}\frac{(E_{k,0}(z))^m}{z^{n+1}}\,dz,$$

we will see that it suffices to consider only on the portion of Ω corresponding to $-\theta_0 \leq \theta \leq \theta_0$ for (say) $\theta = n^{-4/9}$. The remainder of the contribution from the rest of the contour Ω will have asymptotically lower order. Moreover, $|E_{k,r}(\rho e^{\mathbf{i}\theta})|$ is bounded above by $|E_{k,0}(\rho e^{\pm\pi\mathbf{i}}|$, i.e., by the size of $|E_{k,0}(z)|$ at the very endpoints of Ω. Therefore, the contribution from the $E_{k,r}(z)$ for $r \neq 0$ can safely be ignored, when computing the first-order asymptotic growth of $b_{m,k}(n)$.

For this reason, we will only use $E_{k,0}$ (and not the other $E_{k,r}$'s) when calculating the asymptotic behavior of $b_{m,k}(n)$. In other words, equation (4) can be simplified to

$$b_{m,k}(n) \sim \frac{1}{2\pi\mathbf{i}}\int_{\Omega}\frac{(E_{k,0}(z))^m}{z^{n+1}}\,dz.$$

Now we use the Saddle point technique. See Good [9] or Flajolet and Sedgewick [8] for a description of this technique. Let ρ be the Saddle point and Ω the circle $\rho e^{\mathbf{i}\theta}$. By Cauchy's theorem,

$$\begin{aligned} b_{m,k}(n) &\sim \frac{1}{2\pi\mathbf{i}}\int_{\Omega}\frac{(E_{k,0}(z))^m}{z^{n+1}}\,dz \\ &= \frac{1}{\rho^n}\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left[m\left\{\ln(E_{k,0}(\rho)) + \mathbf{i}L_{k,1}(\rho)\theta - \frac{1}{2}L_{k,2}(\rho)\theta^2 - \frac{\mathbf{i}}{6}L_{k,3}(\rho)\theta^3 + \dots\right\} - n\mathbf{i}\theta\right]d\theta, \end{aligned} \tag{5}$$

where we have defined

$$L_{k,j}(\rho) := \partial_u^j \ln(E_{k,0}(\rho e^u))|_{u=0}.$$

To find the location ρ of the saddle point, we need:

$$\frac{\partial}{\partial\rho}\Big(m\ln(E_{k,0}(\rho)) - n\ln(\rho)\Big) = 0,$$

and, therefore,

$$m\frac{E'_{k,0}(\rho)}{E_{k,0}(\rho)} - \frac{n}{\rho} = 0.$$

Rearranging, we see that the saddle point ρ is the root (of smallest modulus) of

$$m\rho E'_{k,0}(\rho) - nE_{k,0}(\rho) = 0. \tag{6}$$

We recall that $L_{k,1}(\rho) := \partial_u \ln(E_{k,0}(\rho e^u))|_{u=0}$, and it follows that the saddle point ρ satisfies:

$$L_{k,1}(\rho) = \frac{\rho\, E'_{k,0}(\rho)}{E_{k,0}(\rho)}. \tag{7}$$

Combining (6) and (7), we see that the saddle point ρ satisfies

$$m\, E_{k,0}(\rho)L_{k,1}(\rho) - nE_{k,0}(\rho) = 0,$$

but $E_{k,0}(\rho) \neq 0$. So dividing by $E_{k,0}(\rho)$ and multiplying by $\mathbf{i}\theta$, we obtain

$$mi\, L_{k,1}(\rho)\theta - n\mathbf{i}\theta = 0.$$

Therefore, equation (5) for $b_{m,k}(n)$ simplifies, when ρ is chosen at the saddle point, to become

$$b_{m,k}(n) \sim \frac{1}{\rho^n}\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left[m\left\{\ln(E_{k,0}(\rho)) - \frac{1}{2}L_{k,2}(\rho)\theta^2 - \frac{\mathbf{i}}{6}L_{k,3}(\rho)\theta^3 + \dots\right\}\right] d\theta,$$

or, more simply,

$$b_{m,k}(n) \sim \frac{1}{\rho^n}\frac{(E_{k,0}(\rho))^m}{2\pi}\int_{-\pi}^{\pi} \exp\left[m\left\{-\frac{1}{2}L_{k,2}(\rho)\theta^2 - \frac{\mathbf{i}}{6}L_{k,3}(\rho)\theta^3 + \dots\right\}\right] d\theta. \tag{8}$$

