

Optimal Location of Transportation Devices

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Abstract. We consider algorithms for finding the optimal location of a simple transportation device, that we call a *moving walkway*, consisting of a pair of points in the plane between which the travel speed is high. More specifically, one can travel from one endpoint of the walkway to the other at speed $v > 1$, but can only travel at unit speed between any other pair of points. The travel time between any two points in the plane is the minimum between the actual geometric distance, and the time needed to go from one point to the other using the walkway. A location for a walkway is said to be optimal with respect to a given finite set of points if it minimizes the maximum travel time between any two points of the set. We give a simple linear-time algorithm for finding an optimal location in the case where the points are on a line. We also give an $\Omega(n \log n)$ lower bound for the problem of computing the travel time diameter of a set of n points on a line with respect to a given walkway. Then we describe an $O(n \log n)$ algorithm for locating a walkway with the additional restriction that the walkway must be horizontal. This algorithm is based on a recent generic method for solving quasiconvex programs with implicitly defined constraints. It is used in a $(1 + \varepsilon)$ -approximation algorithm for optimal location of a walkway of arbitrary orientation.

1 Introduction

Finding the optimal location of a facility modeled as a geometric object is an old, well-studied topic in computational geometry. One of the simplest instances of such a problem is finding the smallest disk covering a given set of points, which can be done in linear time, or its min-sum counterpart, the Fermat-Weber problem, which is difficult to solve exactly in usual models of computation. Surveys on both combinatorial and geometric facility location problems can be found in [12, 13, 22].

The optimality of a location is often defined with respect to a distance measure. Recently, interest has emerged in the definition and analysis of time distances, measuring the travel time between points in the plane given a geometric transportation facility, such as a highway or a geometric network made of line

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segments [1–3, 6, 17]. In general, the transportation facility is a subset of the plane where the travel speed is high, and the travel time is naturally defined as the minimum time needed to reach a point. Primary concerns in previous works on these notions include in particular efficient constructions of the Voronoi diagrams for these distances.

In this work, we consider optimal location of geometric transportation facilities on a line and in the plane. The transportation facilities we define are probably the simplest one can think about, and consist of a pair of points between which one can travel faster than anywhere else in the plane. This simple definition can be used to model moving walkways in airports, shuttle connections or tunnels in a city. For simplicity, we refer to these as *moving walkways*.

A moving walkway is defined by two points a and b , and customers can go from a point s to a point t either by going from s to t directly, or by entering the walkway at one of its endpoint and exiting at the other. The time to go from s to t without using the walkway is simply the distance $d(s, t)$ between s and t . The walkway is parameterized by a real number v so that the time to go from a to b is $\frac{1}{v}d(a, b)$. The distance measure $d(\cdot, \cdot)$ must be convex and satisfy the triangular inequality. For simplicity we assume it is the Euclidean distance, although our results can be generalized to other distance measures. The travel time $d_{\text{time}}(s, t, a, b)$ is the minimum time needed to go from s to t , when a walkway is available between points a and b :

$$d_{\text{time}}(s, t, a, b) = \min \left\{ d(s, t), d(s, a) + \frac{1}{v}d(a, b) + d(b, t), d(s, b) + \frac{1}{v}d(a, b) + d(t, a) \right\}.$$

Note that the moving walkways are bidirectional, hence the travel time is a symmetric distance function. Given a set of points and a moving walkway, we define the travel time diameter as the maximum travel time between two points in the set. A moving walkway is said to be optimal if it minimizes the travel time diameter. Note that we do not restrict the endpoints of the walkway to be in the given set, thus our location problems are geometric optimization problems.

In section 2 we consider the one-dimensional case, in which all the points are on the real line. We first provide an $\Omega(n \log n)$ lower bound on the computation of the travel time diameter for a given walkway in the algebraic computation tree model. Then we give a simple linear-time algorithm for finding the optimal moving walkway on a line.

In section 3, we define moving walkways in the plane, and provide an $O(n \log n)$ deterministic algorithm for deciding whether the travel time diameter is above a given threshold. We show, using Chan’s technique for implicit quasiconvex programs [10], that this implies an $O(n \log n)$ algorithm for finding an optimal horizontal walkway. We then derive an algorithm for computing a $(1 + \varepsilon)$ -approximation of an optimal moving walkway with arbitrary orientation and constant speed in time $O(\varepsilon^{-1}n \log n)$.

Related works. We are not aware of many previous contributions on the topic of optimal transportation facility location based on time distances.

In [8], we presented a family of geometric optimization problems in which the objective function was in a min-max-min form. As an example, we proposed the problem of finding the location of a moving walkway on a line such that the maximum travel time between n given pairs of source-destination points is minimized. The current paper presents extensions and generalizations of this result.

Recently, an effort has been made by several authors to solve the problem of locating a highway that minimizes the maximum travel time among a set of points [5]. In this work, a highway is modeled as a line in the plane on which the travel speed is higher, and customers can enter and exit the highway at any point on the line. In particular, they give an algorithm for the case where the travel speed on the highway is infinite,

and another algorithm for the case where the highway is restricted to be vertical. An $O(n^2 \log n)$ algorithm for the general problem is also given. Various other models are studied by Korman and Tokuyama [19]. The goal of this work is similar, but the transportation model that we study is different.

Another problem related to our work is the well-known *2-center problem*. Given a planar point set S , the goal is to find a pair of disks of minimum radius having every point of S in their union. The question received much attention over the last 15 years. The first results were obtained by Agarwal and Sharir [4], and the current best known algorithm is due to Eppstein [14] and solves the problem in $O(n \log^2 n)$ randomized time. The objective function in this problem is distinct from ours, and it seems that we cannot directly apply these results to our problem.

2 Optimal location of moving walkways in one dimension

Suppose a new moving walkway is to be installed in a long corridor, for instance in the concourse of an airport. The moving walkway is used by customers to go from source to destination points in the corridor. We model the corridor as the real line, source and destination points as real numbers, and the moving walkway by an interval $[a, b]$ on the line. As the travel time distance is symmetric, we generally assume that every source-destination pair s, t is such that $s \leq t$.

We consider the computation of the travel time diameter and the optimal location of such a moving walkway.

2.1 Travel time diameter computation

A first problem is, given a set of points and a walkway $[a, b]$, to compute the travel time diameter, defined as the maximum travel time between two points of the set.

Theorem 1. *The travel time diameter for a moving walkway on a line can be computed in $O(n \log n)$ time.*

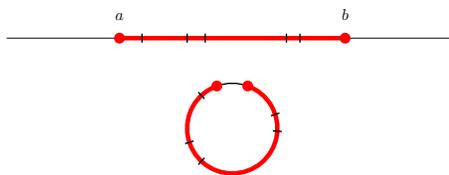


Fig. 1. Computing the travel time diameter for points in the interval $[a, b]$ amounts to computing the angular diameter of points on a circle.

Proof. Without loss of generality, we assume that $\max\{x \in P\} = 1$ and $\min\{x \in P\} = 0$. We distinguish the four intervals $[0, a]$, $(a, (a + b)/2]$, $((a + b)/2, b]$ and $(b, 1]$, and call these I, II, III and IV, respectively. First, the array is sorted in $O(n \log n)$ time.

If a pair s, t is not separated by the point $(a + b)/2$, then the travel time between s and t is $|t - s|$, thus we never use the walkway. From this, the pairs of points between which the moving walkway is used for

traveling can only be of four types: II-III, I-III, II-IV and I-IV. For all the other cases, the travel time is equal to the maximum absolute difference within the corresponding sets of points.

In the case II-III the computation of the diameter is essentially equivalent to computing the maximum angle between two points on a circle (see figure 1), where a and b are mapped to two points on the circle separated by an arc of length $(b - a)/v$. This can be achieved in linear time, once the input is sorted, using rotating calipers [25].