We have

$$E_{k,0}(z) = \frac{\exp(kz^{1/k})}{\sqrt{k}\,(2\pi)^{(k-1)/2}z^{(k-1)/(2k)}}\big(1 + \Theta(z^{-1/k})\big), \tag{9}$$

as $z \to \infty$ along the positive real axis. Returning to equation (6), this leads us to

$$\rho \sim (n/m)^k, \quad n \to \infty \tag{10}$$

as the location of the saddle point. We recall $L_{k,j}(\rho) = \partial_u^j \ln(E_{k,0}(\rho e^u))|_{u=0}$. Again using (9), we obtain, for all $j \geq 1$,

$$L_{k,j}(\rho) = \partial_u^j \ln(E_{k,0}(\rho e^u))|_{u=0} \sim \frac{\rho^{1/k}}{k^{j-1}}. \tag{11}$$

Next, we choose a splitting value θ_0 such that $mL_{k,2}(\rho)\theta_0^2 \to \infty$ and $mL_{k,3}(\rho)\theta_0^3 \to 0$ as $n \to \infty$ (or, equivalently, as $\rho \to \infty$). In this case, these two conditions translate to:

$$m\Big(\frac{\rho^{1/k}}{k}\Big)\theta_0^2 \to \infty \qquad \text{and} \qquad m\Big(\frac{\rho^{1/k}}{k^2}\Big)\theta_0^3 \to 0.$$

As we saw in (10), the location of the saddle point is $\rho \sim (n/m)^k$, so the previous equations become

$$m\Big(\frac{n/m}{k}\Big)\theta_0^2 \to \infty \qquad \text{and} \qquad m\Big(\frac{n/m}{k^2}\Big)\theta_0^3 \to 0$$

so it suffices to have

$$n\theta_0^2 \to \infty \qquad \text{and} \qquad n\theta_0^3 \to 0.$$

So we need an angle θ_0 like $\theta_0 = n^\alpha$ for $-1/2 < \alpha < -1/3$. For instance, we can use $\theta_0 = n^{-4/9}$.

To prune the tails, we compute

$$\left|\frac{1}{\rho^n}\frac{1}{2\pi}\int_{(-\pi,-\theta_0)\cup(\theta_0,\pi)}\frac{(E_{k,0}(\rho e^{\mathbf{i}\theta}))^m}{e^{n\mathbf{i}\theta}}\,d\theta\right| = \frac{1}{\rho^n}\frac{1}{2\pi}O\Big(\Big(\frac{\exp\big(k\rho^{1/k}\cos(\theta_0/k)\big)}{\rho^{(k-1)/(2k)}}\Big)^m\Big) \tag{12}$$

but the saddle point is located at distance $\rho = (n/m)^k$ and we are using $\theta_0 = n^{-4/9}$, so $\cos(\theta_0/k) = O(1-(\theta_0/k)^2/2)$. Therefore we can rewrite equation (12) as

$$\left|\frac{1}{\rho^n}\frac{1}{2\pi}\int_{(-\pi,-\theta_0)\cup(\theta_0,\pi)}\frac{(E_{k,0}(\rho e^{\mathbf{i}\theta}))^m}{e^{n\mathbf{i}\theta}}\,d\theta\right| = \frac{1}{(n/m)^{kn}}\frac{1}{2\pi}O\Big(\frac{\exp\big(kn\big(1-(n^{-4/9}/k)^2/2\big)\big)}{(n/m)^{m(k-1)/2}}\Big) \tag{13}$$

or even more simply as

$$\left|\frac{1}{\rho^n}\int_{(-\pi,-\theta_0)\cup(\theta_0,\pi)}\frac{(E_{k,0}(\rho e^{\mathbf{i}\theta}))^m}{e^{n\mathbf{i}\theta}}\,d\theta\right| = O\Big((em/n)^{kn}\exp(-n^{1/9}/(2k))\Big(\frac{m}{n}\Big)^{(k-1)m/2}\Big). \tag{14}$$

Looking ahead, for comparison to the final asymptotic behavior of $b_{m,k}(n)$ in equation (16), and noting that k and m are held constant, we see that $\exp(-n^{1/9}/(2k))$ decreases much faster than $\sqrt{1/n}$, and thus

$$\left|\frac{1}{\rho^n}\frac{1}{2\pi}\int_{(-\pi,-\theta_0)\cup(\theta_0,\pi)}\frac{(E_{k,0}(\rho e^{\mathbf{i}\theta}))^m}{e^{n\mathbf{i}\theta}}\,d\theta\right| = o\left(\left|\frac{1}{\rho^n}\frac{1}{2\pi}\int_{-\theta_0}^{\theta_0}\frac{(E_{k,0}(\rho e^{\mathbf{i}\theta}))^m}{e^{n\mathbf{i}\theta}}\,d\theta\right|\right), \tag{15}$$

so the region $(-\pi,-\theta_0)\cup(\theta_0,\pi)$ of Ω can be safely ignored.