The case I-III is solved by computing the farthest point from a on the same circle, which can also be achieved by scanning the sorted array. The case II-IV is symmetric to the case I-III. Finally we know that all pairs of type I-IV use the walkway, so the largest travel time is $d_{\text{time}}(0, 1, a, b)$, which can be found in constant time. \square

Theorem 2. *There is an $\Omega(n \log n)$ lower bound on the computation of the travel time diameter for a moving walkway on a line in the algebraic computation tree model, even when $v = +\infty$.*

Proof. The proof of the lower bound is by reduction from the problem of checking whether two sets of numbers A and B are disjoint. We let $v = +\infty$, $a = 0$, $b = 2$ and scale the elements of A and B so that they belong to $[0, 1]$. Let $P = A \cup \{x + 1 : x \in B\}$. Then the travel diameter of the points in P with respect to the moving walkway a, b is equal to 1 iff $A \cap B \neq \emptyset$. The lower bound also holds for the decision version of the problem. \square

2.2 Minimum travel time diameter location

We now consider the following moving walkway location problem.

Problem 1 *Given a set P of n real numbers, a real number $v > 1$, find a moving walkway $[a, b]$ of speed v that minimizes the travel time diameter of P .*

There is a very simple deterministic algorithm solving this problem in $O(n)$ time. It is a surprising observation that we are able to find a moving walkway that minimizes the diameter faster than we are able to compute the diameter for a given walkway. This observation has already been made for the construction of optimal vertical highways [5]. We assume without loss of generality that $\min\{x \in P\} = 0$ and $\max\{x \in P\} = 1$, and denote by OPT the optimum value $\min_{a,b} \max_{s,t \in P: s \leq t} d_{\text{time}}(s, t, a, b)$.

Lemma 1.

$$\text{OPT} \leq UB = \frac{v}{2v - 1}.$$

Proof. Let $c_1 = 1 - UB$ and $c_2 = UB$. We define a moving walkway by $a = c_1/2$ and $b = (1 + c_2)/2$. As c_1 is symmetric to 0 with respect to a and c_2 is symmetric to 1 with respect to b , going from any point in $[0, c_1]$ to a point in $[c_2, 1]$ using the walkway cannot take more time than going from 0 to 1.

To go from 0 to 1, the time needed is

$$a + (1 - b) + \frac{1}{v}(b - a) = UB$$

For every pair in $[0, c_2]$, the time distance is at most UB (the length of the interval). The same observation holds for every pair in $[c_1, 1]$. \square

In what follows, we assume we are given a set of points P and we try to find properties of its optimal moving walkway $[a, b]$. Let s be the smallest point in P such that, to go from 0 to s , it is better to use the moving walkway. We know that $s \leq b$, as to go from 0 to any point larger than or equal to b it is always better to use the moving walkway. Symmetrically, let r be the largest point in P such that to go from 1 to r , it is better to use the moving walkway; we know that $r \geq a$.

The following facts are easy to verify:

Fact 1 For every pair (p, q) enclosing the walkway, i.e., $p \leq a \leq b \leq q$, the walkway is used.

Fact 2 If we have $p \leq p' \leq a$, and the walkway is used for the pair (p', q) , it is also used for the pair (p, q) .

Fact 3 If the walkway is used for the pair (p, q) (with $p \leq q$), and p' is between p and a , and q' is between q and b , then the walkway is used for the pair (p', q') .

Lemma 2. Let P be a set of points and $[a, b]$ an optimal moving walkway. The moving walkway is used only for pairs (p, q) where $p \in [0, r]$ and $q \in [s, 1]$. For every other pair, the facility is not needed.

Proof. We first show that for a pair (p, q) in the interval $[0, s)$, the facility is never used. We distinguish three cases.

1. if (p, q) belongs to $[0, a]$, the walkway is trivially not used.
2. if $p \leq a$ and q belongs to $[a, s)$, then using the walkway for going from p to q implies that we should use it for going from 0 to q (see Fact 2). This contradicts the definition of s .
3. if both p and q belong to $[a, s)$, then using the walkway for going from p to q implies that we should use it also for going from p' to q , where p' is the symmetric of p with respect to a . We are then back to case 2, leading to a contradiction.

In conclusion, the walkway is never used for pairs in the interval $[0, s)$. The same reasoning holds for pairs in $(r, 1]$. \square

Lemma 3. The optimal moving walkway is such that $a = \frac{r}{2}$ and $b = \frac{s+1}{2}$.

Proof. We have not proved that *all* pairs (p, q) with $p \in [0, r]$ and $q \in [s, 1]$ require the use of the moving walkway. What we know however is that the moving walkway will be used for the pairs $(0, 1)$, $(0, s)$ and $(r, 1)$. What we aim to show is that finding the optimal moving walkway is equivalent to finding the optimal moving walkway for these pairs.

First note that the optimal walkway for $(0, 1)$, $(0, s)$ and $(r, 1)$ is such that $a = \frac{r}{2}$ and $b = \frac{s+1}{2}$ (setting a and b differently increases the distance for one of the pairs). Moreover, it implies that $d_{\text{time}}(0, 1, a, b) = d_{\text{time}}(0, s, a, b) = d_{\text{time}}(r, 1, a, b)$. We also have $d_{\text{time}}(0, 1, a, b) \geq d_{\text{time}}(r, s, a, b)$, meaning that we also optimize the distance for (r, s) .

For every pair with $p \in [0, r]$ and $q \in [s, 1]$, we have one of the following:

- if $p \leq a$ and $q \geq b$, the walkway is always used (Fact 1),
- if $p \leq a$ and $q \leq b$, the walkway is always used since it is used for $(0, s)$ (Fact 3),
- symmetrically, if $p \geq a$ and $q \geq b$, the walkway is used since it is used for $(r, 1)$,
- if $p \geq a$ and $q \leq b$ and the walkway is not used, then the time distance for (p, q) is no greater than the shortest distance from p to q using the walkway, which is no greater than the time distance for (r, s) . If it is used, it is no greater than the time distance for (r, s) .

So, in every case, the time distance is upper bounded by that of $(0, 1)$, $(0, s)$, $(r, 1)$ or (r, s) , the maximum of which is minimized with the solution described above. \square

This lemma showed that the optimal moving walkway is uniquely determined by the points r and s .

Lemma 4. *Let*

- r_1 be the largest point of P whose value is less than or equal to $1 - \frac{v}{2v-1}$
- s_1 be the smallest point of P whose value is more than or equal to $\frac{r_1(v-1)+v+1}{3v-1}$
- s_2 be the smallest point of P whose value is more than or equal to $\frac{v}{2v-1}$.
- r_2 be the largest point of P whose value is less than or equal to $\frac{(v-1)(s_2+1)}{3v-1}$

Then, either $r = r_1$ and $s = s_1$, or $r = r_2$ and $s = s_2$.

Proof. By Lemmas 1 and 3 we know that either $r \leq 1 - UB$ or $s \geq UB$, for otherwise the walkway ($a = r/2, b = (s+1)/2$) leads to a diameter greater than UB . We treat the first case, the other one being symmetric.

If $r \leq 1 - UB$, we prove that $r = r_1$ and $s = s_1$. We proceed in two steps: first we will show that $r = r_1$ is optimal, and then prove that $s = s_1$ independently of the choice of r .

By contradiction, imagine that the optimal r is not r_1 . Because we assumed that $r \leq 1 - UB$, this means that there exists a point q between r and $1 - UB$, as r is not the largest. Since $q > r$, and r is, by definition, the largest point for which we should use the walkway to go to 1, we know that the walkway is *not* used from q to 1, leading to a diameter greater than UB , and contradicting Lemma 1. To conclude, we know that in the case where $r \leq 1 - UB$, r is the largest point of P satisfying that constraint.

Now we prove that $s = s_1$. By definition of s , it is the first point in P such that, to go from 0 to s , it is better to use the walkway. Thus we obtain the following equation:

$$s \geq a + \frac{b-a}{v} + b - s = \frac{r}{2} + \left(\frac{1+s}{2} - \frac{r}{2} \right) \frac{1}{v} + \frac{1-s}{2}$$

$$s \geq \frac{r(v-1) + v + 1}{3v-1}$$

By definition, s is the *smallest* point in P satisfying that constraint, and thus, as claimed in the statement of the Lemma, $s = s_1$. \square

Theorem 3. *Problem 1 can be solved in $O(n)$ deterministic time.*

Proof. By Lemma 4, we know that there are two possibilities. So first, we determine the values of r_1, r_2, s_1 and s_2 in linear time (we scan the set twice). By Lemma 3, the optimal moving walkway is given by $a = \frac{r}{2}$ and $b = \frac{s+1}{2}$, so we have two candidates.