Now we need to make a Gaussian approximation for the central region of Ω, i.e., for the integral over the region $(-\theta_0,\theta_0)$. We compute

$$\begin{aligned} b_{m,k}(n) &\sim \frac{1}{\rho^n}\frac{(E_{k,0}(\rho))^m}{2\pi}\int_{-\theta_0}^{\theta_0}\exp\Big[m\Big\{-\frac{1}{2}L_{k,2}(\rho)\theta^2 - \frac{\mathbf{i}}{6}L_{k,3}(\rho)\theta^3 + \dots\Big\}\Big]\,d\theta \\ &\sim \frac{1}{\rho^n}\frac{(E_{k,0}(\rho))^m}{2\pi}\int_{-\theta_0}^{\theta_0}\exp\Big(-\frac{m}{2}L_{k,2}(\rho)\theta^2\Big)d\theta\Big(1+O\Big(\frac{\rho^{1/k}}{k^2}\theta_0^3\Big)\Big). \end{aligned}$$

We recall $L_{k,3}(\rho) \sim \frac{\rho^{1/k}}{k^2}$, as proved in (11). As before, we have $\rho = (n/m)^k$, and we are working on the $(-\theta_0,\theta_0)$ portion of Ω. Putting these together, we get $\frac{\rho^{1/k}}{k^2}\theta_0^3 = \frac{(n/m)}{k}(n^{-4/9})^3 = O(n^{-1/3})$. Therefore, we get

$$b_{m,k}(n) \sim \frac{1}{\rho^n}\frac{(E_{k,0}(\rho))^m}{2\pi}\int_{-\theta_0}^{\theta_0}\exp\Big(-\frac{m}{2}L_{k,2}(\rho)\theta^2\Big)d\theta.$$

Again using (11), we have $L_{k,2}(\rho) \sim \rho^{1/k}/k$, and thus

$$b_{m,k}(n) \sim \frac{1}{\rho^n}\frac{(E_{k,0}(\rho))^m}{2\pi}\int_{-\theta_0}^{\theta_0}\exp\Big(-\frac{m\rho^{1/k}}{2k}\theta^2\Big)d\theta.$$

Finally, we do not change the first order asymptotics if we also include the tails of this integral in the region $(-\infty,-\theta_0)\cup(\theta_0,\infty)$, because $\int_{\theta_0}^{\infty}\exp\Big(-\frac{m\rho^{1/k}}{2k}\theta^2\Big)d\theta = O(\exp(-\frac{m\rho^{1/k}}{2k}\theta_0^2)) = O(\exp(-\frac{n^{-1/9}}{2k}))$, so these tails are exponentially small. So we extend the contour from $(-\theta_0,\theta_0)$ to $(-\infty,\infty)$, and we get

$$b_{m,k}(n) \sim \frac{1}{\rho^n}\frac{(E_{k,0}(\rho))^m}{2\pi}\int_{-\infty}^{\infty}\exp\Big(-\frac{m\rho^{1/k}}{2k}\theta^2\Big)d\theta = \frac{(E_{k,0}(\rho))^m}{\rho^n}\frac{\sqrt{k}}{\sqrt{2\pi m\rho^{1/k}}}.$$

Utilizing the form of $E_{k,0}(\rho)$, which was already given in (9), it follows that, for any fixed m and k, as $n \to \infty$, we have

$$b_{m,k}(n) \sim (em/n)^{kn} \sqrt{\frac{1}{2\pi n k^{m-1}} \left(\frac{m}{2\pi n}\right)^{(k-1)m}}. \tag{16}$$

This concludes the proof of Theorem 3.1.

5 Future Directions

Theorems 3.1 and 3.2 characterize the asymptotic properties of a general family of integer sequences, $a_{m,k}(n)$. We naturally view m and k as fixed, and we study the asymptotic analysis as $n \to \infty$. For a future direction of study, it would be natural to view the m and k as various kinds of functions of n, and to determine the asymptotic growth of $a_{k,m}(n)$ as m and k also grow (at various rates) with n. Such an investigation is beyond our scope in this relatively short treatment, but the recent work of Pemantle and Wilson [13] might be utilized for such a purpose.

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