By Lemma 2 and the arguments developed in the proof of Lemma 3, we know that the largest time distance for pairs (p, q) with $p \in [0, r]$ and $q \in [s, 1]$ is achieved by one of the pairs $(0, 1)$, $(0, s)$, $(r, 1)$ or (r, s) . The largest time distance for pairs where both points lie in $[r, s]$ only depends on the smallest and largest point in that ranges and can also be found by scanning the set. For the pairs in $[0, r]$ or in $[s, 1]$, the worst distance achieved is $\max(r, 1 - s)$.

In other words, we compute the worst time distance for both candidates (for pairs using and not using the walkway), and keep the best. \square

3 Optimal location of moving walkways in the plane

In this section, we consider the problem of locating a moving walkway among points in the plane. We first solve a location problem with n source-destination pairs, under the restriction that the moving walkway must be horizontal. Then we give a decision algorithm for the travel time diameter for a given moving walkway (not necessarily horizontal), and from this derive an algorithm for finding optimal horizontal walkways. We show that this algorithm can be used for obtaining an approximation algorithm for locating moving walkways with arbitrary orientation.

3.1 Horizontal moving walkway location for n source-destination pairs

As a preliminary, we consider the problem of minimizing the maximum travel time between a number of designated source-destination pairs, with the restriction that the moving walkway must be horizontal. We let (p_x, p_y) denote the x and y -coordinates of a point p .

Problem 2 Given n pairs $\{(s_i, t_i)\}_{i=1}^n$ of points, and a real number $v > 1$, find a horizontal moving walkway $[a^*, b^*]$ of speed v minimizing the maximum travel time between a pair:

$$\min_{a, b: a_y = b_y} \max_i d_{\text{time}}(s_i, t_i, a, b).$$

The following lemma states that if one travels from left to right, then the moving walkway can only be used from left to right. This can be easily proved by contradiction.

Lemma 5. Given a pair of points a, b such that $a_y = b_y$ and $a_x < b_x$, and a pair of points s, t such that $s_x < t_x$, we have:

$$d_{\text{time}}(s, t, a, b) = \min\{d(s, t), d(s, a) + \frac{1}{v}d(a, b) + d(b, t)\}.$$

In order to show that Problem 2 can be solved efficiently, we use the following definition: A function f is *quasiconvex* if all its (lower) levels are convex, that is, if the set $\{x : f(x) \leq t\}$ is convex for all value of t . Alternatively, a quasiconvex function is such that $f((x + x')/2) \leq \max\{f(x), f(x')\}$ for any pair x, x' . Quasiconvex programming is the problem of finding a point that minimizes the maximum of a set of quasiconvex functions [15, 11].

Lemma 6. Problem 2 can be solved in $O(n)$ expected time.

Proof. We consider the functions $f_i(a, b) = d_{\text{time}}(s_i, t_i, a, b)$. From Lemma 5, f_i is defined as the minimum between a constant $d(s_i, t_i)$ independent of (a, b) , and a sum of three convex functions, which is a convex function. All the levels of f_i are therefore necessarily convex.

Hence Problem 2 is a quasiconvex program, with quasiconvex constraints f_i . Amenta, Bern and Eppstein [7] showed that any quasiconvex program of dimension d forms an LP-type problem of dimension at most $2d+1$. As a direct consequence, any quasiconvex program with n constraints can be solved in $O(n)$ randomized expected time using a randomized algorithm for LP-type problems [20]. \square

The restriction on the direction of the moving walkway is useful in making the functions f_i quasiconvex. Otherwise, it is easy to show that the levels of the functions are not even always connected.

3.2 Travel time diameter computation

As a next step toward a location algorithm, we give an $O(n \log n)$ time decision procedure for the travel time diameter given a moving walkway (not necessarily horizontal). Note that the travel time diameter for a moving walkway in the plane generalizes the travel time diameter for a moving walkway on a line, so the $\Omega(n \log n)$ lower bound given in theorem 2 still holds and our algorithm is optimal.

Problem 3 Given a set P of n points, a real number $v > 1$, a real number $y > 0$, and a pair of points a, b , decide whether

$$\max_{s,t \in P} d_{\text{time}}(s, t, a, b) \leq y.$$

Lemma 7. Problem 3 can be solved in optimal $O(n \log n)$ deterministic time.

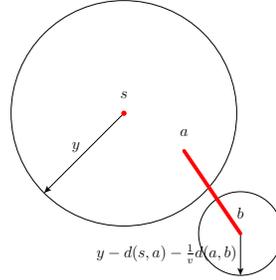


Fig. 2. The travel time disk.

Proof. We first observe that the pairs of points $s, t \in P$ such that $d_{\text{time}}(s, t, a, b) < d(s, t)$ are all separated by the bisector of the line segment $[ab]$. If both points are on the same side of the bisector, then it is always more advantageous to go directly from s to t than to use the walkway. So as a first step, we can partition the point set in two subsets $R = \{s \in P : d(s, a) \leq d(s, b)\}$ and $B = \{t \in P : d(t, a) > d(t, b)\}$, say of red and blue points, according to which side of the bisector the points lie. Then we have :

$$\max_{s,t \in P} d_{\text{time}}(s, t, a, b) = \max\left\{\max_{s,t \in R} d(s, t), \max_{s,t \in B} d(s, t), \max_{s \in R, t \in B} d_{\text{time}}(s, t, a, b)\right\}.$$

So by first computing, in time $O(n \log n)$, the Euclidean diameter of the red and blue sets, the travel time diameter problem boils down to computing the red-blue travel time diameter.

The red-blue travel time diameter is not greater than y if and only if each point $t \in B$ is contained in the intersection of the *travel time disks* of the points in R , where the travel time disk of radius y and center $s \in R$ is the set of points reachable in time y from s . We observe that for a red-blue pair $s \in R, t \in B$, we have $d_{\text{time}}(s, t, a, b) = \min\{d(s, t), d(s, a) + \frac{1}{v}d(a, b) + d(b, t)\}$, which means that for going from a red to a blue point, the walkway is used from a to b . So the travel time disk of radius y and center $s \in R$ is either (see figure 2):

- the disk of radius y centered at s if $y < d(s, a) + \frac{1}{v}d(a, b)$, or

- the union of a disk of radius y centered at s and a disk of radius $y - d(s, a) - \frac{1}{v}d(a, b)$ centered at b , otherwise.

To check the intersection condition, we proceed by first sorting the points in R in nondecreasing order of their distance to a . We denote by $s_1, s_2, \dots, s_{|R|}$ the points in R , such that $d(s_1, a) \leq d(s_2, a) \leq \dots \leq d(s_{|R|}, a)$.

For each point $t \in B$, we identify, in $O(\log |R|)$ time, the smallest index i such that $\forall j \geq i : d(s_j, a) > y - \frac{1}{v}d(a, b) - d(b, t)$. All the red points s_j for $j \geq i$ are such that we cannot go from s_j to t in time less than y by using the walkway. So for all these points, we must be able to reach t directly, which means that t must lie in the intersection of the Euclidean unit disks $\mathcal{D}(s_j, y)$ with centers s_j and radius y for $j \geq i$.

To check the latter condition, we can use a point location structure for disk intersections. We first need a technical lemma.

Lemma 8. *Consider the intersection I of i unit disks, and an additional, distinct, unit disk \mathcal{D} . The boundary of \mathcal{D} intersects the boundary of I in at most two points.*

We create a planar map in an incremental fashion: the map is at first composed of a single cell corresponding to the disk $\mathcal{D}(s_{|R|}, y)$. Then we consider the disks $\mathcal{D}(s_i, y)$ for $i = |R| - 1, |R| - 2, \dots, 1$, one at a time, while maintaining a cell corresponding to $C_i = \bigcap_{j=i}^{|R|} \mathcal{D}(s_j, y)$ in the map. In other words, given a planar map with a cell corresponding to C_i , we compute the arc of $\mathcal{D}(s_{i-1}, y)$ intersecting C_i , and we obtain the new cell corresponding to C_{i-1} . Note that by the above lemma, the map has complexity $O(n)$. We label each cell of the map with the smallest index i for which the cell is contained in C_i . From this, we can apply a known point location algorithm for planar maps [23], yielding a query time of $O(\log n)$.

So, for each point $t \in B$, we can check in $O(\log n)$ time whether $t \in C_i$ for the computed index i . Hence the whole algorithm takes $O(n \log n)$ deterministic time. \square

Using a simple reduction due to Chan [9], it is not difficult to show that this algorithm can be converted into an evaluation algorithm that computes the travel time diameter in $O(n \log n)$ randomized expected time. We leave the details to the reader.

3.3 Minimum travel time diameter location of horizontal moving walkways

Now given our decision algorithm for the travel time diameter and the observation that the location problem for n source-destination pairs is LP-type, we are able to solve the minimum diameter location problem.

Problem 4 *Given a set P of n points, and a real number $v > 1$, find a horizontal moving walkway $[a^*, b^*]$ of speed v minimizing the travel time diameter of P .*

Recently, Chan introduced a generic technique for solving some LP-type problems with a large number of constraints [10]. This technique is relevant in situations in which the input is not an explicit list of constraints, but rather some data P that can be mapped to a set of constraints $g(P)$ that is much larger. such problems are called *implicit quasiconvex programs*. Our minimum-diameter problem falls in that class, since the diameter is defined by a pair of points in a n -point set, yielding $\binom{n}{2}$ constraints.

The introduction of this method was motivated by the problem of computing the Tukey depth of a set of points. It has since been used in a number of other contributions, including a new algorithm from Morin for computing the Zonoid depth of a set of points [21], and a facility location algorithm from Eppstein and Wortman [16]. We give the main result applied to quasiconvex programming, using notations from Eppstein [15].

Theorem 4 ([10]). Let \mathcal{P} be a set of input values and \mathcal{H} a set of quasiconvex functions. If there exists a function $g : 2^{\mathcal{P}} \mapsto 2^{\mathcal{H}}$ such that:

1. the quasiconvex program defined by the constraints $g(B)$ with $B \subseteq \mathcal{P}$ can be solved in $O(1)$ time whenever $|B| = O(1)$,
2. there exists an algorithm running in $O(D(|P|))$ time for deciding whether a given pair (x, y) is a feasible solution for the program defined by $P \subseteq \mathcal{P}$, i.e. whether $y \geq f(x) \forall f \in g(P)$, and such that $D(n)/n^\varepsilon$ is monotone increasing for a constant $\varepsilon > 0$,
3. there exist two constants $\alpha < 1$ and $r \in \mathbb{N}$ such that we can, in time $O(D(|P|))$, decompose the problem $P \subseteq \mathcal{P}$ into r subproblems P_1, P_2, \dots, P_r of size at most $\alpha|P|$ and such that $g(P) = \bigcup_{i=1}^r g(P_i)$

then for any $P \in \mathcal{P}$, we can solve the quasiconvex program defined by $g(P)$ in randomized expected time $O(D(|P|))$.

Thus under some simple conditions, we are able to solve a quasiconvex program in essentially the same time as to decide the feasibility of a solution. This is achieved by an extension of the standard randomized incremental algorithm [20]. In this extension, the bases are evaluated recursively, and the satisfaction-violation test is handled by the algorithm in requirement 2.

Theorem 5. Problem 4 can be solved in $O(n \log n)$ randomized expected time.

Proof. Applying Theorem 4 requires to be able to decompose a set of points P into r groups P_i , each of size at most $\alpha|P|$, such that the set of constraints encoded by P (the pairs $s, t \in P$) is the union of the corresponding sets for the P_i . Here, we can partition the set P in three equal-sized subsets, say Q, R and S . We form the following three groups of points : $P_1 = Q \cup R, P_2 = R \cup S$ and $P_3 = Q \cup S$, each group containing $2n/3$ points. We have $r = 3$ and $\alpha = 2/3$. This decomposition satisfies requirement 3 of the lemma, because each pair of points is included in at least one group.

Another thing we need from requirement 2 of the lemma is an algorithm for the satisfaction-violation test for a group of constraints. This corresponds to a decision algorithm for the travel time diameter, which, from the previous lemma, can run in $O(n \log n)$ time.

Hence from Theorem 4, the optimization problem can be solved within the same time bounds, in the randomized expected sense. \square

In view of the results in one dimension, one may wonder whether this algorithm is optimal. We leave this question open.

3.4 An approximation algorithm for minimum travel time diameter location in the plane

So far we only considered the constrained problem in which the moving walkway must be horizontal. The unconstrained problem is as follows.

Problem 5 Given a set P of n points and a real number $v > 1$, find a moving walkway $[a^*, b^*]$ of speed v minimizing the travel time diameter of P .

It is easy to verify that the constraints are not quasiconvex anymore, so we are not able to apply the previous randomized techniques. However, we can solve the problem approximately.

Theorem 6. A $(1+\varepsilon)$ -approximate solution for problem 5 can be found in $O(\frac{v}{\varepsilon} n \log n)$ randomized expected time.

Proof. For each angle $i\varepsilon/v$, with $i = 0, 1, \dots, \lfloor 2\pi v/\varepsilon \rfloor$, we rotate the set of points of this angle and solve Problem 4. Among all the solutions computed, the one minimizing the diameter is retained.

The angle of the walkway returned by this procedure and the angle of an optimal moving walkway a^*, b^* differs by at most $\alpha = \varepsilon/v$. Also, the returned solution is not worse than a walkway a, b with the same orientation and such that $a = a^*$ and $d(a, b) = d(a^*, b^*)$. We remark that $d(b, b^*) \leq \alpha d(a^*, b^*) = \frac{\varepsilon}{v} d(a^*, b^*)$. Thus for any pair s, t we have

$$\begin{aligned} d(s, a) + \frac{1}{v}d(a, b) + d(b, t) &\leq d(s, a^*) + \frac{1}{v}d(a^*, b^*) + d(b^*, t) + d(b, b^*) \\ &\leq (1 + \varepsilon) \left(d(s, a^*) + \frac{1}{v}d(a^*, b^*) + d(b^*, t) \right) \end{aligned}$$

□

Note that by combining the above trick with a grid technique (see for instance Har-Peled [18]), we can obtain a location algorithm whose complexity is of the form $O(n + \text{poly}(\varepsilon^{-1}, v))$. The main idea is to snap the points in P to a grid of size $\varepsilon^{-1} \times \varepsilon^{-1}$, and work on the point set formed by taking the centers of all nonempty cells. We can then run the above algorithm for $n = O(\varepsilon^{-2})$.

Finding an efficient exact algorithm for problem 5 is the natural extension of this work. It is easy to see that the problem is polynomial. In the unconstrained setting, every pair of points of the set defines a constraint consisting of pieces of 4-variate functions in 5-space. The pieces are algebraic functions of bounded degree. Thus, the complexity of the arrangement is polynomial, as well as the time to compute the upper envelope (see, for instance [24]) and to find the minimum value over the envelope. We leave open the question of finding a more efficient exact algorithm.

4 Conclusions

A number of variants of the moving walkway model can be considered, by combining a number of restrictions on the way the walkway is used.

In the case of *unidirectional walkways*, for instance, we assume that the walkway can only be used for going from a to b , and not for going from b to a . If we simply assume this, the diameter of a point set is trivially the geometric diameter, since for each pair of points, the walkway cannot be useful in both directions. To make the model interesting, we can for instance partition the point set in red and blue points, and suppose that we always travel from a red point to a blue point. In this model, the constraints defining the problem are all quasiconvex, whatever the orientation of the walkway is. And since the diameter algorithm can be adapted, we immediately have an $O(n \log n)$ randomized expected time algorithm for the optimal location of a (unidirectional, red-blue) walkway of arbitrary orientation.

If the use of the walkway is mandatory, and we always travel from a red to a blue point, then the problem can be shown to be a convex programming problem, solvable in linear expected time [20].

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